# A Cucker-Smale inspired deterministic Mean Field Game with velocity interactions

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# Motivation

Figure: Collective behavior of birds

Figure: Lane formation in bidirectional pedestrian flows

# Static Game and Differential Game

**Optimization problem**

One individual wants to minimize the cost function $\phi(x)$ for input $x \in X$.

$\Longrightarrow$

**Static game (one-shot game)**

Each individual $i$ wants to minimize its own cost function $\phi_i(x_1, \cdots, x_N)$ for input $(x_1, \cdots, x_N) \in X_1 \times \cdots \times X_N$.

Rock–paper–scissors, Prisoner's Dilemma, Matching pennies,...

- Nash equilibrium: a state $(x_1^*, \cdots, x_N^*)$ such that for each $i = 1, \cdots, N$, $x_i^*$ is a minimizer of $\phi_i(x_1^*, \cdots, x_{i-1}^*, -, x_{i+1}^*, \cdots, x_N^*)$.
- Nash's theorem: for noncooperative games with a finite action set has a Nash equilibrium in mixed strategies (probability distribution of strategies).

# Static Game and Differential Game

## Optimal control problem

Given initial condition $x(0) = x_0$, 'one' individual wants to minimize the cost function $J(u)$ of the form

$$J(u) = \psi(x(T)) + \int_0^T L(t, x(t), u(t))dt,$$

where the state $x$ evoles in time, according to an ODE

$$x'(t) = f(t, x(t), u(t)),$$

with the control function $t \mapsto u(t) \in U$.

# Static Game and Differential Game

## Differential game

Given initial condition $x(0) = x_0$, 'each' individual $i$ wants to minimize the cost function $J_i(u_1, \cdots, u_N)$ of the form

$$J_i(u_1, \cdots, u_N) = \psi_i(x(T)) + \int_0^T L_i(t, x(t), u_1(t), \cdots, u_N(t))dt,$$

where the state $x$ evoles in time, according to an ODE

$$x'(t) = f(t, x(t), u_1(t), \cdots, u_N(t)),$$

with the control functions $t \mapsto u_i(t) \in U_i$.

# Mean-Field Game (MFG)

- MFGs focus on a case which greatly simplifies the study: the continuous case, where agents are supposed to be indistinguishable and negligible. In this case, only the distribution of mass on the set of trajectories plays a role, and if a single agent decides to deviate, it does not affect this distribution, which means that the other agents will not react to its change.
- As a result, MFG theory explains that one just needs to implement strategies based on the distribution of the other players.

# Mean-Field Game: Eulerian point of view

- In the (deterministic) Eulerian framework, the equilibrium found in the mean field limit turns out to be a solution of the forward-backward system of PDEs which couples a Hamilton-Jacobi-Bellman (HJB) equation with a continuity equation:

$$\begin{cases} -\partial_t u + H(x, \nabla u) = F(x, m) \quad \text{in } [0, T] \times \Omega, \\ \partial_t m - \nabla \cdot (m \nabla_p H(x, \nabla u)) = 0 \quad \text{in } [0, T] \times \Omega, \\ m(0) = m_0 \quad u(x, T) = F(x, m(T)). \end{cases} \tag{1}$$

- In **Larsy-Lions** (2006, 2007), the well-posedness of (1) was developed when $\Omega = \mathbb{T}^d$ (or $\mathbb{R}^d$) by using a fixed point argument which uses in an essential way the fact that viscosity solutions of the Hamilton-Jacobi equation $(1)_1$ in $[0, T] \times \mathbb{T}^n$ are smooth on a sufficiently large set to allow the continuity equation $(1)_2$ to be solvable.

