Invariants in restriction of admissible representations of p-adic groups

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France-Korea IRL webinar in Number Theory

December 20, 2021

Outline

Part I:

Objective – conjectural statements in LLC for p-adic groups

Part II:

Methodology – restriction and lifting in a certain pair (G, H)

Achievements – successful cases (G, H)

Part III:

Obstacles – issues towards general cases (G, H)

Some recent developments – resolutions of some obstacles

Objective – conjectural statements in LLC for p-adic groups

- F is a p-adic field of characteristic 0.
- W_F is the Weil group of F, and Γ is the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$.
- G is a connected, reductive, linear, algebraic group over F.
- ${}^LG := \widehat{G} \rtimes \Gamma$.

LLC for tempered representations of p-adic groups

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- W_F is the Weil group of F, and Γ is the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$.
- G is a connected, reductive, linear, algebraic group over F.
- ${}^L G := \widehat{G} \rtimes \Gamma$.

e.g.)

G	GL_n	$GL_m(D)$	SL_n	$SL_m(D)$	SO_{2n+1}	SO_{2n}
$\widehat{G}$	$GL_n(\mathbb{C})$	$GL_{md}(\mathbb{C})$	$PSL_n(\mathbb{C})$	$PSL_{md}(\mathbb{C})$	$Sp_{2n}(\mathbb{C})$	$SO_{2n}(\mathbb{C})$

...

- $\mathrm{Irr}_{\mathrm{temp}}(G)$ is the set of equivalence classes of irreducible, tempered, complex representations of $G(F)$.
- $\Phi_{\mathrm{temp}}(G)$ is the set of $\widehat{G}$-conjugacy classes of tempered L-parameters (an L-parameter $\varphi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ is tempered if $\varphi(W_F)$ is bounded).

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Tempered local Langlands conjecture of *p*-adic groups

There is a **surjective, finite-to-one map**

$$\mathscr{L}_{\mathrm{temp}} : \mathrm{Irr}_{\mathrm{temp}}(G) \longrightarrow \Phi_{\mathrm{temp}}(G).$$

This map is supposed to satisfy a number of natural properties. $\mathscr{L}_{\mathrm{temp}}$ preserves γ-factors, L-factors, and ε-factors, if they are available in both sides

Example $G = \mathrm{GL}_2(F)$

$$\mathscr{L}_{\mathrm{temp}} : \mathrm{Irr}_{\mathrm{temp}}(GL_2) \overset{bijection}{\longrightarrow} \Phi_{\mathrm{temp}}(GL_2).$$

More precisely, given $[F : \mathbb{Q}_p] < \infty$, $G = GL_2$, $Art_F : F^\times \overset{\simeq}{\to} W_F^{ab}$, and $\chi, \chi_i \in Hom_{cont}(F^\times, \mathbb{C}^\times)$, the above bijection provides:

$$\chi \circ \det \longleftrightarrow (\chi|\cdot|_F^{1/2} \circ Art_F^{-1}) \oplus (\chi|\cdot|^{-1/2} \circ Art_F^{-1})$$

$$i_B^G(\chi_1 \otimes \chi_2) \longleftrightarrow (\chi_1 \circ Art_F^{-1}) \oplus (\chi_2 \circ Art_F^{-1})$$

...

Internal Structure of L-packets

- For any $\varphi_G \in \Phi_{\mathrm{temp}}(G)$, $\Pi_{\varphi_G}(G) := \mathscr{L}_{\mathrm{temp}}^{-1}(\varphi_G)$ denotes a **tempered L-packet**.

Internal Structure of L-packets

- For any $\varphi_G \in \Phi_{\text{temp}}(G)$, $\Pi_{\varphi_G}(G) := \mathscr{L}_{\text{temp}}^{-1}(\varphi_G)$ denotes a **tempered L-packet**.
- $\text{Irr}(\mathscr{S}_{\varphi,\text{sc}}(\widehat{G}), \zeta_G)$ denotes the set of irreducible representations of $\mathscr{S}_{\varphi,\text{sc}}(\widehat{G})$ whose restriction to $\widehat{Z}_{\varphi,\text{sc}}(G)$ equals ζ_G. Here, $\mathscr{S}_{\varphi,\text{sc}}(\widehat{G})$ fits into a central extension (the version of discrete φ)

$$1 \longrightarrow \widehat{Z}_{\varphi,\text{sc}}(G) \longrightarrow \mathscr{S}_{\varphi,\text{sc}}(\widehat{G}) \longrightarrow \mathscr{S}_{\varphi}(\widehat{G}) \longrightarrow 1,$$

