Chebyshev's bias and sums of two squares

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Prime numbers in congruence classes

We denote $\pi(x) = \#\{p \leq x : p \text{ prime }\}$,
and for $(a,q)=1$, $\pi(x;q,a) = \#\{p \leq x : p \text{ prime }, p \equiv a \bmod q\}$.

Theorem (Prime Number Theorem in Arithmetic Progressions)

Let $a, q \in \mathbf{N}$, *with* $(a,q)=1$, *then*

$$\pi(x;q,a) \sim \frac{1}{\phi(q)}\pi(x),$$

where $\phi(q) = \#(\mathbf{Z}/q\mathbf{Z})^{\times}$.

Idea of proof of the PNTAP for $q = 4$

Let χ_4 be the non-principal Dirichlet character modulo 4,

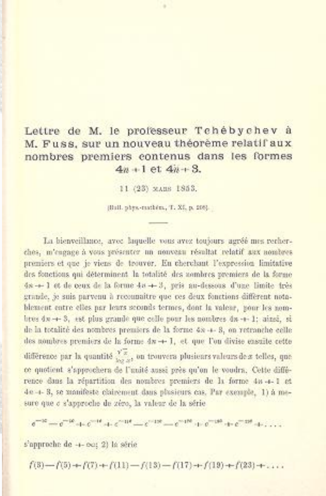
$$\chi_4(m) = \begin{cases} 0 & \text{if } m \text{ is even,} \\ 1 & \text{if } m \equiv 1 \pmod 4, \\ -1 & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

We have

$$\begin{aligned} \pi(x;4,\pm 1) &= \sum_{2<p\le x} \frac{1 \pm \chi_4(p)}{2} \\ &= \frac{1}{2} \underbrace{\sum_{2<p\le x} 1}_{\zeta \text{ has a pole at } s=1} \pm \frac{1}{2} \underbrace{\sum_{2<p\le x} \chi_4(p)}_{L(s,\chi_4) \text{ does not have a pole or zero on } \operatorname{Re}(s)=1} \\ &= \frac{1}{2}\pi(x) + o(\pi(x)). \end{aligned}$$

Chebyshev's observation

Pafnuty Lvovich Chebyshev



Lettre de M. le professeur Tchébychev à M. Fuss, sur un nouveau théorème relatif aux nombres premiers contenus dans les formes $4n+1$ et $4n+3$.

11 (23) mars 1853.

(Bull. phys-mathém., T. XI, p. 208).

La bienveillance, avec laquelle vous avez toujours agréé mes recherches, m'engage à vous présenter un nouveau résultat relatif aux nombres premiers et que je viens de trouver. En cherchant l'expression limitative des fonctions qui déterminent la totalité des nombres premiers de la forme $4n+1$ et de ceux de la forme $4n+3$, pris au-dessous d'une limite très grande, je suis parvenu à reconnaître que ces deux fonctions diffèrent notablement entre elles par leurs seconds termes, dont la valeur, pour les nombres $4n+3$, est plus grande que celle pour les nombres $4n+1$; ainsi, si de la totalité des nombres premiers de la forme $4n+3$, on retranche celle des nombres premiers de la forme $4n+1$, et que l'on divise ensuite cette différence par la quantité $\frac{\sqrt{x}}{\log x}$, on trouvera plusieurs valeurs de x telles, que ce quotient s'approchera de l'unité aussi près qu'on le voudra. Cette différence dans la répartition des nombres premiers de la forme $4n+1$ et $4n+3$, se manifeste clairement dans plusieurs cas. Par exemple, 1) à mesure que c s'approche de zéro, la valeur de la série

$$e^{-3c}-e^{-5c}+e^{-7c}+e^{-11c}-e^{-13c}-e^{-17c}+e^{-19c}+e^{-23c}+\ldots$$

s'approche de $+\infty$; 2) la série

$$f(3)-f(5)+f(7)+f(11)-f(13)-f(17)+f(19)+f(23)+\ldots$$

Chebyshev's observation

“ En cherchant l'expression limitative des fonctions qui déterminent la totalité des nombres premiers de la forme $4n+1$ et de ceux de la forme $4n+3$, pris au dessous d'une limite très grande, je suis parvenu à reconnaître que ces deux fonctions diffèrent notablement par leur second termes, dont la valeur, pour les nombres $4n+3$, est plus grande que celle pour les nombres $4n+1$. ”

We have

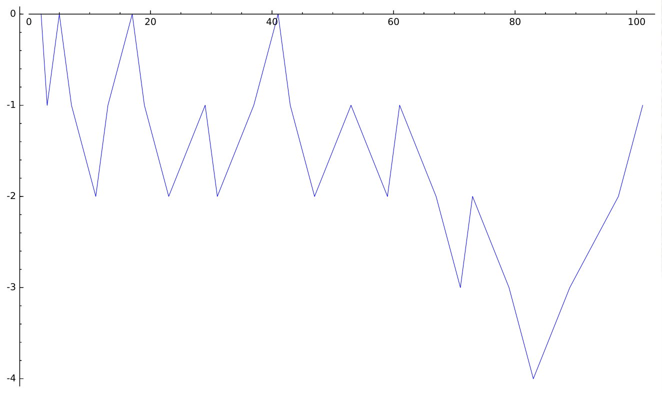
$$\pi(x;4,3) \sim \pi(x;4,1)$$

as $x \to \infty$, but, studying the secondary terms, we *observe* that,

$$\pi(x;4,3) \geq \pi(x;4,1).$$

Looking at the first prime numbers

Graphically : $\pi(x;4,1) - \pi(x;4,3)$ (go up each time you meet a prime in the team $4n+1$ and go down if it is in the team $4n+3$).



