Adelic Rogers integral formula

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Let us start by looking at the celebrated Siegel integral formula.

Theorem (Siegel)

Let $G = SL(n, \mathbb{R}), \Gamma = SL(n, \mathbb{Z})$*. Also let* $e_n = (0, \ldots, 0, 1) \in \mathbb{R}^n$*, and* $P \subseteq G$ *be the stabilizer of* e_n*. Then for measurable* $f : \mathbb{R}^n \to \mathbb{R}$*,*

$$\frac{1}{vol(\Gamma \backslash G)} \int_{\Gamma \backslash G} \sum_{\gamma \in (\Gamma \cap P) \backslash \Gamma} f(e_n \gamma g) dg = \frac{1}{\zeta(n)} \int f(x) dx.$$

There is a lot of mathematics that springs from this theorem, and the goal of this talk is to introduce some of it.

First, some re-wording: $\Gamma\backslash G$ can be identified with the moduli space X_n of all covolume 1 lattices in $\mathbb{R}^n$ by the correspondence

$$g \in G \leftrightarrow (\mathbb{Z}\text{-span of the rows of } g).$$

Let μ_n be the unique right G-invariant probability measure on X_n. Then Siegel's formula can be rephrased as follows.

Theorem

For any $f : \mathbb{R}^n \to \mathbb{R}$ measurable,

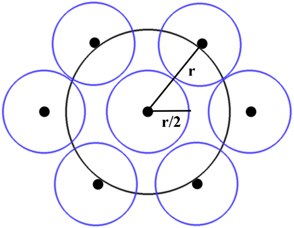
$$\int_{X_n} \sum_{\substack{x \in L \\ x \neq 0}} f(x) d\mu_n(L) = \int f(x) dx.$$

Seen this way, it's an average of a lattice point counting formula.

Siegel's original motivation was to provide insight on the following theorem.

Theorem (Minkowski-Hlawka)

There exists a lattice $L \in X_n$ such that $B(1) \cap L = \{0\}$, where $B(V)$ here is a ball at center of volume V. In other words, there exists a lattice packing of $\mathbb{R}^n$ by spheres of density 2^{-n}.



Note (density) $= vol(ball)/cov(lattice)$.

Proof (rough): let $f(x)$ be the characteristic function of $B(1)$. By Siegel's formula,

$$\int_{X_n} \sum_{\substack{x \in L \\ x \neq 0}} f(x) d\mu_n(L)$$
$$= \int_{X_n} |B(1) \cap L \backslash \{0\}| d\mu_n(L)$$
$$= \int f(x) dx = 1,$$

and thus there exists $L \in X_n$ such that $B(1) \cap L \backslash \{0\}$ is empty.

This theorem is not as trivial as it might seem. I know of no construction achieving 2^{-n} density for arbitrary n.

Natural question: can we compute the higher moments of $|B \cap L|$?

Theorem (Rogers)

Let $1 \leq k < n$ and $f\colon (\mathbb{R}^n)^k \to \mathbb{R}$ be measurable. Then

$$\int_{X_n} \sum_{\substack{x_1,\ldots,x_k \in L \\ \text{indep.}}} f(x_1,\ldots,x_k)\, d\mu_n(L) = \int f(x_1,\ldots,x_k)\, dx_1 \ldots dx_k.$$

There exists a similar formula for the independence condition being dropped.

This allows us to compute the moments of $|B \cap L|$ or use some other tricks to prove a result, such as

Theorem (W. Schmidt, very roughly paraphrased)

For every $n \geq 13$, there exists about $e^{-0.1n}$ μ_n-measure of lattices in X_n that attains the packing density of at least $0.283n2^{-n}$.

Notice the factor of n improvement from Minkowski-Hlawka. There have been numerous attempts over the last 70 years to improve on this bound, but it has been at most by a constant term.

Current best record, for your information:

Theorem (Venkatesh)

(i) For $n \gg 0$, there exists a lattice in X_n with packing density $\geq 65963n2^{-n}$.
(ii) For infinitely many n, there exists a lattice with packing density $0.5n \log\log n2^{-n}$.

Both are done by averaging techniques, but (i) is more of the Siegel mass formula. (ii) uses the Siegel integral formula generalized to cyclotomic fields.

More recent applications in dynamics

Although my personal motivation aligns more with the story so far...

...recently, there has been a flurry of applications and extensions of the Rogers integral formula, many in homogeneous dynamics. It has become a staple tool in the field, playing a role in a number of influential works — Eskin-Margulis-Mozes, Kleinbock-Margulis, etc. I cannot do justice to all of them, but would like to introduce a few.

Logarithm laws.

Theorem (Kleinbock-Margulis, Athreya-Margulis, Kelmer-Yu, ...)