# Mean-Field Game: Lagrangian point of view

- In the Lagrangian framework, the optimization problem considered by each agent is of the form

$$\min \left\{ \int_0^T L(t, \gamma(t), \gamma'(t), Q) dt + \Psi(x(T)) : \gamma(0) = x_0 \right\},$$

  where $Q \in \mathcal{P}(\Gamma)$ is the distribution of mass of the players on the space $\Gamma$ of possible paths in $\Omega$.
- Typical examples for $L$ is to penalize passing through regions with high concentration of players (e.g. Benamou-Carlier-Santambrogio (2017)):

$$L(t, x, v, Q) = \frac{1}{2}|v|^2 + g(\rho_t(x)), \quad \rho_t(x)^\alpha |v|^\beta, \cdots$$

- Sometimes it is also possible to penalize a first exit time from a given bounded domain through a part of its boundary (e.g. Mazanti-Santambrogio (2019)).

# Cucker-Smale model

Cucker-Smale model (2007): Each bird adjusts its velocity by a weighted sum of relative velocities:

$$\ddot{x}_i = \frac{1}{N}\sum_{j=1}^{N}\eta(|x_i - x_j|)(x_j' - x_i'), \quad i = 1, \cdots, N. \tag{2}$$

Then, a solution of (2) satisfies

$$\int_0^t \sum_{i,j=1}^{N} \eta(|x_i(s) - x_j(s)|)|x_j'(s) - x_i'(s)|^2 ds$$
$$= \frac{1}{2}\left(\sum_{i,j=1}^{N} |x_i'(0) - x_j'(0)|^2\right) - \frac{1}{2}\left(\sum_{i,j=1}^{N} |x_i'(t) - x_j'(t)|^2\right).$$

# Cucker-Smale inspired interaction cost

- Then, a very simple MFG model could be built upon the assumption that the cost for the agent $i$ following the trajectory $x_i(t)$ should include a term of the form

$$\int_0^T \sum_j \frac{1}{2}\eta(x_i(t) - x_j(t))|x_i'(t) - x_j'(t)|^2 dt, \tag{3}$$

  where $\eta(z)$ is a decreasing function of $|z|$ (e.g. $\eta(z) = e^{-\frac{|z|}{\varepsilon}}$).
- Lack of compactness: translation invariant.

# Differential game with velocity interaction cost

- We set a finite time horizon $T$ and a compact set $\Omega \subset \mathbb{R}^d$(or $\mathbb{T}^d$ or $\mathbb{R}^d$).
- We assume each agent chooses a trajectory $x : [0, T] \to \Omega$ which minimizes the final cost $\Psi(x(T))$ with less 'effort' $\int_0^T |x'(t)|^2 dt$ as possible.
- For given $(x_j)_{j \neq i}$, we set the total cost of $i$-th agent for the strategy(path) $x_i : [0, T] \to \Omega$ as

$$\int_0^T \left( \frac{\delta}{2} |x_i'|^2 + \frac{\lambda}{2N} \sum_j \eta(x_i - x_j) |x_i' - x_j'|^2 \right) dt + \Psi(x_i(T)), \qquad (4)$$

where $\delta, \lambda > 0$ are scale parameters.

# Mean-Field Game with velocity interaction cost

In order to define our game and the notion of equilibrium, we define

- $\Gamma := (C([0,T];\Omega), \|\cdot\|_\infty), \quad H^1 := (H^1([0,T];\Omega), \|\cdot\|_{H^1}).$
- $e_t : \Gamma \to \Omega, \quad e_t(\gamma) = \gamma(t).$
- $K : \Gamma \to \mathbb{R}, \quad K(\gamma) = \begin{cases} \frac{1}{2}\int_0^T |\gamma'(t)|^2 dt & \gamma \in H^1 \\ \infty & \gamma \notin H^1 \end{cases}.$
- $K_{\delta,\Psi} : \Gamma \to \mathbb{R}, \quad K_{\delta,\Psi}(\gamma) = \delta K(\gamma) + \Psi(\gamma(T)).$
- $V(\gamma,\tilde{\gamma}) := \begin{cases} \frac{1}{2}\int_0^T |\gamma' - \tilde{\gamma}'|^2 \eta(\gamma - \tilde{\gamma}) dt & \gamma - \tilde{\gamma} \in H^1 \\ \infty & \gamma - \tilde{\gamma} \notin H^1 \end{cases}.$
- $V_Q : \Gamma \to \mathbb{R}, \quad V_Q(\cdot) = \int_\Gamma V(\cdot,\tilde{\gamma})dQ(\tilde{\gamma}) \quad (Q \in \mathcal{P}(\Gamma)).$