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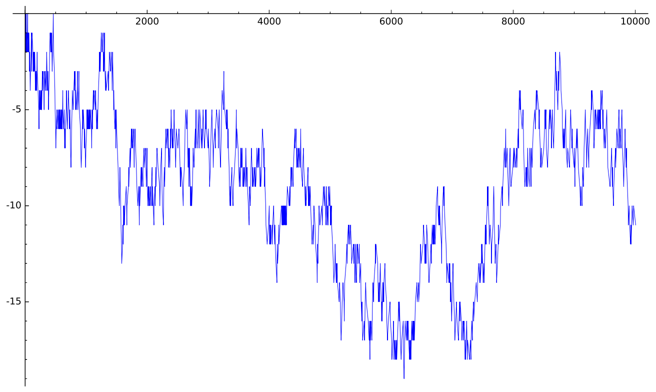

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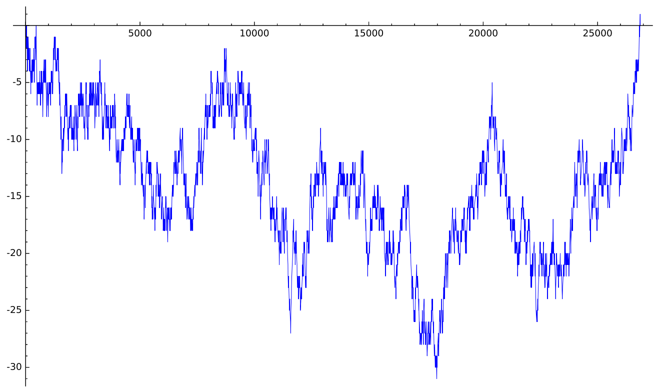

The team $4n+1$ takes the lead for the first time at 26861.

Looking at the first prime numbers

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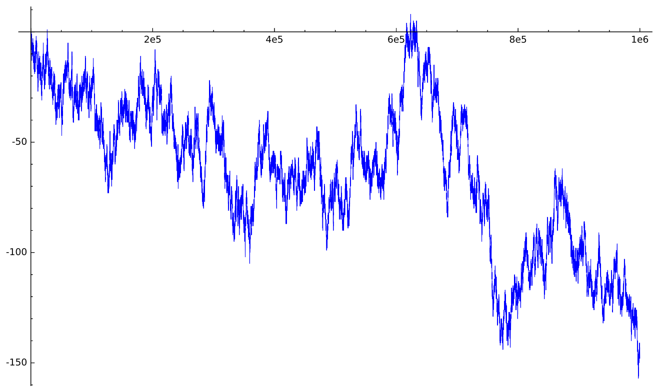


The team $4n+1$ takes the lead for the first time at 26861.
The team $4n+1$ takes the lead for the second time at 616841.
Littlewood (1914) : there exist arbitrarily large x such that $\pi(x;4,1) > \pi(x;4,3)$.

Why this bias ?

Let us study more precisely the function $\pi(x;4,1) - \pi(x;4,3) = \sum_{p\leq x} \chi_4(p)$. Using Mellin transform (Perron's formula), we obtain the explicit formula for the Dirichlet L-function $L(s,\chi_4) = \prod_p (1-\chi_4(p)p^{-s})^{-1}$:

$$\sum_{k\geq 1}\sum_{p^k\leq x} \chi_4(p^k)\log(p) = -\sum_{\substack{\rho, L(\rho,\chi_4)=0 \\ |\mathrm{Im}(\rho)|\leq T}} \frac{x^\rho}{\rho} + R(x,T)$$

$$\begin{aligned}\sum_{p\leq x} \chi_4(p)\log(p) &= -x^{\frac{1}{2}} \sum_{\substack{\rho, L(\rho,\chi_4)=0 \\ |\mathrm{Im}(\rho)|\leq T}} \frac{x^{\rho-\frac{1}{2}}}{\rho} - \sum_{k\geq 2}\sum_{p^k\leq x} \chi_4(p^k)\log(p) \\ &\quad + R(x,T) \\ &= -x^{\frac{1}{2}} \sum_{\substack{\rho, L(\rho,\chi_4)=0 \\ |\mathrm{Im}(\rho)|\leq T}} \frac{x^{\rho-\frac{1}{2}}}{\rho} - x^{\frac{1}{2}} + R(x,T)\end{aligned}$$

where $R(x,T)$ is small in average for a good choice of T.

