Equivariant BSD conjecture over global function fields

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*[C]elui qui m'a fasciné plus que tout autre et continue à me fasciner, c'est la **structure** cachée dans les choses mathématiques.*

(A. Grothendieck)

Plan

Review of BSD over global function fields

Artin-Hasse-Weil L-function

Equivariant BSD conjecture

Selmer complexes and the equivariant BSD formula

Review of BSD over global function fields

Setting

- K finite extension of $\mathbb{F}_p(T)$.
- X smooth projective curve with function field K.
- $\forall v \in |X|$ (places of K), K_v completion of K at v, $\mathscr{O}_v$, $\mathfrak{m}_v$, $k_v := \mathscr{O}_v/\mathfrak{m}_v$, $q_v := |k_v|$.
- A abelian variety over K
- $U \subset X$: open contained in good reduction locus.

Examples

Example I (*cf.* Milne): Constant abelian varieties

- i.e., those ab var's A/K defined over $\mathbb{F}_q \subset K$.
- Then A/K has good reduction everywhere, so any U is allowed (including $U = X$).

Example II: Some explicit equations (Ulmer)

- Let $K = \mathbb{F}_q(t)$ of characteristic p, so $X = \mathbb{P}^1_{\mathbb{F}_q}$.
- $A : y^2 + xy = x^3 - t^d$ (with $d = p^n + 1$),
  U away from zeros & poles of $\Delta = t^d(1 - 2^4 3^3 t^d)$.

Geometric analogue of the BSD conjecture

Definition: (partial) Hasse-Weil L-function of A/K

$$L_U(A;s) := \prod_{v \in |U|} \det\left(1 - q_v^{1-s}\varphi_v | V_\ell(A)\right)^{-1} \in \mathbb{Q}(p^{-s}).$$

Rank part of the BSD conjecture

$$(r_{an} :=) \operatorname{ord}_{s=1} L_U(A,s) = \operatorname{rank}_{\mathbb{Z}} A(K) (=: r_{MW}).$$

Theorem (Néron, Lang)

$A(K)$ is a finitely generated abelian group.

Geometric analogue of the BSD conjecture, cont'd

Conjectural BSD formula of the leading term

$$L_U^*(A,1) = \frac{|Ш(A/K)| \cdot \mathrm{discr}(h_{NT})}{|A(K)_{tor}| \cdot |A^t(K)_{tor}|} \cdot \mathrm{vol}\Big(\prod_{v \notin U} A(K_v)\Big),$$

where

- $Ш(A/K)$ Tate-Shafarevich group, conj'd to be finite.
- $h_{NT} : A(K)_{\mathbb{Q}} \times A^t(K)_{\mathbb{Q}} \to \mathbb{R}$ Néron-Tate ht pairing.
- $\mathrm{vol}(\prod_{v \notin U} A(K_v)) = c^{-1} \prod_{v \notin U} \mu_v(A(K_v))$ "vol term" (missing Euler factors, Tamagawa numbers,...).

Known results (in characteristic $p > 0$)

Theorem (Tate,..., Schneider,..., Kato-Trihan)

$$\mathrm{ord}_{s=1} L_U(A; s) \geqslant \mathrm{rank}_{\mathbb{Z}} A(K).$$

Furthermore, TFAE:

1. $Ш(A/K)[\ell^\infty]$ *is finite for some prime* ℓ,
2. $Ш(A/K)$ *is finite.*
3. $\mathrm{ord}_{s=1} L_U(A; s) = \mathrm{rank}_{\mathbb{Z}} A(K)$.

If these conditions hold, then the BSD formula for the leading term also holds.

Known results (in characteristic $p > 0$)

Finiteness of Ш

- **(Artin–Tate)** If A is a jacobian, then finiteness of $Ш(A/K)$ is implied by the Tate conjecture for a certain surface over a finite field.
- **(Milne)** If A is a constant abelian variety then $Ш(A/K)$ is finite.
- There are some explicit examples of elliptic curves over rational function fields where BSD can be explicitly verified. **(Ulmer, Griffon, ...)**

Artin-Hasse-Weil L-function

“Equivariant” setting

- L/K: finite Galois extension with $G = Gal(L/K)$.
- $\pi : X' \to X$ corresponding to L/K.
- A an abelian variety over K (often viewed over L)
- $U \subset X$ open, away from bad reduction locus of A and the ramification locus of π.
- $U' := \pi^{-1}(U)$ fin étale G-cover of U.