# Mean-Field Game with velocity interaction cost

## MFG with velocity interactions

We consider the behavior of a population of agents with following rules:

- Initially distributed according to $m_0 \in \mathcal{P}(\Omega)$.
- Trying to minimize the function

$$\gamma \mapsto F(\gamma, Q) := K_{\delta,\Psi}(\gamma) + \lambda V_Q(\gamma),$$

  according to their trajectory distribution $Q \in \mathcal{P}(\Gamma)$.

Their interaction gives rise to a game that we will call $MFG(\Omega, \Psi, \delta, \eta, \lambda, m_0)$.

## Definition(Equilibrium)

A probability measure $Q \in \mathcal{P}(\Gamma)$ is called an equilibrium of $MFG(\Omega, \Psi, \delta, \eta, \lambda, m_0)$ if

1. $e_{0\#}Q = m_0 \in \mathcal{P}(\Omega)$.
2. $\int_\Gamma F(\gamma, Q) dQ(\gamma) < \infty$.
3. $F(\gamma, Q) = \inf_{\substack{w \in \Gamma \\ \omega(0)=\gamma(0)}} F(\omega, Q), \quad \forall\, \gamma \in \mathrm{spt}(Q)$.

# Kakutani's fixed point theorem

## Kakutani's fixed point theorem

Let $C$ be a nonempty, compact and convex subset of a locally convex space $E$, and assume the set-valued map $S : C \to 2^C$ satisfies

1. $\{x|Sx \subset W\}$ is open in $C$ for each open subset $W \subset C$.
2. For each $x \in C$, $Sx$ is nonempty, compact and convex.

Then, one can find $x_0 \in C$ satisfying $x_0 \in Sx_0$.

In usual MFG for the set of admissible curves $\mathcal{A}$, initial distribution $m_0 \in \mathcal{P}(\Omega)$ and cost function $F(\gamma, Q)$, we define

$$C := \left\{Q \in \mathcal{P}(\mathcal{A}) : e_{0\#}Q = m_0\right\},$$

$$S : Q \mapsto \left\{\tilde{Q} \in C : F(\gamma, Q) = \inf_{\substack{w \in \mathcal{A} \\ \omega(0)=\gamma(0)}} F(\omega, Q), \quad \forall\, \gamma \in \operatorname{spt}(\tilde{Q})\right\}.$$

But in our case every curve in $\Gamma$ is admissible, and the set

$$\mathcal{Q}_{m_0} := \left\{Q \in \mathcal{P}(\Gamma) : e_{0\#}Q = m_0 \in \mathcal{P}(\Omega)\right\}$$

is nonempty, convex, but not compact.

# Variational framework

It is possible to prove that minimizers of a suitable functional $\mathcal{J} = \mathcal{J}(Q)$ are necessarily equilibrium which always exist:

Define $J : \Gamma \times \Gamma \to \mathbb{R} \cup \{+\infty\}$ and $\mathcal{J} : \mathcal{Q}_{m_0} \to \mathbb{R} \cup \{+\infty\}$, which are given by

$$J(\gamma, \tilde{\gamma}) := K_{\delta,\Psi}(\gamma) + K_{\delta,\Psi}(\tilde{\gamma}) + \lambda V(\gamma, \tilde{\gamma}),$$
$$\mathcal{J}(Q) := \int_{\Gamma\times\Gamma} J(\gamma, \tilde{\gamma}) d(Q \otimes Q)(\gamma, \tilde{\gamma}).$$

Then, we have

$$\frac{\delta}{\delta Q}\left(\int_\Gamma J(\gamma, \tilde{\gamma}) dQ(\tilde{\gamma})\right) = 2F(\gamma, Q) = 2(K_{\delta,\Psi}(\gamma) + \lambda V_Q(\gamma)),$$

and any $\mathcal{J}$-minimizer $Q_0$ satisfies

$$\int_\Gamma \frac{\delta}{\delta Q}\left(\int_\Gamma J(\gamma, \tilde{\gamma}) dQ(\tilde{\gamma})\right)\Big|_{Q=Q_0} d(Q - Q_0) \geq 0, \quad \forall\ Q \in \mathcal{Q}_{m_0}.$$

# Existence of equilibrium-Sketch of proof

- Step 1: Every $\mathcal{J}$-minimizer is MFG equilibrium.