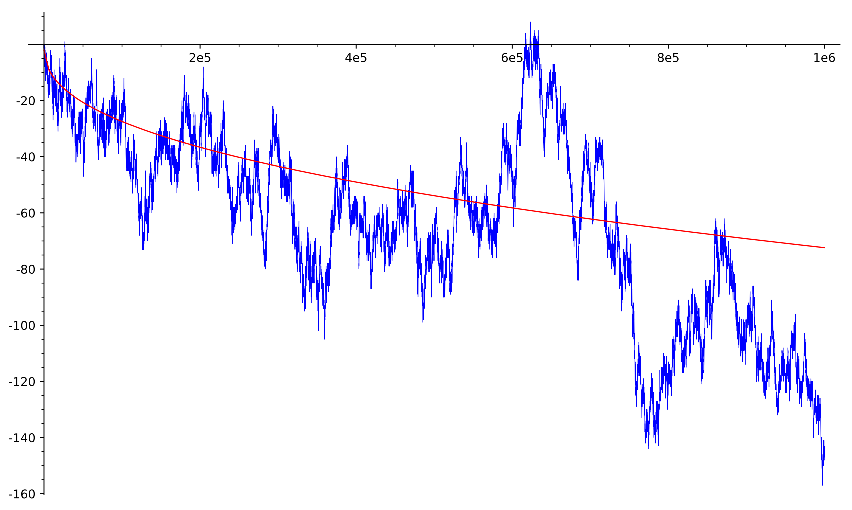
Why this bias ?

In the end, assuming the Generalized Riemann Hypothesis, we obtain that

$$\sum_{p \leq x} \chi_4(p) = \frac{x^{\frac{1}{2}}}{\log x} \left(-1 - \sum_{\substack{|\gamma| \leq T \\ L(\frac{1}{2}+i\gamma, \chi_4)=0}} \frac{e^{i\gamma \log x}}{\frac{1}{2} + i\gamma} + R(x,T) \right)$$

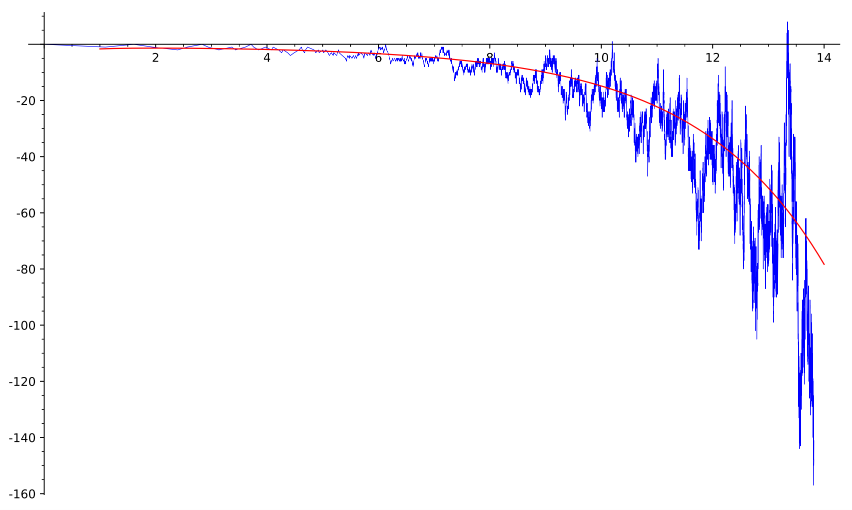
where $R(x,T)$ is small in average for a good choice of T.

Comparison with the mean value and change of variable



$\sum_{p\leq x}\chi_4(p)$, $\frac{-\sqrt{x}}{\log x}$

Comparison with the mean value and change of variable



$\sum_{p\leq e^y} \chi_4(p)$, $\frac{-e^{\frac{y}{2}}}{y}$

Limiting distribution

Theorem (Rubinstein–Sarnak, 1994)

Under the Generalized Riemann Hypothesis, the function $E : y \mapsto \frac{y}{e^{y/2}} \sum_{p \leq e^y} \chi_4(p)$ *admits a limiting distribution, with mean value equal to* -1.

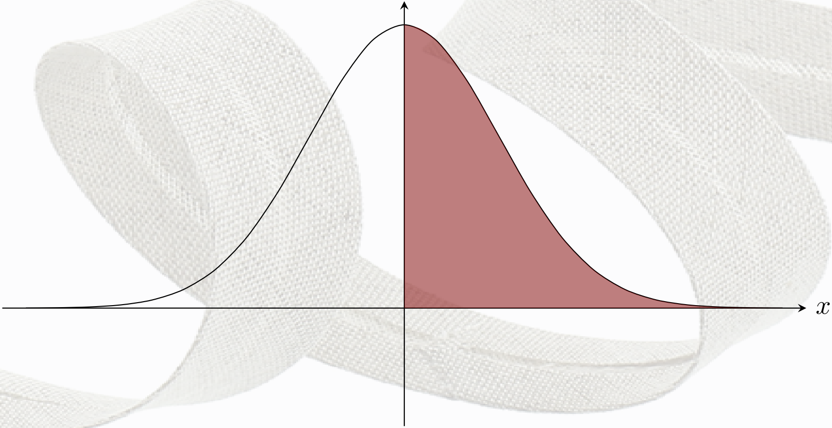
Definition

$E : \mathbf{R} \to \mathbf{R}$ has a limiting distribution μ if

$$\lim_{Y \to \infty} \frac{1}{Y} \int_2^Y g(E(y))dy = \int_{\mathbf{R}} g(t)d\mu(t)$$

