Some new applications of the duality method for cross diffusion equations

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Classical duality lemma

Lemma (M. Pierre, D. Schmitt): We consider Ω a bounded smooth open subset of $\mathbb{R}^N$, and $T > 0$. We also consider $\tilde{C} > 0$ a constant, and $M := M(t,x)$ an $L^1([0,T] \times \Omega)$ function.

Let $u \geq 0$ be a solution of the inequality

$$\partial_t u(t,x) - \Delta_x (M(t,x)\, u(t,x)) \leq \tilde{C},$$

satisfying the Neumann homogeneous boundary conditions.

Then

$$\int_0^T \int_\Omega u^2\, M \leq 4\,(\tilde{C}\, T + ||u(0,\cdot)||_{L^2(\Omega)})^2 \left(Cst(\Omega, T) + \int_0^T \int_\Omega M \right).$$

Classical duality lemma: perspective

Assumption: $u(t,x) \geq 0, \qquad 0 < c_1 \leq M(t,x) \leq c_2$.

$$\text{If} \qquad \partial_t u(t,x) - M(t,x)\,\Delta_x u(t,x) \leq \tilde{C},$$

multiplying by $\Delta_x u$, one gets $u \in H^2$.

$$\text{If} \qquad \partial_t u(t,x) - \nabla_x \cdot (M(t,x)\,\nabla_x u(t,x)) \leq \tilde{C},$$

multiplying by u, one gets $u \in H^1$.

$$\text{If} \qquad \partial_t u(t,x) - \Delta_x (M(t,x)\,u(t,x)) \leq \tilde{C},$$

multiplying by the solution of the dual problem (close to $\Delta_x^{-1} u$) , one gets $u \in L^2$.

Classical duality lemma: domains of application (I)

Reaction-diffusion systems: A priori estimate and existence of weak solutions for the system (LD, Fellner, Pierre, Vovelle 07)

$$\partial_t u_i - d_i \Delta_x u_i = (-1)^i (u_1 u_3 - u_2 u_4), \qquad i = 1, .., 4.$$

$$\partial_t (u_1 + u_2 + u_3 + u_4) - \Delta_x (d_1 u_1 + d_2 u_2 + d_3 u_3 + d_4 u_4) = 0,$$

$$\partial_t (u_1 + u_2 + u_3 + u_4) - \Delta_x \left(\left[\frac{d_1 u_1 + d_2 u_2 + d_3 u_3 + d_4 u_4}{u_1 + u_2 + u_3 + u_4} \right] (u_1 + u_2 + u_3 + u_4) \right) = 0,$$

$$\int_0^T \int_\Omega (u_1 + u_2 + u_3 + u_4) (d_1 u_1 + d_2 u_2 + d_3 u_3 + d_4 u_4) \le Cst_T.$$

Classical duality lemma: domains of application (II)

Coagulation-fragmentation-diffusion systems: A priori estimate and non gelation for the system (Canizo, LD, Fellner 10)

$$\partial_t u_i - d_i \Delta_x u_i = \frac{1}{2} \sum_{j=1}^{i-1} \sqrt{i-j}\, \sqrt{j}\, u_{i-j}\, u_j - \sum_{j=1}^{\infty} \sqrt{i}\, \sqrt{j}\, u_i\, u_j, \qquad i \geq 1.$$

$$\partial_t \Big(\sum_{i\geq 1} i\, u_i \Big) - \Delta_x \Big(\sum_{i\geq 1} d_i\, i\, u_i \Big) = 0,$$

$$\partial_t \Big(\sum_{i\geq 1} i\, u_i \Big) - \Delta_x \Big(\Big[\frac{\sum_{i\geq 1} d_i\, i\, u_i}{\sum_{i\geq 1} i\, u_i} \Big] \sum_{i\geq 1} i\, u_i \Big) = 0,$$

$$\int_0^T \int_\Omega \big(\sum_{i\geq 1} i\, u_i \big) \big(\sum_{i\geq 1} d_i\, i\, u_i \big) \leq Cst_T .$$

A typical cross diffusion system: Shigesada-Kawasaki-Teramoto (SKT) model (1979)