where

$$\begin{aligned} \mathscr{S}_{\varphi}(\widehat{\mathbf{G}}) &:= \pi_0(S_{\varphi}(\widehat{\mathbf{G}})), \\ \mathscr{S}_{\varphi,\text{sc}}(\widehat{\mathbf{G}}) &:= \pi_0(S_{\varphi,\text{sc}}(\widehat{\mathbf{G}})), \\ \widehat{Z}_{\varphi,\text{sc}}(\mathbf{G}) &:= Z(\widehat{\mathbf{G}}_{\text{sc}})/(Z(\widehat{\mathbf{G}}_{\text{sc}}) \cap S_{\varphi,\text{sc}}(\widehat{\mathbf{G}})^{\circ}), \end{aligned}$$

Internal structure of L-packets (Arthur version)

Fixing a Whittaker datum, there is a **one-to-one** correspondence

$$\Pi_\varphi(G) \xleftrightarrow{1-1} \mathrm{Irr}(\mathscr{S}_{\varphi_G,\mathrm{sc}}(\widehat{G}), \zeta_G)$$

along with **endoscopic character identity**.

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Note: Originally formulated by Arthur / **Refined** by Kaletha **using a slightly different group** than $\mathscr{S}_{\varphi_G,\mathrm{sc}}(\widehat{G})$, and there is a **bijection between two formulations**.

$G = \mathrm{GL}_1(D)$, D a quaternion division algebra over $\mathbb{Q}_p$

$$1 \longrightarrow \widehat{Z}_{\varphi_G,\mathrm{sc}}(G) = \mu_2(\mathbb{C}) \longrightarrow \mathscr{S}_{\varphi,\mathrm{sc}}(\widehat{G}) \simeq \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathscr{S}_{\varphi_G}(\widehat{G}) = \pi_0(Cent(\varphi_G, \widehat{G})/Z(\widehat{G})^{\Gamma}) = 1 \longrightarrow 1.$$

$Hom(\mu_2(\mathbb{C}), \mathbb{C}^{\times}) = \{\mathbb{1}, \mathrm{sgn}\}$

- $G = GL_2$, $\zeta_{GL_2} = \mathbb{1}$

$$\Pi_{\varphi_G}(GL_2(F)) \overset{1-1}{\longleftrightarrow} \mathrm{Irr}(\mu_2(\mathbb{C}), \mathbb{1}) = \{\mathbb{1}\}.$$

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Known Cases

- GL_n (Harris-Taylor 2000, Henniart 2003, Scholze 2013);
- SL_n (Gelbart-Knapp 1982);
- non-quasi-split F-inner forms of GL_n and SL_n (Labesse-Langlands 1979, Hiraga-Saito 2012);
- GSp_4, Sp_4 (Gan-Takeda 2010,2011);
- non-quasi-split F-inner form $GSp_{1,1}$ of GSp_4 (Gan-Tantono 2014);
- Sp_{2n}, SO_n, SO^*_{2n} (Arthur 2013);
- U_n (Rogawski 1990, Mok 2015), non quasi-split F-inner forms of U_n (Rogawski 1990, Kaletha-Minguez-Shin-White 2014);
- non-quasi-split F-inner form $Sp_{1,1}$ of Sp_4 (C. 2017);
- $GSpin_4, GSpin_6$ and their inner forms (Asgari-C. 2017);
- GSp_{2n}, GO_{2n} (Xu 2017).
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Methodology – restriction and lifting in a certain pair (G, H)

Achievements – successful cases (G, H)

Restriction and lifting in a certain pair (G, H)

- G connected reductive group over a p-adic field F.
- H closed F-subgroup of G such that

$$H_{\mathrm{der}} = G_{\mathrm{der}} \subseteq H \subseteq G.$$

e.g.)

G	GL	$GL(D)$	GSp	GSO	U
H	SL	$SL(D)$	Sp	SO	SU

...