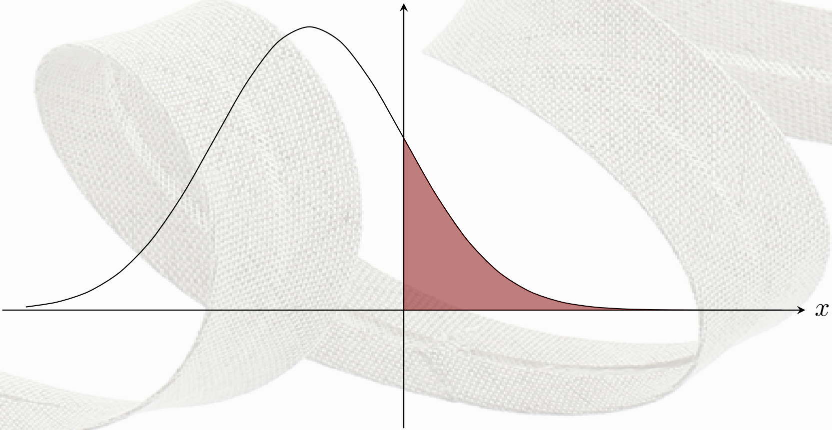
for all bounded Lipschitz-continuous function g on $\mathbf{R}$.
Then the probability of E to be positive (called *the bias* of E towards positive values) is measured by $\text{bias}(\{E(y) > 0\}) = \mu((0, \infty))$.

Mean value of the limiting distribution



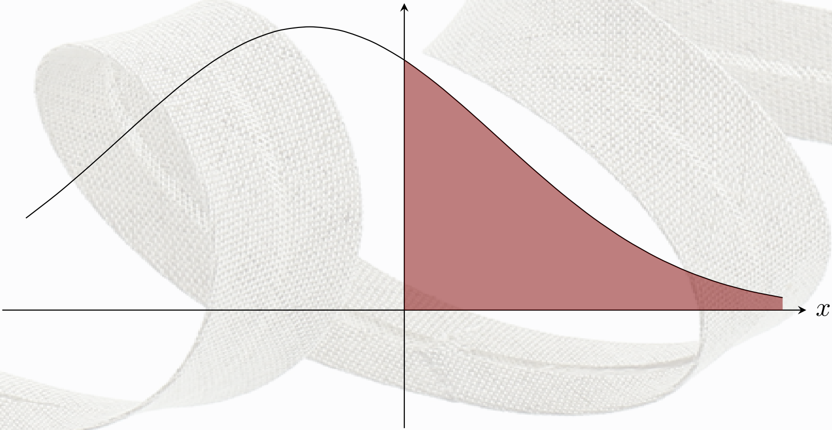
mean value $\approx 0 \rightsquigarrow \text{bias}(\{E(y) > 0\}) \approx \frac{1}{2}$.

Mean value of the limiting distribution



$$\text{mean value} < 0 \rightsquigarrow \text{bias}(\{E(y) > 0\}) < \tfrac{1}{2}.$$

Mean value of the limiting distribution



mean value < 0, and large variance $\rightsquigarrow$ bias($\{E(y) > 0\}$) $< \frac{1}{2}$ but close to $\frac{1}{2}$.

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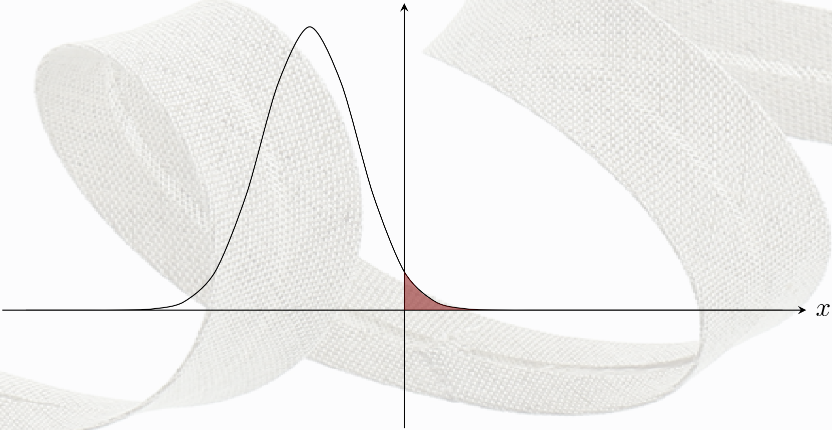

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Under the Generalized Riemann Hypothesis, the function $E : y \mapsto \frac{y}{e^{y/2}} \sum_{p \leq e^y} \chi_4(p)$ *admits a limiting distribution, with mean value equal to* -1*. Assuming also the Linear Independence of the zeros, the bias of* E *towards positive values is* ≈ 0.0041*.*

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Then the bias of E towards positive values is measured by $\mathrm{bias}(\{E(y) > 0\}) = \mu((0, \infty))$.

More properties of this limiting distribution and on the bias under weaker hypotheses : Kaczorowski, Rubinstein–Sarnak, Kaczorowski–Ramaré, Martin–Ng, D.