1. The $\mathcal{J}$-minimizer $Q_0$ satisfies $\int_\Gamma F(\gamma, Q_0) dQ_0(\gamma) < \infty$ and

$$\int_\Gamma F(\gamma, Q_0) dQ(\gamma) \geq \int_\Gamma F(\gamma, Q_0) dQ_0(\gamma), \quad \forall\ Q \in \mathcal{Q}_{m_0}. \tag{5}$$

2. The inequality (5) implies

$$F(\gamma, Q_0) = \inf_{\substack{w \in \Gamma \\ \omega(0)=\gamma(0)}} F(\omega, Q_0), \quad Q_0 - \text{almost every } \gamma. \tag{6}$$

3. The set of all optimal curves, i.e., $\gamma \in \Gamma$ satisfying

$$F(\gamma, Q_0) = \inf_{\substack{w \in \Gamma \\ \omega(0)=\gamma(0)}} F(\omega, Q_0)$$

is closed in $\Gamma$.

# Existence of equilibrium-Sketch of proof

- Step 2: Existence of $\mathcal{J}$-minimizer.

  1. $J$ is lower semicontinuous: if $\{(\gamma_n, \tilde{\gamma}_n)\}_{n\geq 1}$ converges to $(\gamma, \tilde{\gamma})$ in $\Gamma \times \Gamma$, then we have
  $$\liminf_{n\to\infty} J(\gamma_n, \tilde{\gamma}_n) \geq J(\gamma, \tilde{\gamma}).$$
  2. Minimizing sequence of $\mathcal{J}$ is tight. Indeed, $\{Q_n\}_{n\geq 1}$ is tight if $\{\mathcal{J}(Q_n)\}_{n\geq 1}$ is uniformly bounded.
  3. Since $J$ is l.s.c, the associated weak limit $Q_\infty$ satisfies
  $$\int_{\Gamma\times\Gamma} Jd(Q_\infty \otimes Q_\infty) \leq \liminf_{n\to\infty} \int_{\Gamma\times\Gamma} Jd(Q_n \otimes Q_n) = \inf_{Q\in\mathcal{P}(\Gamma)} \mathcal{J}(Q),$$
  and conclude that $Q_\infty$ is the desired minimizer.

# Existence of equilibrium

**Theorem(Existence of equilibrium)**

Let $\Omega$ be a compact subset of $\mathbb{R}^d$, or $\mathbb{R}^d$ or $\mathbb{R}^d/\mathbb{Z}^d$, and $m_0 \in \mathcal{P}(\Omega)$. If $\Psi : \Omega \to \mathbb{R}$ is bounded Lipschitz and $\eta : \Omega \to \mathbb{R}_+$ is a bounded Lipschitz continuous function satisfying $\eta(x) = \eta(-x)$, then for every $\delta, \lambda > 0$, $MFG(\Omega, \Psi, \delta, \eta, \lambda, m_0)$ has at least on equilibrium.

# Regularity of the optimal curves

For given equilibrium measure $Q$, we are also interested in the properties of each optimal curve $\gamma$. We denote the set of all optimal curves associated with measure $Q$ as $\mathcal{O}(Q)$. That is,

$$\mathcal{O}(Q) := \left\{ \gamma \in \Gamma : F(\gamma, Q) = \inf_{\substack{w \in \Gamma \\ \omega(0) = \gamma(0)}} F(\omega, Q) \right\}.$$

Then from the definition of equilibrium, we have

$$\text{spt}(Q) \subset \mathcal{O}(Q) \subset H^1.$$

# Regularity of the optimal curves

For a better understanding of the problem, we define the following quantities.