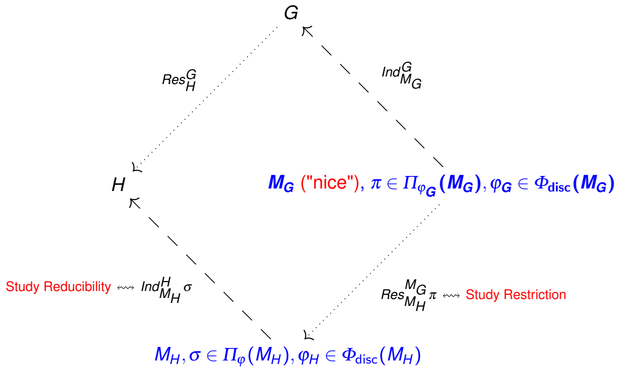
- F-Levi subgroups: $M_G \subseteq G$ and $M_H = M_G \cap H \subseteq H$

$$\Rightarrow (M_H)_{\mathrm{der}} = (M_G)_{\mathrm{der}} \subseteq M_H \subseteq M_G.$$

p-adic group/Representation side:

Big Picture

<u>p-adic group/Representation side:</u>



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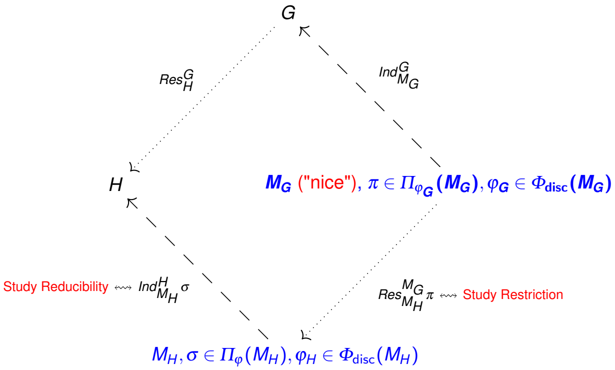

<u>L-group/Galois/L-parameter side:</u>

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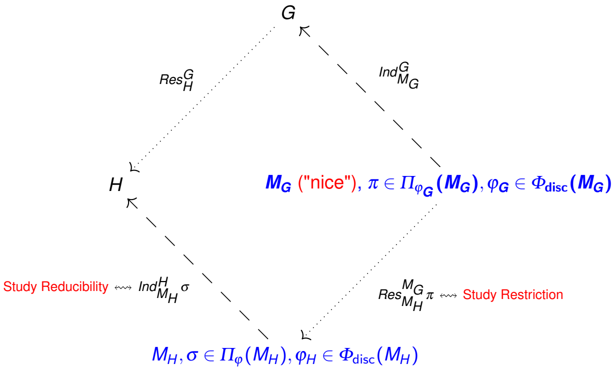

L-group/Galois/*L*-parameter side:

$$\Phi(G) \twoheadrightarrow \Phi(H)$$

Recall(LLC for $\square$):

$$\boxed{\mathscr{L}_{\text{temp}} : \text{Irr}_{\text{temp}}(\square) \longrightarrow \Phi_{\text{temp}}(\square)}$$

LLC for $G \Rightarrow$ LLC for H

Recall(LLC for $\Box$):

$$\boxed{\mathscr{L}_{\text{temp}} : \text{Irr}_{\text{temp}}(\Box) \longrightarrow \Phi_{\text{temp}}(\Box)}$$

- Given $\sigma \in \text{Irr}(H)$, there is a lifting $\widetilde{\sigma} \in \text{Irr}(G)$ such that

$$\sigma \hookrightarrow \text{Res}_H^G(\widetilde{\sigma}),$$

  due to: Labesse-Langlands, Gelbart-Knapp, Tadić, and others.
- LLC for G, $\mathscr{L}_G : \text{Irr}(G) \twoheadrightarrow \Phi(G)$, assigns an L-parameter $\mathscr{L}_G(\widetilde{\sigma})$.

- Due to Weil, Henniart, Labesse:

$$\Phi(G) \twoheadrightarrow \Phi(H)$$

- Under a surjective map $\widehat{G} \twoheadrightarrow \widehat{H}$,
  We define:

$$\begin{array}{rcl} \mathscr{L}_H : \mathrm{Irr}(H) & \longrightarrow & \Phi(H) \\ \sigma & \longmapsto & pr \circ \mathscr{L}_G(\widetilde{\sigma}). \end{array}$$

$\Rightarrow$ $\mathscr{L}_H$ is finite-to-one, surjective as desired for H.

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$\Rightarrow$ $\mathscr{L}_H$ is finite-to-one, surjective as desired for H.

Remarks:

- It is independent of the choice of the lifting $\widetilde{\sigma}$.
- This is a case of the (local) principal of functoriality, as we had $\widehat{G} \twoheadrightarrow \widehat{H}$.