Generalizations

Can be adapted for any L-function

- Riemann ζ function $\rightsquigarrow$ study the difference between $\pi(x)$ and $\mathrm{li}(x)$ (Phragmén, Littlewood, Ingham, Wintner,...) ;
- Dirichlet L-functions (Prime Number Theorem in Arithmetic Progressions) $\rightsquigarrow$ races between any two congruences classes (Shanks, Knapowski–Turán, Kaczorowski, Rubinstein–Sarnak, Martin,...), or any two sets of congruence classes (Fiorilli), and between more than two contestants (Ford, Konyagin, Lamzouri,...) ;
- Artin L-functions (Chebotarev's Theorem) $\rightsquigarrow$ generalizations through Galois theory to primes in congruence classes of the Galois group of some extension (Ng, Fiorilli–Jouve, Bailleul) ;
- Hasse–Weil L-functions (Mazur's bias) $\rightsquigarrow$ bias in the Sato–Tate Conjecture (Sarnak, Fiorilli) ;
- Your favourite L-function $\rightsquigarrow$ distribution of its coefficients at prime values (Akbary–Ng–Shahabi, D.) ;
- Your favourite infinite set of L-functions (Sarnak, D. this talk).

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Writing a prime as a sum of two squares

If $p \equiv 1 \pmod 4$, there us a unique way to write $p = a^2 + 4b^2$ with a, b integers, $a, b > 0$.

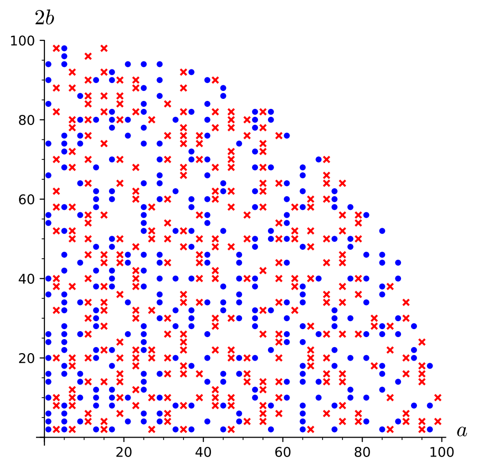
Theorem (Hecke)

We have

$$\#\{p \leq x : p = a^2 + 4b^2, |a| \equiv 1 \pmod 4\} \sim \#\{p \leq x : p = a^2 + 4b^2, |a| \equiv 3 \pmod 4\} \sim \frac{1}{4}\pi(x).$$

What about the secondary term ?

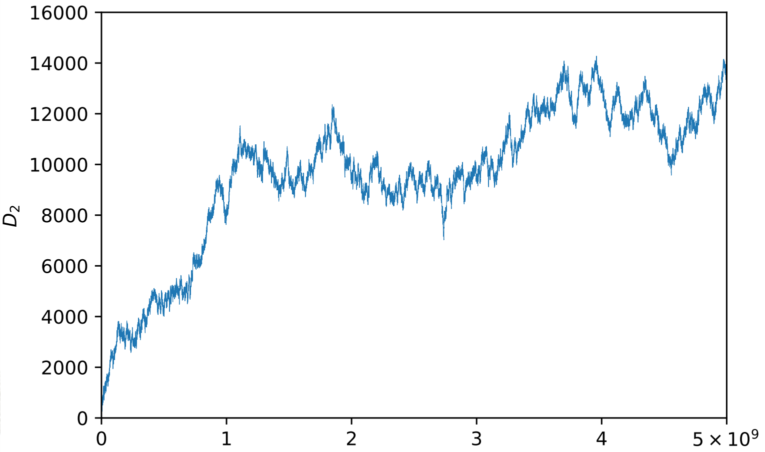
Writing a prime as a sum of two squares



Prime numbers with $p = a^2 + 4b^2 < 10000$,

$a, b > 0$, blue dots : $a \equiv 1 \pmod 4$, red crosses : $a \equiv 3 \pmod 4$.

The race



$$\#\{p \le x : p = a^2 + 4b^2, |a| \equiv 1 \pmod 4\} - \#\{p \le x : p = a^2 + 4b^2, |a| \equiv 3 \pmod 4\}$$

Interpretation

In $\mathbf{Z}[i]$, the prime numbers $p \equiv 1 \pmod 4$ split as $p = (a + i2b)(a - i2b)$. We can choose $b > 0$, $a - 2b \equiv 1 \pmod 4$. Then this defines uniquely the angle of the Gaussian prime :

$$\theta_p = \arg(a + 2ib) \in [0, \pi].$$

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So $|a| \equiv 1 \pmod 4 \Leftrightarrow \begin{cases} p \equiv 1 \pmod 8 & \text{and } a > 0 \text{ i.e. } \theta_p \in (0, \frac{\pi}{2}) \\ p \equiv 5 \pmod 8 & \text{and } a < 0 \text{ i.e. } \theta_p \in (\frac{\pi}{2}, \pi) \end{cases}$