Equations for the densities of population of two competing species:

$$\partial_t u_1 - \Delta_x \left(u_1 \left[D_1 + A_{12}\, u_2 \right] \right) = (r_1 - S_{11}\, u_1 - S_{12}\, u_2)\, u_1,$$

$$\partial_t u_2 - \Delta_x \left(u_2 \left[D_2 + A_{21}\, u_1 \right] \right) = (r_2 - S_{21}\, u_1 - S_{22}\, u_2)\, u_2.$$

Neumann boundary condition (for $t \geq 0$, $x \in \partial\Omega$)

$$\nabla_x u_1(t,x) \cdot n(x) = 0, \quad \nabla_x u_2(t,x) \cdot n(x) = 0.$$

Assumption: $D_i > 0$, $A_{12}, A_{21} > 0$, $r_i > 0$, $S_{ij} > 0$.

Use of the duality lemma for cross diffusion systems of SKT type

Example of SKT-type model (LD-Lepoutre-Moussa 14)

$$\partial_t u_1 - \Delta_x \left(u_1 \left[1 + u_2^{1/2} \right] \right) = (1 - u_1^{1/4})\, u_1,$$

$$\partial_t u_2 - \Delta_x \left(u_2 \left[1 + u_1^{1/2} \right] \right) = (1 - u_2^{1/4})\, u_2.$$

Neumann boundary condition (for $t \geq 0$, $x \in \partial\Omega$)

$$\nabla_x u_1(t,x) \cdot n(x) = 0, \quad \nabla_x u_2(t,x) \cdot n(x) = 0.$$

Duality estimate (leading to existence of weak solutions):

$$\int_0^T \int_\Omega \left(u_1^2\,(1 + u_2^{1/2}) + u_2^2\,(1 + u_1^{1/2}) \right) \leq Cst_T.$$

A model of chemotaxis coming out of the modelling of multiple sclerosis

Model proposed by MC. Lombardo, R. Barresi, E. Bilotta, F. Gargano, P. Pantano, and M. Sammartino:

$$\begin{cases} \partial_t m = \Delta_x m + m - m^a - \chi \nabla_x \cdot (f(m) \nabla_x c), \\ \partial_t c = \varepsilon_0 \Delta_x c + \delta d - c + \beta m, \\ \partial_t d = r\, g(m)(1-d), \\ \nabla_x m \cdot n(x) = \nabla_x c \cdot n(x) = 0, & (t,x) \in [0,T] \times \partial\Omega, \\ m(0,x) = m_0(x), c(0,x) = c_0(x), d(0,x) = d_0(x), & , \end{cases}$$

where the parameters satisfy $a > 1$, and $\chi, \varepsilon_0, \delta, \beta, r > 0$.

Finally,

$$a = 2, \quad f(m) = \frac{m}{1+m} \quad \text{and} \quad g(m) = \frac{m^2}{1+m}.$$

Explication of the modeling

$$\begin{cases} \partial_t m = \Delta_x m + m - m^a - \chi \nabla_x \cdot \left(\dfrac{m}{1+m} \nabla_x c \right), \\ \partial_t c = \varepsilon_0 \Delta_x c + \delta d - c + \beta m, \\ \partial_t d = r \, \dfrac{m^2}{1+m} (1-d), \end{cases}$$

where

m : density of activated macrophages
c : concentration of cytokines
d : density of destroyed oligodendrocytes

Basic a priori estimates

$$\begin{cases} \partial_t m = \Delta_x m + m - m^a - \chi \nabla_x \cdot (f(m) \nabla_x c), \\ \partial_t c = \varepsilon_0 \Delta_x c + \delta d - c + \beta m, \\ \partial_t d = r\, g(m)(1-d), \end{cases}$$

By direct integration of the first equation:

$$m \in L^\infty_t(L^1_x) \cap L^a$$

,
Thanks to maximum principle:

$$d \in L^\infty$$

Thanks to maximal regularity and heat semigroup property:

$$\partial_t c,\ \partial_{x_i x_j} c \in L^a, \qquad \nabla_x c \in L^{\frac{(n+2)\,a}{n+2-a} - 0}$$

A priori estimate based on the duality method

Proposition: Assume that $|f(y)| \leq Cst\, y^b$, $|f'(y)| \leq Cst\, y^\ell$ for some $\ell > 0$ and $b < \min(1 + \frac{a}{n+2}, \frac{2a}{n+2})$.