Internal structure of L-packets for $G \Rightarrow$ that for H

Recall(Internal structure of L-packets for $\square$):

$$\boxed{\Pi_\varphi(\square) \overset{1-1}{\longleftrightarrow} \mathrm{Irr}(\mathscr{S}_{\varphi_\square, \mathrm{sc}}(\widehat{\square}), \zeta_\square)}$$

Recall(Internal structure of L-packets for $\square$):

$$\boxed{\Pi_\varphi(\square) \overset{1-1}{\longleftrightarrow} \mathrm{Irr}(\mathscr{S}_{\varphi_\square,\mathrm{sc}}(\widehat{\square}), \zeta_\square)}$$

We consider:

$$\{a \in H^1(W_F, \widehat{(G/H)}) : a\varphi_G \simeq \varphi_G \text{ in } \widehat{G}\} / \operatorname{Im}\left(Z(\widehat{H})^\Gamma \to H^1(W_F, \widehat{G/H})\right).$$

Denote it by $\boldsymbol{X(\varphi_G)}$.

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Observe:

- $\boldsymbol{X(\varphi_G)}$ is a finite abelian group.

- An exact sequence of finite groups

$$\mathbf{1} \longrightarrow \mathscr{S}_{\varphi_G,\mathrm{sc}} \longrightarrow \mathscr{S}_{\varphi_H,\mathrm{sc}} \longrightarrow X(\varphi_G) \longrightarrow \mathbf{1},$$

equipped in the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Z(\widehat{G}_{\mathrm{sc}}) & \longrightarrow & \mathscr{S}_{\varphi_G,\mathrm{sc}}(\widehat{G}) & \longrightarrow & \mathscr{S}_{\varphi_G}(\widehat{G}) & \longrightarrow & 1 \\
 & & \| & & \downarrow \cap & & \downarrow \cap & & \\
1 & \longrightarrow & Z(\widehat{H}_{\mathrm{sc}}) & \longrightarrow & \mathscr{S}_{\varphi_H,\mathrm{sc}}(\widehat{H}) & \longrightarrow & \mathscr{S}_{\varphi_H}(\widehat{H}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & \longrightarrow & S_{\varphi_H,\mathrm{sc}}(\widehat{H})/S_{\varphi_G,\mathrm{sc}}(\widehat{G}) & \xrightarrow{\simeq} & X(\varphi_G) & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

Successful cases (G, H)

- $G = \mathrm{GL}_n, H = \mathrm{SL}_n$ (Gelbart-Knapp 1982);
- $G = \mathrm{GL}_m(D), H = \mathrm{SL}_m(D)$ (Labesse-Langlands 1979, Hiraga-Saito 2012);
- $G = \mathrm{GSp}_4, H = \mathrm{Sp}_4$ (Gan-Takeda 2010,2011);
- $G = \mathrm{GSp}_{1,1}, H = \mathrm{Sp}_{1,1}$ (C. 2017);
- $G = \mathrm{GL}_2 \times \mathrm{GL}_2, H = \mathrm{GSpin}_4$; $G = \mathrm{GL}_4 \times \mathrm{GL}_1, H = \mathrm{GSpin}_6$, and their inner forms (Asgari-C. 2017),

$G = \mathrm{GL}_2, H = \mathrm{SL}_2$ (Gelbart-Knapp 1982)

- $$\mathscr{S}_{\varphi_H}(\widehat{H}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \ \ \mathbb{Z}/2\mathbb{Z}, 1, \quad |\Pi_\varphi(H)| = 1, 2, 4.$$

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- $$\mathscr{S}_{\varphi_H}(\widehat{H}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},\ \ \mathbb{Z}/2\mathbb{Z}, 1, \quad |\Pi_\varphi(H)| = 1, 2, 4.$$
- For $\varphi_G \in \Phi(G)$ dihedral with respect to three quadratic extensions:
  - $$\mathscr{S}_{\varphi_G}(\widehat{\mathrm{GL}_2}) = \{1\}, \quad \mathscr{S}_{\varphi_H}(\widehat{\mathrm{SL}_2}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$
  - $$\Pi_{\varphi_G}(\mathrm{GL}_2) = \{\widetilde{\sigma}\}, \quad \mathrm{Res}^{\mathrm{GL}_2}_{\mathrm{SL}_2}(\widetilde{\sigma}) = \Pi_{\varphi_H}(\mathrm{SL}_2) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}.$$
  - Thus,
    $$\boxed{\Pi_\varphi \overset{1-1}{\longleftrightarrow} \mathrm{Irr}(\mathscr{S}_\varphi(\widehat{\mathrm{SL}_2}), \mathbb{1}) = \{\chi_1, \chi_2, \chi_3, \chi_4\}}$$