Artin-Hasse-Weil L-functions

(partial) Artin-Hasse-Weil L-function

Given $\rho \in \widehat{G}$ (nec def'd over some number field),

$$L_U(A, \rho; s) := \prod_{v \in |U|} \det(1 - q_v^{1-s} \varphi_v | V_\ell(A) \otimes \rho)^{-1}.$$

Remarks

- $L_U(A, \rho_{\mathrm{triv}}; s) = L_U(A; s)$
- $L_U(A, \rho; s) \in \mathbb{Q}(p^{-s})$ is independent of ℓ.
- (Artin formalism)

$$L_{U'}(A/L; s) = \prod_{\rho \in \widehat{G}} L_U(A, \rho; s)^{\dim(\rho)}.$$

Motivating Question for equivariant BSD conj

Arithmetic invariants of A/L are G-modules:

eg. $A(L)$, $Ш(A/L)$, Selmer groups, etc.

Question (ver 1)

How does the *BSD conjecture* 'interact' with the *Galois module structure* of arithmetic invariants?

Question (ver 2)

How is the leading term of $L_U(A, \rho; s)$ at $s = 1$ related to the *Galois module structure*?

Equivariant BSD conjecture

Leading term formula revisited (sanity check)

$$L_U^*(A,1) = \underbrace{\frac{|\text{Ш}(A/K)| \cdot \mathrm{discr}(h_{NT})}{(|A(K)_{tor}| \cdot |A^t(K)_{tor}|}}_{\text{(A)}} \cdot \underbrace{\mathrm{vol}(\prod_{v \in S} A(K_v))}_{\text{(B)}},$$

- $L_U^*(A,1) \in (\log p)^{r_{an}} \cdot \mathbb{Q}_{>0}^{\times}$ (**NB:** $L_U(A,s) \in \mathbb{Q}(p^{-s})$).
- Since $(\log p)^{-r_{MW}} h_{NT}$ is rat'l, RHS$\in (\log p)^{r_{MW}} \cdot \mathbb{Q}_{>0}^{\times}$.
- This formula has a cohomological meaning:
  - LHS: Lefschetz trace formula.
  - (A)&(B): "Euler char." of Selmer gp and coho of vb.

Equivariant leading term

L-values and $K_1(\mathbb{Q}[G])$

- For any field F, we have $K_1(F) = F^\times$.
- Have $K_1(\mathbb{Q}[G]) \subset Z(\mathbb{Q}[G])^\times$.
- $\rho : G \to GL_n(F)$ with $F \supset \mathbb{Q} \rightsquigarrow \chi_\rho : K_1(\mathbb{Q}[G]) \to F^\times$.

Proposition (Burns-Kakde-K)

Given $(A; L/K)$ as before, there exists a natural element $\mathcal{L}_U(A; L/K) \in K_1(\mathbb{Q}[G])$ ("leading term in $\mathbb{Q}[G]$-coeff.") such that for any $\rho \in \widehat{G}$ we have

$$\chi_\rho : \mathcal{L}_U(A; L/K) \longmapsto L_U^*(A, \rho; 1)/(\log p)^{r_\rho},$$

where r_ρ is the multiplicity of ρ in $A(L)$.

Relative K_0 for group rings

F finite extension of $\mathbb{Q}$ (or $\mathbb{Q}_\ell$); and $\mathscr{O} \subset F$ "ring of int's".

- $K_0(\mathscr{O}, F) = F^\times / \mathscr{O}^\times$; i.e., the set of $\mathscr{O}$-lattices in F.
  E.g., $K_0(\mathbb{Z}, \mathbb{Q}) = \mathbb{Q}^\times / \langle \pm 1 \rangle \cong \mathbb{Q}^\times_{>0}$.
- $\partial : K_1(F) \to K_1(\mathscr{O}, F)$ is just $F^\times \twoheadrightarrow F^\times / \mathscr{O}^\times$.