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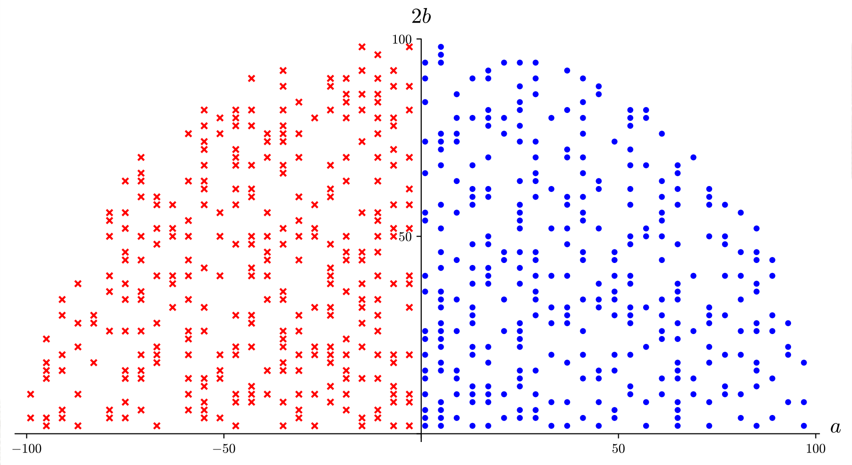
similarly $|a| \equiv 3 \pmod 4 \Leftrightarrow \begin{cases} p \equiv 1 \pmod 8 & \text{and } \theta_p \in (\frac{\pi}{2}, \pi) \\ p \equiv 5 \pmod 8 & \text{and } \theta_p \in (0, \frac{\pi}{2}). \end{cases}$

Define the modified angle

$$\tilde{\theta}_p = \begin{cases} \theta_p & \text{if } p \equiv 1 \pmod 8 \\ \pi - \theta_p & \text{if } p \equiv 5 \pmod 8). \end{cases}$$

Modified angles



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Interpretation – Hecke characters

Let ψ be the Hecke character on the multiplicative groups of fractional ideals of $\mathbf{Z}[i]$ modulo (4) defined by

$$\psi((\alpha)) = \begin{cases} \left(\frac{\alpha}{|\alpha|}\right) & \text{if } \alpha \equiv 1 \bmod (4) \\ -\left(\frac{\alpha}{|\alpha|}\right) & \text{if } \alpha \equiv 3+2i \bmod (4) \\ 0 & \text{if } (\alpha,(4)) \neq 1. \end{cases}$$

So that for any unramified splitting rational prime $(p) = \mathfrak{p}\overline{\mathfrak{p}}$, one has, for any $m \in \mathbf{N}_{>0}$

$$2\cos(m\tilde{\theta}_p) = \psi^m(\mathfrak{p}) + \psi^m(\overline{\mathfrak{p}})$$

and for inert primes $p \equiv 3 \pmod 4$, one has $(p) = \mathfrak{p}$

$$\psi((p)) = -1 \rightsquigarrow \tilde{\theta}_p = \pi.$$

Explicit formula

The L-function associated to ψ^m is

$$L(s,\psi^m) = \prod_{\mathfrak{p}} (1-\psi^m(\mathfrak{p})N\mathfrak{p}^{-s})^{-1}$$

$$= \prod_{p\equiv 1 \pmod 4} (1 - 2\cos(m\tilde{\theta}_p)p^{-s} + p^{-2s})^{-1} \prod_{p\equiv 3 \pmod 4} (1-(-1)^m p^{-2s})^{-1}.$$

We have the explicit formula

$$\sum_{k\geq 1}\sum_{N\mathfrak{p}^k\leq x} \psi^m(\mathfrak{p}^k)\log(N\mathfrak{p}) = -\sum_{\substack{\rho, L(\rho,\psi^m)=0 \\ |\mathrm{Im}(\rho)|\leq T}} \frac{x^\rho}{\rho} + R_m(x,T)$$

$$\sum_{\substack{p\leq x \\ p\equiv 1 \pmod 4}} 2\cos(m\tilde{\theta}_p)\log(p) = -x^{\frac{1}{2}} \sum_{\substack{\rho, L(\rho,\psi^m)=0 \\ |\mathrm{Im}(\rho)|\leq T}} \frac{x^{\rho-\frac{1}{2}}}{\rho}$$

$$- \sum_{\substack{p\leq\sqrt{x} \\ p\equiv 1 \pmod 4}} 2\cos(2m\tilde{\theta}_p)\log(p) - \sum_{\substack{p\leq\sqrt{x} \\ p\equiv 3 \pmod 4}} (-1)^m 2\log(p) + R_m(x,T)$$

where $R_m(x,T)$ is small in average for a good choice of T.

First approximation

In the end, assuming the Generalized Riemann Hypothesis, we obtain that

$$\sum_{\substack{p \leq x \\ p \equiv 1 \pmod 4}} \cos(m\tilde{\theta}_p) = \frac{x^{\frac{1}{2}}}{\log x} \left(-\frac{(-1)^m}{2} - \frac{1}{2} \sum_{\substack{|\gamma| \leq T \\ L(\frac{1}{2}+i\gamma, \psi^m)=0}} \frac{e^{i\gamma \log x}}{\frac{1}{2} + i\gamma} + R_m(x,T) \right)$$