Let $\{g_t\}_{t\in\mathbb{R}}$ be an unbounded one-parameter flow on G and d be the distance function in X_n. Fix $\Lambda_0 \in X_n$. Then for almost every $\Lambda \in X_n$,

$$\limsup_{t\to\infty} \frac{d(\Lambda g_t, \Lambda_0)}{\log t} = \frac{1}{n}.$$

This is in fact true in much greater generality — see Kelmer-Yu, K.-Skenderi, etc. The Rogers formula gives the upper bound for free for most statements of this kind, since $d(\Lambda, \Lambda_0) \sim$ (shortest nonzero vector length of Λ).

Oppenheim conjecture-ish statements.

Theorem (Athreya-Margulis)

For an indefinite quadratic form Q in $n \geq 3$ variables, $-\infty < a \leq b < \infty$ and $T > 0$, let

$$N(Q, a, b, T) = |Q^{-1}(a, b) \cap \mathbb{Z}^n \cap B_T|;$$

here $B_T = (\text{ball at center of radius } T)$. Then for every $\delta > 0$, and almost every Q

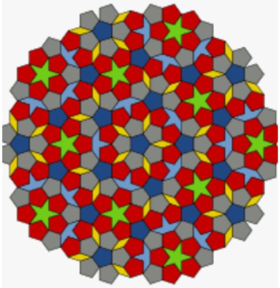
$$N(Q, a, b, T) = c_Q(b - a)T^{n-2} + o(T^{(n-1)/2+\delta})$$

for some constant c_Q.

For proof, they use the Rogers integral formula to show that the probability that $N(\Lambda \cdot Q, a, b, T)$ deviates from the norm by $T^{(n-1)/2+\delta}$ is $\ll T^{-const \cdot \delta}$, and applies the Borel-Cantelli lemma on an appropriate sequence of T.
There are numerous generalizations, to inhomogeneous quadratic forms (Ghosh-Kelmer-Yu) and to homogeneous higher-degree polynomials (Kelmer-Yu).

Counting different objects.

- Affine lattices (El-Baz-Marklöf-Vinogradov, Alam-Ghosh-Han, etc.)
- Over function fields (Thunder)
- Rational points on Grassmannian/flags (Thunder, K.)
- Saddle connections of translation surfaces (Veech)
- Cut-and-project sets e.g. Penrose tiling (Rühr-Smilansky-Weiss)



The latter two appeals to ergodic theoretic methods for proof, e.g. Ratner's theorem.

Adelic version

Motivations

- Venkatesh's work mentioned above; his idea is that if we consider the moduli space of lattices with structures, perhaps a stronger sphere packing bound can be finessed.
- Generalization of the dynamics results to all number fields and all levels.

Theorem (K.)

Let F be a number field, $1 \leq k < n$, and $f : (\mathbb{A}_F^n)^k \to \mathbb{R}$ be integrable. Write

$$G_n = \{g \in GL(n, \mathbb{A}_F) : \| \det g \|_{\mathbb{A}} = 1\},$$

and $\mathcal{X}_n = GL(n, F)\backslash G_n$.

$$\int_{\mathcal{X}_n} \sum_{\substack{x_1, \ldots, x_k \in F^n \\ \text{indep.}}} f(x_1 g, \ldots, x_k g) d\mu_n(g) = \int f \, d\alpha,$$

where μ_n is the right G_n-invariant probability measure on $\mathcal{X}_n$, and α is the Tamagawa measure on $(\mathbb{A}_F^n)^k$.

There are similar formulas where the independence condition is dropped or modified.

Why are statements of this kind true?

Siegel's formula can be viewed as a constant term computation of the pseudo-Eisenstein series

$$\sum_{\substack{x \in \mathbb{Z}^n g \\ x \neq 0}} f(x).$$

But

$$\sum_{\substack{x_1, \ldots, x_k \in F^n \\ indep.}} f(x_1 g, \ldots, x_k g)$$

for $k > 1$ doesn't seem to have a pretty interpretation as a pseudo-Eisenstein series. It's a sum over $(\Gamma \cap P_0)\backslash\Gamma$ where P_0 is a "similar-looking" but much smaller subgroup of a parabolic.

There are also arguments from an ergodic theoretic perspective. One uses the fact that, for instance, the only $SL(n, \mathbb{R})$-invariant measure on $\mathbb{R}^n$'s are linear combinations of the Lebesgue measure and the Dirac delta at zero. In this line of argument, the hard part is to show that

$$\sum_{\substack{x \in \mathbb{Z}^n g \\ x \neq 0}} f(x)$$

is integrable in X_n.

Another idea is to replace the integration over X_n to the integration over a unipotent orbit, since it tends to be equidistributed in X_n. In fact, this is what Rogers seemed to have exploited back in 1955. However, his proof contains an error (which was discovered less than a year ago), though we now know his claims are correct by hindsight.