$$
\begin{aligned}
M_1(t) &:= \int_\Gamma |\omega'(t)| dQ(\omega) \in L^2[0,T],\\
M_2(t) &:= \int_\Gamma |\omega'(t)|^2 dQ(\omega) \in L^1[0,T],\\
a(t,x) &:= \int_\Gamma \eta(x-\omega(t)) dQ(\omega),\\
u(t,x) &:= \frac{1}{a(t,x)} \int_\Gamma \omega'(t)\eta(x-\omega(t)) dQ(\omega),\\
\sigma(t,x) &:= \int_\Gamma |\omega'(t)-u(t,x)|^2 \eta(x-\omega(t)) dQ(\omega).
\end{aligned}
$$

This allows to re-write the optimization problem for $\gamma$ using $a, u, \sigma$:

$$
\begin{aligned}
V_Q(\gamma) &= \frac{1}{2}\int_\Gamma \int_0^T |\gamma' - \tilde{\gamma}'|^2 \eta(\gamma - \tilde{\gamma}) dt dQ(\tilde{\gamma})\\
&= \frac{1}{2}\int_0^T \left( a(t,\gamma(t))|\gamma'(t) - u(t,\gamma(t))|^2 + \sigma(t,\gamma(t)) \right) dt.
\end{aligned} \tag{7}
$$

# Regularity of the optimal curves

Then, by applying classical calculus of variation argument, we can say that for $\gamma \in \mathcal{O}(Q)$ there exists an absolutely continuous function $z_\gamma$ satisfying

- $z_\gamma$ coincides a.e. with
$$\delta\gamma'(t) + \lambda a(t, \gamma(t))(\gamma'(t) - u(t, \gamma(t)));$$
- $z'_\gamma$ coincides a.e. with:
$$\frac{1}{2}\Big[\nabla_x a(t,\gamma)|\gamma' - u(t,\gamma)|^2 + 2a(t,\gamma)(u(t,\gamma) - \gamma')\nabla_x u(t,\gamma) + \nabla_x \sigma(t,\gamma)\Big];$$
- the transversality condition $z_\gamma(T) = -\nabla\Psi(\gamma(T))$ is satisfied;

# Regularity of the optimal curves-Lipschitz

**Lemma**

*We have the following inequalities on the values of $a, u, \sigma$, where $C$ denotes a universal constant, only depending on a strictly positive, bounded and Lipschitz continuous function $\eta$:*

$$a \leq C, \quad a|u| \leq CM_1, \quad a|u|^2 \leq CM_2, \quad \sigma \leq CM_2.$$

*Moreover, if there is a constant $C$ such that the inequality $|\nabla\eta(y)| \leq C\eta(y)$ holds for every $y \in \mathbb{R}^d$, then we have the following inequalities on the values of the gradients of $a, u, \sigma$, where $C$ denotes a universal constant, only depending on $\eta$:*

$$|\nabla_x a| \leq Ca \leq C, \quad |\nabla_x(au)| \leq Ca|u| \leq CM_1,$$
$$a|\nabla_x u| \leq Ca|u| \leq CM_1, \quad a|\nabla_x u|^2 \leq CM_2, \quad |\nabla_x\sigma| \leq CM_2.$$

# Regularity of the optimal curves-Lipschitz

Then, we compare the optimal curve $\gamma$ to the constant curve $\tilde{\gamma} \equiv x_0$ and obtain

$$\frac{\delta}{2}\int_0^T |\gamma'(t)|^2 dt + \frac{\lambda}{2}\int_0^T a(t,\gamma(t))|\gamma'(t) - u(t,\gamma(t))|^2 dt$$
$$\leq \frac{\lambda}{2}\int_0^T \left(a(t,x_0)|u(t,x_0)|^2 + \sigma(t,x_0)\right) dt + \left(\Psi(x_0) - \Psi(\gamma(T))\right) \leq C.$$

Therefore, since $z'_\gamma$ coincides a.e. with

$$\frac{1}{2}\Big[\nabla_x a(t,\gamma)|\gamma' - u(t,\gamma)|^2 + 2a(t,\gamma)(u(t,\gamma) - \gamma')\nabla_x u(t,\gamma) + \nabla_x \sigma(t,\gamma)\Big],$$

we have the uniform boundedness of $z_\gamma$ in $t$ and $\gamma$.