For any (smooth) solution of the system

$$\begin{cases} \partial_t m = \Delta_x m + m - m^a - \chi \nabla_x \cdot (f(m) \nabla_x c), \\ \partial_t c = \varepsilon_0 \Delta_x c + \delta d - c + \beta m, \\ \partial_t d = r\, g(m)(1 - d), \end{cases}$$

and any $p \geq 1$,

$$||m||_{L^p([0,T]\times\Omega)} \leq Cst,$$

where the constant depends only on p, T and the parameters of the problem (Ω, $||m_0||_{L^\infty}$, $||c_0||_{W^{1,\infty}}$, $||d_0||_{L^\infty}$, a, b, ℓ, g, χ, ε_0, δ, β, r).

A priori estimate based on the duality method: idea of the proof (I)

Presented in the special case: $a = 2,\ n = 2,\ f(m) = (1+m)^{1/4}$

$$m \in L^a \qquad \Rightarrow \qquad m \in L^2$$

$$\nabla_x c \in L^{\frac{(n+2)\,a}{n+2-a}-0} \qquad \Rightarrow \qquad \nabla_x c \in L^{4-0}$$

Equation for m:

$$\partial_t m = \Delta_x m + m - m^2 - \chi \nabla_x \cdot ((1+m)^{1/4} \nabla_x c).$$

A priori estimate based on the duality method: idea of the proof (II)

We already know that $m \in L^2$, $\nabla_x c \in L^{4-0}$, and

$$\partial_t m = \Delta_x m + m - m^2 - \chi \nabla_x \cdot ((1+m)^{1/4} \nabla_x c), \qquad m(0,\cdot) = m_0.$$

One introduces $\theta := \theta(t,x) \geq 0$ such that $||\theta||_{L^{4/3}} = 1$ and $\phi := \phi(t,x) \geq 0$ defined by

$$\partial_t \phi + \Delta_x \phi = -\theta,$$

$$\nabla_x \phi \cdot n(x) = 0 \quad \text{on} \quad [0,T] \times \partial\Omega, \qquad \phi(T,\cdot) = 0.$$

Then

$$\int_0^T \int_\Omega m\,\theta = -\int_0^T \int_\Omega m\,(\partial_t \phi + \Delta_x \phi) = \int_0^T \int_\Omega \phi\,(\partial_t m - \Delta_x m) + \int_\Omega \phi(0,\cdot)\,m_0$$

$$= \int_0^T \int_\Omega \phi \left[m - m^2 - \chi \nabla_x \cdot ((1+m)^{1/4} \nabla_x c) \right] + \int_\Omega \phi(0,\cdot)\,m_0$$

$$= \int_0^T \int_\Omega \phi\,(m - m^2) + \int_\Omega \phi(0,\cdot)\,m_0 + \chi \int_0^T \int_\Omega \nabla_x \phi \cdot \left[(1+m)^{1/4} \nabla_x c \right].$$

A priori estimate based on the duality method: idea of the proof (III)

We already know that $m \in L^2$, $\quad \nabla_x c \in L^{4-0}$, and

$$\int_0^T \int_\Omega m\,\theta = \int_0^T \int_\Omega \phi\,(m - m^2) + \int_\Omega \phi(0,\cdot)\, m_0 + \chi \int_0^T \int_\Omega \nabla_x \phi \cdot \left[(1+m)^{1/4} \nabla_x c\right]$$

$$\leq \frac{1}{4}\, ||\phi||_{L^1} + ||m_0||_{L^\infty} \int_\Omega \int_0^T |\partial_t \phi| + \chi\, ||\nabla_x \phi||_{L^{2-0}}\, ||\nabla_x c||_{L^{4-0}}\, ||(1+m)^{1/4}||_{L^{4+0}}$$

$$\leq Cst \left(||\phi||_{L^1} + ||\partial_t \phi||_{L^1} + ||\nabla_x \phi||_{L^{2-0}} \right),$$

so that

$$||m||_{L^4} \leq Cst \left(||\phi||_{L^1} + ||\partial_t \phi||_{L^1} + ||\nabla_x \phi||_{L^{2-0}} \right).$$