$G = \mathrm{GL}_1(D), H = \mathrm{SL}_1(D)$ (Labesse-Langlands 1979, Hiraga-Saito 2012)

D quaternion division algebra over F

- $$\mathscr{S}_{\varphi_H,\mathrm{sc}}(\widehat{H}) \simeq Q_8,\ \ \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z},\ \ \mathbb{Z}/2\mathbb{Z},\quad |\Pi_{\varphi_H}(H)| = 1, 2, 4.$$

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- For $\varphi_G \in \Phi(G)$ dihedral with respect to three quadratic extensions:
  - $$1 \longrightarrow \mu_2(\mathbb{C}) \longrightarrow \mathscr{S}_{\varphi_H,\mathrm{sc}}(\widehat{\mathrm{SL}_1(D)}) \simeq Q_8 \longrightarrow \mathscr{S}_{\varphi_H}(\widehat{\mathrm{SL}_1(D)}) \simeq \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} \longrightarrow 1,$$
    where Q_8 denotes the quaternion group of order 8.
  - $$\mathrm{Irr}(Q_8) = \{\chi_1,\chi_2,\chi_3,\chi_4,\rho'\},$$
    where χ_i's are distinct 1-dimensional representations, and ρ' is the 2-dimensional representation of Q_8.
  - $$\Pi_{\varphi_G}(\mathrm{GL}_1(D)) = \{\widetilde{\sigma}'\},\ \ \mathrm{Res}^{\mathrm{GL}_1(D)}_{\mathrm{SL}_1(D)}(\widetilde{\sigma}') = \Pi_{\varphi}(\mathrm{SL}_1(D)) = \{\sigma'\},$$
  - Thus,
    $$\boxed{\Pi_{\varphi}(\mathrm{SL}_1(D)) \overset{1-1}{\longleftrightarrow} \mathrm{Irr}(\mathscr{S}_{\varphi,\mathrm{sc}}(\widehat{\mathrm{SL}_1(D)}),\mathrm{sgn}) = \{\rho'\},}$$
    where we correspond $\sigma' \leftrightarrow \rho'$.

The group $\mathscr{S}_{\varphi,\mathrm{sc}}(\widehat{\mathrm{Sp}_{1,1}})$ is isomorphic to one of the following **seven** groups:

1. $\mathbb{Z}/2\mathbb{Z}$,
2. $(\mathbb{Z}/2\mathbb{Z})^2$,
3. $\mathbb{Z}/4\mathbb{Z}$,
4. $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$,
5. the dihedral group $\mathscr{D}_8$ of order 8,
6. the Pauli group $\{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$, where $i = \sqrt{-1}$,

$$I = I_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \text{ and } Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

7. the central product of $\mathscr{D}_8$ and the quaternion group of order 8.

By-product: L-packet size $|\Pi_\varphi(\mathrm{Sp}_{1,1})| = 1, 2, 4$.

$G = \mathrm{GSp}_{1,1}, H = \mathrm{Sp}_{1,1}$ (C.2017)

Consider $\varphi_G = \varphi_0 \oplus (\varphi_0 \otimes \chi) \in \Phi(\mathrm{GSp}_4)$

- χ is a quadratic character,
- $\varphi_0 \in \Phi(\mathrm{GL}_2)$ is primitive (i.e., $\varphi_0 \neq \mathrm{Ind}_{W_E}^{W_F} \rho$ for a finite extension E/F and some irreducible ρ)
- $\varphi_0 \not\simeq \varphi_0 \otimes \chi$.
- The projection φ of φ_G onto $\widehat{\mathrm{Sp}_4} = \mathrm{SO}_5(\mathbb{C})$ is

$$\varphi = \mathbb{1} \oplus \chi \oplus Ad(\varphi_0)\chi \in \Phi(\mathrm{Sp}_4),$$