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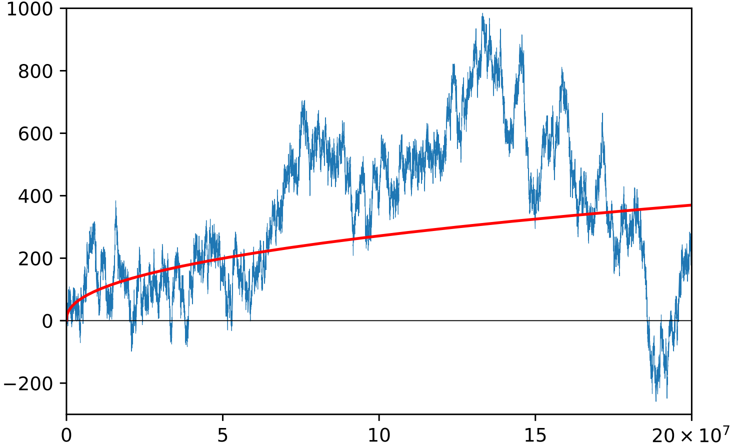
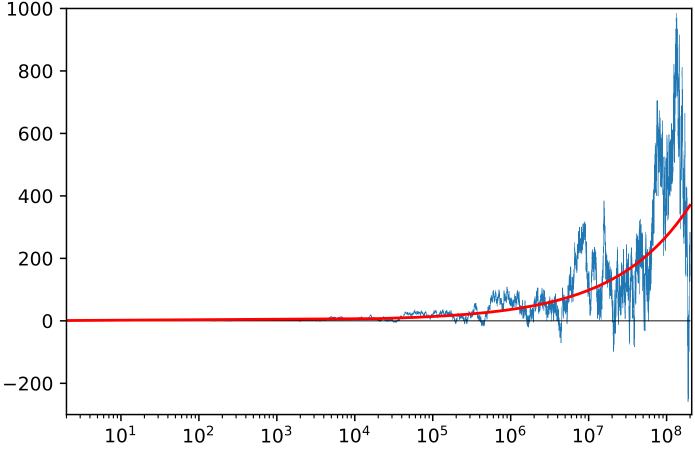
Comparison with the mean value and change of variable



$\frac{\log x}{\sqrt{x}} \sum_{p \leq x} \cos(\tilde{\theta}_p)$, $\frac{\sqrt{x}}{2 \log x}$

Comparison with the mean value and change of variable



$\frac{y}{e^{y/2}} \sum_{p \leq e^y} \cos(\tilde{\theta}_p), \; \frac{e^{\frac{y}{2}}}{2y}$

Limiting distribution

As in [Rubinstein–Sarnak, 1994], we deduce that :

Theorem

Under the Generalized Riemann Hypothesis, the function $y \mapsto \frac{y}{e^{y/2}} \sum_{p \leq e^y} \cos(\tilde{\theta}_p)$ *admits a limiting distribution with mean value equal to* $\frac{1}{2}$.

Definition

$E : \mathbf{R} \to \mathbf{R}$ has a limiting distribution μ if

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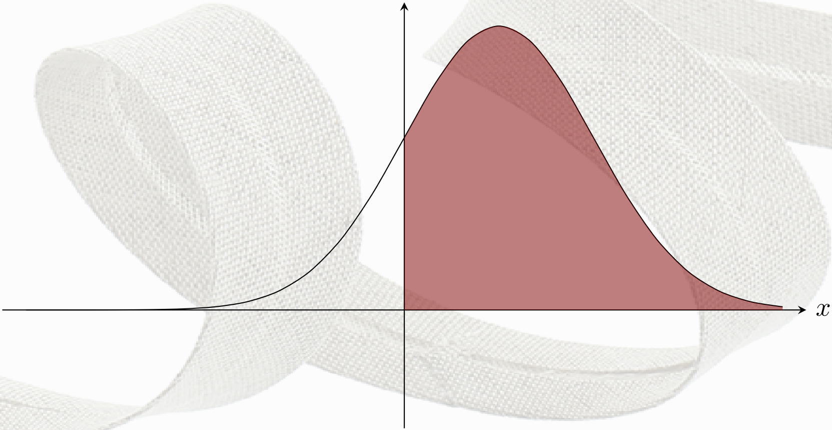
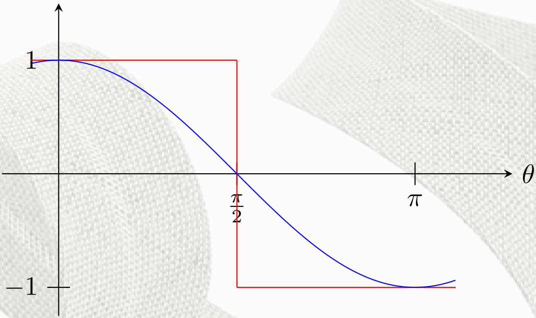
Mean value of the limiting distribution



mean value $> 0 \rightsquigarrow \text{bias}(\{E(y) > 0\}) > \frac{1}{2}$.

Approximation of the step function



$\mathbf{1}_{(0,\frac{\pi}{2})} - \mathbf{1}_{(\frac{\pi}{2},\pi)}$, $\cos(x)$.

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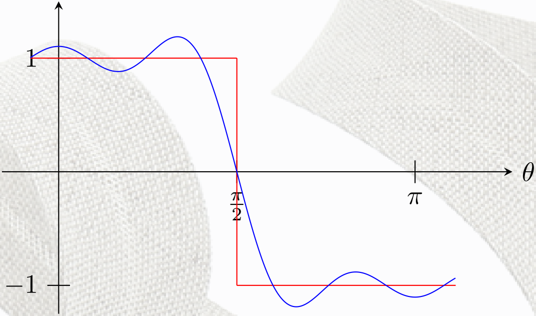


$\mathbf{1}_{(0,\frac{\pi}{2})} - \mathbf{1}_{(\frac{\pi}{2},\pi)}$, $\frac{4}{\pi}\cos(x) - \frac{4}{3\pi}\cos(3x) + \frac{4}{5\pi}\cos(5x)$.