A priori estimate based on the duality method: idea of the proof (IV)

$$||m||_{L^4} \leq Cst \left(||\phi||_{L^1} + ||\partial_t \phi||_{L^1} + ||\nabla_x \phi||_{L^{2-0}} \right).$$

Remembering that

$$\partial_t \phi + \Delta_x \phi = -\theta,$$

$$\nabla_x \phi \cdot n(x) = 0 \quad \text{on} \quad [0, T] \times \partial\Omega, \qquad \phi(T, \cdot) = 0,$$

where $||\theta||_{L^{4/3}} = 1$, we get

$$||\partial_t \phi||_{L^{4/3}} + ||\phi||_{L^{4-0}} + ||\nabla_x \phi||_{L^{2-0}} \leq Cst,$$

so that

$$||m||_{L^4} \leq Cst.$$

Existence, uniqueness and smoothness

Theorem: Assume that $|f(y)| \leq Cst\, y^b$, $|f'(y)| \leq Cst\, y^\ell$ for some $\ell > 0$ and $b < \min(1 + \frac{a}{n+2}, \frac{2a}{n+2})$. Assume also that g is C^1 and strictly positive.

Then, there exists a unique strong solution to the system

$$\begin{cases} \partial_t m = \Delta_x m + m - m^a - \chi \nabla_x \cdot (f(m) \nabla_x c), \\ \partial_t c = \varepsilon_0 \Delta_x c + \delta d - c + \beta m, \\ \partial_t d = r\, g(m)(1 - d), \end{cases}$$

with homogeneous Neumann boundary conditions and initial data $m(0,\cdot), c(0,\cdot), d(0,\cdot) \in C^2$.

Asymptotic behavior: equilibrium and linear stability

The system

$$\begin{cases} \partial_t m = \Delta_x m + m - m^a - \chi \nabla_x \cdot (f(m) \nabla c), \\ \partial_t c = \varepsilon_0 \Delta_x c + \delta d - c + \beta m, \\ \partial_t d = r\, g(m)(1-d), \end{cases}$$

with homogeneous Neumann boundary conditions has one homogeneous solution $(1, \beta + \delta, 1)$.

This equilibrium is linearly stable for $\chi > 0$ small enough, and linearly unstable when χ is large. This leads to the appearance of Turing-like patterns (cf. E. Bilotta, F. Gargano, V. Giunta, M. C. Lombardo, P. Pantano, M. Sammartino).

Asymptotic behavior: nonlinear stability

Theorem: Under the assumptions of the existence theorem, and assuming moreover that $f(1) > 0$ and

$$\chi < \chi_{crit} := \frac{4\sqrt{\varepsilon_0\,(a-1)}}{\beta\, f(1)},$$

there exists $\varepsilon > 0$ such that the unique strong solution to the system

$$\begin{cases} \partial_t m = \Delta_x m + m - m^a - \chi \nabla_x \cdot (f(m) \nabla c), \\ \partial_t c = \varepsilon_0 \Delta_x c + \delta d - c + \beta m, \\ \partial_t d = r\, g(m)(1-d), \end{cases}$$

with homogeneous Neumann boundary conditions, and initial data $m(0,\cdot), c(0,\cdot), d(0,\cdot) \in C^2$ satisfying

$$||m(0,\cdot) - 1||_{L^2} + ||c(0,\cdot) - (\beta+\delta)||_{L^2} + ||d(0,\cdot) - 1||_{L^2} \leq \varepsilon,$$

one has for all $t \geq 0$ and $p \geq 1$ some cst (independant of time) and

$$||m(t,\cdot) - 1||_{L^p} + ||c(t,\cdot) - (\beta+\delta)||_{L^p} + ||d(t,\cdot) - 1||_{L^p} \leq cst\, e^{-cst\, t}.$$