$$\begin{array}{ccccccccc}
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1 & \longrightarrow & Z(\widehat{G}_{\mathrm{sc}}) & \longrightarrow & \mathscr{S}_{\varphi_G,\mathrm{sc}}(\widehat{G}) & \longrightarrow & \mathscr{S}_{\varphi_G}(\widehat{G}) \simeq \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 1 \\
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1 & \longrightarrow & Z(\widehat{H}_{\mathrm{sc}}) \simeq \mu_2(\mathbb{C}) & \longrightarrow & \mathscr{S}_{\varphi_H,\mathrm{sc}}(\widehat{H}) \simeq \mathscr{D}_8 & \longrightarrow & \mathscr{S}_{\varphi_H}(\widehat{H}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & \longrightarrow & S_{\varphi_H,\mathrm{sc}}(\widehat{H})/S_{\varphi_G,\mathrm{sc}}(\widehat{G}) & \xrightarrow{\simeq} & X(\varphi_G) \simeq \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 1 \\
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\end{array}$$

$$\boxed{\Pi_\varphi(\mathrm{Sp}_{1,1}) = \{\sigma'\} \overset{1-1}{\longleftrightarrow} \mathrm{Irr}(\mathscr{S}_{\varphi,\mathrm{sc}}(\widehat{\mathrm{Sp}_{1,1}}), \mathrm{sgn}) = \mathrm{Irr}(\mathscr{D}_8, \mathrm{sgn})}$$

Obstacles – issues towards general cases (G, H)

Some recent developments – resolutions of some obstacles

$\mathrm{GL}_n, \Pi_{\varphi_G}(G) = \{ Ind_{M_G}^{G} \widetilde{\sigma} \}$

Multiplicity=1 $\quad Res_{\mathrm{SL}_n}^{\mathrm{GL}_n}(Ind_{M_G}^{G} \widetilde{\sigma})$

$Ind_{M_G}^{\mathrm{GL}_n} \widetilde{\sigma} \quad$ $R_{\widetilde{\sigma}} = 1$

$\mathrm{SL}_n, \Pi_{\varphi}(\mathrm{SL}_n) = \{ \pi \subset Res_{\mathrm{SL}_n}^{\mathrm{GL}_n}(Ind_{M_G}^{G} \widetilde{\sigma}) \} / \simeq$

$M_G (\simeq \prod \mathrm{GL}_{n_i}), \Pi_{\varphi_G}(M_G) = \{ \widetilde{\sigma} \}$

$Res_{M_H}^{M_G}$ $\quad$ Multiplicity=1

$M \ (\simeq \mathrm{SL}_n \cap \prod \mathrm{GL}_{n_i}), \sigma \in \Pi_{\mathrm{disc}}(M)$

"compatible" $\Updownarrow$

GL_n

$\mathrm{SL}_n, \Pi_{\varphi}(\mathrm{SL}_n) = \{ \text{irred. const. of } Ind_{M}^{\mathrm{SL}_n}(\tau) \mid \tau \in \Pi_{\varphi}(M) \} / \simeq$

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R_τ *known* $\quad Ind_{M}^{\mathrm{SL}_n}$

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$M \ (\simeq \mathrm{SL}_n \cap \prod \mathrm{GL}_{n_i}), \sigma \in \Pi_{\mathrm{disc}}(M), \Pi_{\varphi}(M) = \{ \tau \subset Res_{M_H}^{M_G}(Res_{M_H}^{M_G} \widetilde{\sigma}) \} / \simeq$

Definition of Multiplicity in restriction

- F a p-adic field of characteristic 0
- G a connected reductive group over F
- H a closed F-subgroup of G such that $H_{\mathrm{der}} = G_{\mathrm{der}} \subseteq H \subseteq G$.

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Given irreducible smooth representations $\sigma \in \text{Irr}(H)$ and $\pi \in \text{Irr}(G)$, the multiplicity $\langle \sigma, \pi \rangle_H$ of σ in the restriction $\text{Res}_H^G(\pi)$ of π to H is defined as follows:

$$\boxed{\langle \sigma, \pi \rangle_H := \dim_{\mathbb{C}} \text{Hom}_H(\sigma, \text{Res}_H^G(\pi)) \quad \in \mathbb{N} \cup \{0\}.}$$

e.g.) $G = GL_n, H = \text{SL}_n$, any $\pi \in \text{Irr}(H), \sigma \in \text{Irr}(H)$, we have $\langle \sigma, \pi \rangle_H = 0$, *or* 1.