For group rings of G, we have

- a notion of $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$, equipped with
  the boundary map $\partial : K_1(\mathbb{Q}[G]) \to K_0(\mathbb{Z}[G], \mathbb{Q}[G])$.
- For $\rho \in \widehat{G}$ we have $\chi_\rho : K_0(\mathbb{Z}[G], \mathbb{Q}[G]) \to K_0(\mathscr{O}, F)$.

Refined Euler characteristic

Let $C^\bullet$ be a perfect complex of $\mathbb{Z}$-modules.
If all $\mathrm{H}^i(C^\bullet)$ are torsion, then

$$\chi^{ref}(C^\bullet) := \prod |\mathrm{H}^i(C^\bullet)|^{(-1)^i} \in \mathbb{Q}_{>0} \cong K_0(\mathbb{Z}, \mathbb{Q})$$

Can generalise χ^{ref} for $C^\bullet$ with "height pairing" h.

Equivariant refined Euler char (Burns, et al.)

Defined $\chi^{ref}(C^\bullet, h) \in K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ for perfect $\mathbb{Z}[G]$-complex with G-equiv "height pairing" h.

Equivariant BSD conjecture: statement

Recap

- $\mathcal{L}_U(A; L/K) \in K_1(\mathbb{Q}[G])$ interpolating the leading terms of Artin-Hasse-Weil L-functions.
- $\partial : K_1(\mathbb{Q}[G]) \to K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ boundary map.

Geometric equiv BSD conj (Burns-Kakde-K)

$\partial(\mathcal{L}_U(A; L/K)) \in K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ can be expressed in terms of the **refined Euler characteristics** of

- "arithmetic cohomology" and
- coherent cohomology of some vector bundle,

interpolating the BSD formula for $L_U^*(A, \rho; 1)$.

Main Result

Theorem (Burns-Kakde-K)

The geometric equivariant BSD holds for $(A; L/K)$ if

- $\text{Ш}(A/L)[\ell^\infty]$ *is finite for some* ℓ,
- *A has semi-stable reduction at all place of* K, *and*
- L/K *is tame at all places.*

Remarks on the conditions

- Finiteness of Ш is equivalent to BSD for A/L, and known for *constant abelian varieties*, etc.
- Without semistable reduction, we cannot define suitable integral p-adic cohomology.
- Tameness of L/K: Galois descent.

More Remarks on Main Result

Theorem (Burns-Kakde-K)

The geometric equivariant BSD holds for $(A; L/K)$ if

- $Ш(A/L)[\ell^\infty]$ *is finite for some* ℓ,
- *A has semi-stable reduction at all place of K, and*
- *L/K is tame at all places.*

Previous and other results

- **(Trihan–Vauclair)** conditional result when L/K is unramified everywhere.
- **(Lai–Longhi–Tan–Trihan)** when A is constant ordinary (and L/K is abelian).

Our proof is by “refining” the argument of Kato-Trihan.

Selmer complexes and the equivariant BSD formula

Slogans

- The “equivariant BSD formula” is roughly of the form $\partial(\mathcal{L}_U(A; L/K)) = \chi^{ref}(\text{Selmer cplx}) - (\text{geom term})$.
- This formula is a consequence of

$$\begin{aligned}(\text{Selmer complex}) &\otimes_{\mathbb{Z}} \mathbb{Z}_\ell \\ &= \text{“}ker\text{”}(1 - \varphi| \text{ integral } \ell\text{-adic geom coho})\end{aligned}$$

and (!) $\mathbb{Z}_\ell[G]$-perfectness of ℓ-adic geom. coho. cplx.

(The key difficulty is to handle $\ell = p$.)

Selmer Complex in the "Toy Case"

Given $(A; L/K)$, assume the following (as a "toy case"):

- A/K has good reduction everywhere, and L/K is unramified everywhere.
- $U = X$, and A/K extends to an ab var $\mathcal{A}/X$ (and similarly, A/L extends to $\mathcal{A}'/X'$).

Theorem (Kato-Trihan, Burns-Kakde-K.)

If $Ш(A/L)$ finite then $\exists\ \mathbb{Z}[G]$-perfect complex $SC := SC(A; L/K)$ ("Selmer complex") satisfying:

- $\mathrm{H}^i(SC) = 0$ if $i \notin \{0, 1, 2\}$.