# Regularity of the optimal curves-Lipschitz

We now set $\|z\|_\infty := \sup_{t,\gamma} |z_\gamma(t)|$.

Since $a|u|(t,\cdot)$ is bounded by $CM_1(t)$, the boundedness of $z_\gamma$ implies that for a.e. $t$, the speed $|\gamma'(t)|$ is uniformly bounded in $\gamma \in \mathcal{O}(Q)$, say $L(t)$.

This bound again gives $|u(x,t)| \leq L(t)$, since $u$ is the weighted average of $\gamma'$ over $\mathcal{O}(Q)$. Therefore, we deduce

$$(\delta + \lambda a)L(t) = \sup_{\gamma \in \mathcal{O}(Q)} (\delta + \lambda a)|\gamma'(t)| \leq \lambda a u + \|z\|_\infty \leq \lambda a L(t) + \|z\|_\infty$$

$$\implies L(t) \leq \frac{\|z\|_\infty}{\delta}.$$

In short,
(Lemma)+(E-L eqn)+(comparison with constant curve)
→ (Lemma)+(boundedness of $z_\gamma$)
→ boundedness of $\gamma'$ by $L$
→ boundedness of $u$ by $L$
→ use (E-L eqn) again to find an upper bound of $L$.

# Regularity of the optimal curves-$C^{1,1}$

Now, we use the uniform boundedness of $M_1$ and $M_2$ in $t$ to obtain the Lipschitz continuity of $z_\gamma = \delta\gamma' + \lambda a(\gamma' - u)$, $a$ and $au$, and this implies the Lipschitz continuity of $\gamma'$ on a set of full measure, i.e., $\gamma \in C^{1,1}$.

**Theorem(Regularity of the optimal curves)**

Suppose that $\Omega, \eta$ and $\Psi$ satisfy

- (H$\eta$) $\eta$ is strictly positive, bounded and Lipschitz continuous, and there is a constant $C$ such that the inequality $|\nabla\eta(y)| \leq C\eta(y)$ holds for every $y \in \mathbb{R}^d$;
- (H$\Omega$) $\Omega$ has no boundary (i.e. it is either the torus or the whole space $\mathbb{R}^d$) ;
- (H$\Psi$) $\Psi$ is Lipschitz continuous.

Then, if $Q$ is an equilibrium of $MFG(\Omega, \Psi, \delta, \eta, \lambda, m_0)$, every curve $\gamma \in \mathcal{O}(Q)$ is in $C^{1,1}[0, T]$, and their derivatives are bounded uniformly in $\mathcal{O}(Q)$, only depending on $Q$ and on the parameters of the MFG.

# Monokineticity

- The $C^{1,1}$ result implies monokineticity in the following sense: if we take two curves $\gamma_1, \gamma_2 \in \mathrm{spt}(Q)$, a time $t \in (0, T]$, and we suppose $\gamma_1(t) = \gamma_2(t)$, then we also have $\gamma_1'(t) = \gamma_2'(t)$.
- Hence, for each time $t$ which is not the initial time $t = 0$, the velocity of all particles at a same point is the same, thus defining a velocity field $v(t, x)$ such that the curves $\gamma \in \mathrm{spt}(Q)$ follow $\gamma'(t) = v(t, \gamma(t))$.
- This allows to re-write our optimization problem using an Eulerian formulation in terms $\rho = \rho(t, x)$ and $v = v(t, x)$: the problem becomes the minimization of

$$\frac{\delta}{2}\int_0^T \int_\Omega |v_t|^2 d\rho_t(x) dt + \frac{\lambda}{2}\int_0^T \int_\Omega \int_\Omega \eta(x - x')|v_t(x) - v_t(x')|^2 d\rho_t(x) d\rho_t(x') dt$$

among all $(\rho, v)$ satisfying

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \ \ \rho_0 = m_0.$$

Thank you very much for attention.