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- H a subgroup of a finite group $\widetilde{H}$

Given $\tilde{\delta} \in \mathrm{Irr}(G)$ and $\gamma \in \mathrm{Irr}(H)$, the multiplicity $\langle \tilde{\delta}, \gamma \rangle_H$ of γ in the restriction $\mathrm{Res}_H^G(\tilde{\delta})$ of γ to H is defined as follows:

$$\langle \tilde{\delta}, \gamma \rangle_H := \dim_{\mathbb{C}} \mathrm{Hom}_H(\tilde{\delta}, \mathrm{Res}_H^G(\gamma)) \quad \in \mathbb{N} \cup \{0\}.$$

e.g.) $H < G$ with index 2, any $\tilde{\delta} \in \mathrm{Irr}(G), \gamma \in \mathrm{Irr}(H)$, we have $\langle \tilde{\delta}, \gamma \rangle_H = 0$, *or* 1, *or* 2.

Example in $\mathrm{SL}(1,D)$

(Labesse-Langlands, Shelstad 1979)

- $G = \mathrm{GL}(1,D)$, $H = \mathrm{SL}(1,D)$, D is the quaternion division algebra over F.
- $\varphi : W_F \times SL(2,\mathbb{C}) \to \widehat{H} = PGL(2,\mathbb{C})$ is an L-parameter for H and $\Pi_\varphi(H)$ is the L-packet.
- For any $\sigma \in \mathrm{Irr}(H)$ and $\pi \in \mathrm{Irr}(G)$,

$$\langle \sigma, \pi \rangle_H = \begin{cases} 2, & \text{if } \sigma \in \Pi_\varphi(H), pr \circ \varphi_\pi = \varphi, Cent(\varphi, \widehat{H}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\ 1, & \text{if } \sigma \in \Pi_\varphi(H), pr \circ \varphi_\pi = \varphi, Cent(\varphi, \widehat{H}) \not\simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\ 0, & \text{otherwise.} \end{cases}$$

Definition of three R-groups

- $P = MN$: a standard F-parabolic subgroup of G.
- A_M : the split component of M.
- $W(G,M) = N_G(A_M)/Z_G(A_M)$.
- $\Phi(P, A_M)$: the set of reduced roots of P with respect to A_M.
- Given $\sigma \in \Pi_{\text{disc}}(M)$, $W(\sigma) := \{w \in W(G,M) : {}^w\sigma \simeq \sigma\}$
- W_σ° is the subgroup of $W(\sigma)$ generated by the reflections in the roots of $\{\alpha \in \Phi(P, A_M) : \mu_\alpha(\sigma) = 0\}$, where $\mu_\alpha(\sigma)$ is the rank one Plancherel measure for σ attached to α.

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Note (Knapp-Stein (1972); Silberger (1978)) :

- $\text{End}_G(i_{G,M}(\sigma)) \simeq \mathbb{C}[R_\sigma]_\eta$ as algebras, where $\eta \in H^2(R_\sigma, \mathbb{C}^\times)$.

$\Rightarrow$ **Reducibility of the parabolic induction $i_M^G(\sigma)$**
$\leftrightsquigarrow$ **Knapp-Stein R–group** $\subset W(G,M)$
$\leftrightsquigarrow$ **Tempered non-discrete spectra and L-packets**

Let $\phi : W_F \times SL_2(\mathbb{C}) \to \widehat{M} \hookrightarrow \widehat{G}$ be an *elliptic tempered* L-parameter for M.

- $C_\phi(\widehat{G})$ is the centralizer of the image of ϕ in $\widehat{G}$ and $C_\phi(\widehat{G})^\circ$ is its identity component.
- T_ϕ is a fixed maximal torus in $C_\phi(\widehat{G})^\circ$.

Set $W_\phi^\circ := N_{C_\phi(\widehat{G})^\circ}(T_\phi)/Z_{C_\phi(\widehat{G})^\circ}(T_\phi), \quad W_\phi := N_{C_\phi(\widehat{G})}(T_\phi)/Z_{C_\phi(\widehat{G})}(T_\phi)$.
Note that W_ϕ can be identified with a subgroup of $W(G, M)$.
The **endoscopic *R*-group** R_ϕ is defined by

$$R_\phi := W_\phi / W_\phi^\circ.$$

Given $\sigma \in \Pi_\phi(M)$, the L-packet associated to the L-parameter ϕ,
Set $W_{\phi,\sigma}^\circ := W_\phi^\circ \cap W(\sigma), \quad W_{\phi,\sigma} := W_\phi \cap W(\sigma)$.
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Arthur Conjecture for R-groups : For $\sigma \in \Pi_\phi(M)$, we have

$$R_\sigma \simeq R_{\phi,\sigma} \hookrightarrow R_\phi.$$

Example in $SL(2,\mathbb{Q}_p)$

- $F = \mathbb{Q}_p$, where p is a prime number.
- $G(F) = SL(2,\mathbb{Q}_p)$, $M(F) = \left\{ \left(\begin{smallmatrix} a & o \\ o & a^{-1} \end{smallmatrix} \right) : x \in \mathbb{Q}_p^\times \right\}$.
- $\sigma = \chi$: a unitary unramified character on $M(F)$ given by

$$\chi \left(\begin{smallmatrix} a & o \\ o & a^{-1} \end{smallmatrix} \right) = |a|_p^{\pi\sqrt{-1}/\log p}.$$