Properties of Selmer cplx $SC(A; L/K)$ (con'd)

Theorem on $SC(A; L/K)$ (con'd)

If $\text{Ш}(A/L)$ finite then $\exists\ \mathbb{Z}[G]$-perfect complex SC ("Selmer complex") satisfying:

- $\mathrm{H}^i(SC) = 0$ if $i \notin \{0, 1, 2\}$.
- $\mathrm{H}^2(SC) = A(L)^\vee_{\mathrm{tor}}$, $\mathrm{H}^0(SC) = A^t(L)$ and $\mathrm{H}^1(SC) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} = \mathrm{Sel}_{\mathbb{Q}/\mathbb{Z}}(A/L)^\vee$
- $SC \otimes \mathbb{Z}_\ell =$"$\ker(1 - \varphi)$" of some geometric coho
    - If $\ell \neq p$ then $R\Gamma_{\text{ét},c}(\overline{X}', T_\ell(\mathcal{A}')(-1))$.
    - If $\ell = p$ then some integral crystalline cohomology.

Remarks on Selmer complex $SC(A; L/K)$

- Constr of $SC(A; L/K) \otimes \widehat{\mathbb{Z}}$ is due to Kato-Trihan.
- We show it is $\widehat{\mathbb{Z}}[G]$-perfect if $Ш(A/L)[\ell^\infty]$ is finite, and extract a $\mathbb{Z}[G]$-lattice.
- If G is triv and A has good red'n everywhere,

$$\chi^{ref}(SC, h) = \frac{|Ш(A/K)| \cdot \mathrm{discr}(h)}{|A(K)_{tor}| \cdot |A^t(K)_{tor}|} \in \mathbb{Q}^\times_{>0} \cong K_0(\mathbb{Z}, \mathbb{Q}).$$

- In general, need to modify the constr of $SC(A; L/K)$ to ensure $\mathbb{Z}[G]$-perfectness (**cf.** Kato-Trihan).

Equiariant BSD formula: "Toy Case"

Main Result in the "Toy Case"

If $U = X$ (in part, A good red'n and L/K unram), then

$$\partial(\mathcal{L}_X(A; L/K)) = \chi^{\mathrm{ref}}(SC(A; L/K), h) - \chi^{\mathrm{ref}}(R\Gamma(X', \mathrm{Lie}(\mathcal{A}'))) + (\text{sign correction term}).$$

Main ingredients: Etale descent

- $\mathbb{Z}_\ell[G]$-perfectness of $R\Gamma_{\text{ét},c}(\overline{X}', T_\ell(\mathcal{A}')(-1))$ ($\ell \neq p$).
- $\mathbb{Z}_p[G]$-perfectness of the crystalline and coherent cohomology complexes.

  (Consequence of étale descent as L/K unramified.)

Equivariant BSD formula: semi-stable tame case

Theorem (Burns-Kakde-K.)

If $Ш(A/L)$ is finite, A/K is semi-stable and L/K is tame, then in $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ we have

$$\partial(\mathcal{L}_U(A; L/K)) = \chi^{\mathrm{ref}}(SC_U(A; L/K), h)$$
$$- \chi^{\mathrm{ref}}(R\Gamma(X', \mathrm{Lie}(\mathcal{A}')(-E')) + (\text{sign correction term}).$$

Remarks

- $SC_U(A; L/K)$: $SC(A, L/K)$ modified at $X \setminus U$.
- **(Easy)** $R\Gamma_{\text{ét},c}(\overline{X}', V_\ell(\mathcal{A}'))$ is $\mathbb{Z}_\ell[G]$-perfect $\forall \ell \neq p$.
- **(Main technical result)** $\mathbb{Z}_p[G]$-perf of some int log crys coho if A/K semistable & L/K is tame(!).

Wild speculation on wild ramification

Work in progress

If A/K is ordinary (but possibly admitting non-ord reduction) and L/K arbitrary, *what can we say about equiv BSD for* $(A; L/K)$ *from unit-root L-functions?*

Further wild questions

Assume A/K has semi-stable reduction everywhere, and L/K is arbitrary (possibly with wild ramification).

How can one to extract $\mathbb{Z}_p[G]$*-perfect complex from integral p-adic cohomology?*

Thank you for your attention!