Theorem

Under the Generalized Riemann Hypothesis, the function $y \mapsto \frac{y}{e^{y/2}} \sum_{p \leq e^y} \left(\frac{4}{\pi}\cos(\tilde{\theta}_p) - \frac{4}{3\pi}\cos(3\tilde{\theta}_p) + \frac{4}{5\pi}\cos(5\tilde{\theta}_p) \right)$ *admits a limiting distribution with mean value equal to* $\frac{1}{2}\left(\frac{4}{\pi} - \frac{4}{3\pi} + \frac{4}{5\pi}\right) > 0$.

Summing the explicit formulas

We have

$$\sum_{\substack{p\le x\\ p\equiv 1 \pmod 4}} \sum_{m\ge 1} C_m \cos(m\tilde{\theta}_p)$$

$$= \frac{x^{\frac{1}{2}}}{\log x} \sum_{m\ge 1} C_m \left(-\frac{(-1)^m}{2} - \frac{1}{2} \sum_{\substack{|\gamma|\le T\\ L(\frac{1}{2}+i\gamma,\psi^m)=0}} \frac{e^{i\gamma \log x}}{\frac{1}{2}+i\gamma} + R_m(x,T) \right)$$

where $R_m(x,T)$ are small in average for a good choice of T.

Summing the explicit formulas

We have

$$\sum_{\substack{p\le x\\ p\equiv 1 \pmod 4}} \sum_{m\ge 1} C_m \cos(m\tilde{\theta}_p)$$

$$= \frac{x^{\frac12}}{\log x} \sum_{m\ge 1} C_m \left(-\frac{(-1)^m}{2} - \frac12 \sum_{\substack{|\gamma|\le T\\ L(\frac12+i\gamma,\psi^m)=0}} \frac{e^{i\gamma\log x}}{\frac12 + i\gamma} + R_m(x,T) \right)$$

where $R_m(x,T)$ are small in average for a good choice of T.

We have $L(\frac12, \psi^m) = 0$ for $m \equiv 3 \pmod 4$.

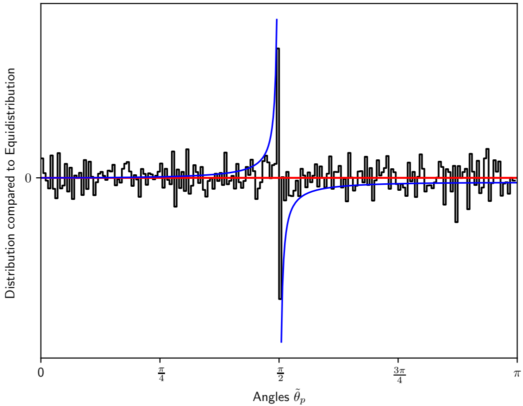
Even, 2π-periodic smooth functions

Theorem (D.)

Assume the Generalized Riemann Hypothesis, and that for all $m \in \mathbf{N}_{>0}$ we have $L(\frac{1}{2}, \psi^m) = 0$ with order 1 if and only if $m \equiv 3 \pmod 4$.
Then, for any ϕ smooth, even, 2π-periodic with $\int_0^\pi \phi = 0$ the function $y \mapsto \frac{y}{e^{y/2}} \sum_{p \le e^y} \phi(\tilde{\theta}_p)$ admits a limiting distribution with mean value equal to

$$-\frac{\phi(0) + \phi(\pi)}{4} + \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(t) \frac{1}{2\cos(t)} \, \mathrm{d}t.$$

Distribution of the angles of unfolded Gaussian primes

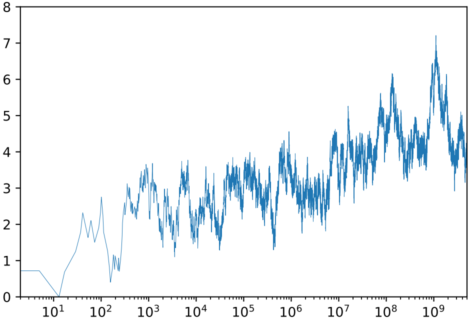


Distribution of the angles $\tilde{\theta}_p$ in intervals $[\frac{k\pi}{200}, \frac{(k+1)\pi}{200})$ for $p \equiv 1 \pmod 4$, $p \leq 10^7$. equidistribution, with secondary term : $\frac{1}{2\cos(t)}$

Wrapping up

Take $\phi = \mathbf{1}_{(0,\frac{\pi}{2})} - \mathbf{1}_{(\frac{\pi}{2},\pi)}$ in the mean value :

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(t)\frac{1}{2\cos(t)}\,\mathrm{d}t = +\infty$$



$\frac{\log x}{\sqrt{x}}\Big(\#\{p \le x : p = a^2 + 4b^2, |a| \equiv 1 \pmod 4\} - \#\{p \le x : p = a^2 + 4b^2, |a| \equiv 3 \pmod 4\}\Big)$,
normalised, logarithmic scale