- $\phi : W_{\mathbb{Q}_p} \longrightarrow \mathbb{C}^1$ is given by χ from the local class field theory.

Then, we have

- $W(G,M) = \{I, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)\} = W_\phi = W_{\phi,\chi}$.
- $W_\chi^\circ = \{I\} = W_\phi^\circ = W_{\phi,\chi}^\circ$.

Therefore,

$$\boxed{R_\chi \simeq R_\phi \simeq R_{\phi,\chi} \simeq \mathbb{Z}/2\mathbb{Z}.}$$

Some obstacles towards general (G, H)

Among many obstacles, we here single out the following:

1. How to control the case of $\sigma_{G,1}, \sigma_{G,2} \in \Pi_{\varphi_G}(G)$ such that

$$\mathrm{Res}_H^G(\sigma_{G,1}) \simeq \mathrm{Res}_H^G(\sigma_{G,2}) \ ?$$

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5. In general, the structure of R-groups for p-adic groups remains open.
   e.g.) $H =$ exceptional groups, etc.

Theorem (C. 2019)

- Assuming LLC for both G and H and further two technical arguments,
- Given $\varphi_H \in \Phi_{\mathrm{temp}}(H), \varphi_G \in \Phi_{\mathrm{temp}}(G)$ such that $\varphi_H = pr \circ \varphi_G$ with $pr : \widehat{G} \twoheadrightarrow \widehat{H}$,
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where $\Pi_{\rho_H}(\mathscr{S}_{\varphi_G,\mathrm{sc}}) := \{\delta \subset \mathrm{Res}^{\mathscr{S}_{\varphi_H,\mathrm{sc}}}_{\mathscr{S}_{\varphi_G,\mathrm{sc}}} \rho_H\} / \simeq .$

Ideals of proof: some Galois cohomological arguments + Clifford theory for finite groups.

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Remark: [Obstacle 2.] How to control the so-called multiplicity $m \in \mathbb{N}$ such that given $\sigma_G \in \Pi_{\varphi_G}(G)$,

$$\text{Res}^G_H(\sigma_G) \simeq m \bigoplus_{\tau_H \in \Pi_{\sigma_G}(H)} \tau_H \ ?$$

by means of ingredients (all finite) in the parameter side.

Corollary (C. 2019)

Assume as in Theorem above, One can control $\Pi_{\sigma_G}(H) \subset \Pi_{\varphi_H}(H)$ *in terms of* $\mathscr{S}_{\varphi_H}$*-groups or* $\mathscr{S}_{\varphi_H,\mathrm{sc}}$*-groups:*

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Remark: [Obstacle 3.] Is there any way to characterize $\Pi_{\sigma_G}(H)$? by means of ingredients (all finite) in the parameter side.

Restriction in Pseudo-z-embedding

Following T. Kaletha 2018, given a connected reductive group G over a non-archmedean local field F, an embedding from G to another connected reductive group G_z over F is called to be **a pseudo-z-embedding** of G if:

(1) the cokernel G_z/G is a torus;

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Basic Facts:

- Such embedding G_z always exists for any G.
- There is a bijection between F-Levi subgroups M_z of G_z and M of G, via $M = M_z \cap G$.
- $G_z(F) = Z(G)G(F)$ which yields

$$\mathrm{Res}^{G_z}_{G}(\sigma_z) \in \mathrm{Irr}(G).$$

- There exists a bijection:

$$\mathrm{Irr}(\mathscr{S}_{\varphi_{G_z},\mathrm{sc}}, \zeta_{G_z}) \overset{1-1}{\longleftrightarrow} \mathrm{Irr}(\mathscr{S}_{\varphi_G,\mathrm{sc}}, \zeta_G)$$

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Remark: [Obstacle 4.] In general, not easy to find G and M_G with such "nice" properties, by means of a setting with some cohomological conditions on groups.

Merry Christmas & Happy New Year!