My idea was inspired by Rogers' but I replaced the unipotent orbit by certain Hecke operators to fill in his gap. Writing

$$\hat{f} = \sum_{\substack{x_1,\ldots,x_k \in F^n \\ indep.}} f(x_1 g, \ldots, x_k g)$$

for short, I show that along a certain sequence of primes p of F,

$$T_p \hat{f} \to \int f \, d\alpha$$

as $Np \to \infty$. Then the theorem follows by some measure theoretic argument essentially due to Rogers.

To be precise, I do this for f for which $\hat{f}$ is fixed by $\prod_{\nu \nmid \infty} GL(n, \mathcal{O}_\nu)$, i.e. the level 1 case; the general case follows from this by some unfolding trick.

Some details of the computation:

For example, consider the case $k = 1$, $f = f_{fin} f_\infty$ where f_{fin} is the characteristic function of $\prod_{\nu \nmid \infty} \mathcal{O}_\nu$. Then

$$T_p \hat{f}(g) = \frac{1}{\omega_p(K_p a_p K_p)} \int_{K_p a_p K_p} \hat{f}(x(gN(p)_\infty^{\frac{1}{nd}} h)^*) d\omega_p(h).$$

where $K_p = GL(n, \mathcal{O}_p)$, ω_p is the Tamagawa measure on $GL(n, F_p)$, $a_p = diag(\pi_p, 1, \ldots, 1)$, $*$ is the inverse transpose, d is the degree of $F/\mathbb{Q}$, and

$$N(p)_\infty^{\frac{1}{nd}} = (N(p)^{\frac{1}{nd}}, \ldots, N(p)^{\frac{1}{nd}}) \in \mathbb{A}_\infty$$

is a normalizing factor.

For principal $p = (\pi_p)$, the main term of this roughly looks like

$$\frac{1}{Np^{n-1}} \sum_{x_1,\ldots,x_{n-1}} \sum_{\substack{A\in\mathcal{O}_F^n \\ A\neq 0}} f_\infty(Np^{-\frac{1}{nd}} \left(\textstyle\sum_{i=1}^{n-1} a_i x_i + \pi_p a_0, \quad a_1, \quad \ldots, \quad a_{n-1}\right) g),$$

where each x_i runs over a set of representatives of $\mathbb{F}_p := \mathcal{O}_F/p\mathcal{O}_F$, and $A = (a_0, \ldots, a_{n-1})$. The idea is that $\sum_{i=1}^{n-1} a_i x_i$ is a surjection $\mathbb{F}_p^{n-1} \to \mathbb{F}_p$ for most A, and thus $\sum_{i=1}^{n-1} a_i x_i + \pi_p a_0$ is "equidistributed" in $\mathcal{O}_F$. Thus this becomes, up to vanishing errors,

$$\frac{1}{Np} \sum_{\substack{A\in\mathcal{O}_F^n \\ A\neq 0}} f_\infty(Np^{-\frac{1}{nd}} Ag),$$

which one can use standard lattice-point counting estimate to show approaches $\int f \, d\alpha$ as $Np \to \infty$.

Second moment estimate: this suffices for most applications in dynamics.

For simplicity, take $f\colon \mathbb{A}_F^n \to \mathbb{R}$ to be of form $f_{fin}f_\infty$, where f_{fin} is as earlier, and f_∞ is a characteristic function of a ball or an annulus at origin. Then

Theorem (K.)

$$\int_{X_n} \left(\sum_{x \in F^n \setminus \{0\}} f(xg) \right)^2 d\mu_n - (\alpha^n(f))^2 = O_F(\alpha^n(f)).$$

The right-hand side is actually

$$\sum_{c \in F^*} \int_{\mathbb{A}_F^n} f(x) f(cx) d\alpha^n,$$

which is

$$= \sum_u \int_{\mathbb{A}_F^n} f(x) f(ux) d\alpha^n + 2 \sum_q \sum_p \sum_u \int_{\mathbb{A}_F^n} f(x) f(I(pq^{-1})ux) d\alpha^n$$
$$\propto \sum_u \int_{\mathbb{A}_\infty^n} f(x) f(ux) d\alpha^n + 2 \sum_q \sum_p \sum_u \frac{1}{Nq^n} \int_{\mathbb{A}_\infty^n} f(x) f(I(pq^{-1})ux) d\alpha^n$$

where q ranges over integral ideals of F, p over integral ideals in the class of q such that $Np < Nq$ and $(p, q) = 1$, u over units, and $I(pq^{-1})$ indicates a choice of an element of F generating the principal fractional ideal pq^{-1}.

Pesky units!

As perhaps anticipated from the previous slide, units are the biggest nuisance in evaluating higher moments, e.g. terms like

$$\sum_{u_1,u_2,u_3} \int f(x_1)f(x_2)f(x_3)f(u_1x_1 + u_2x_2 + u_3x_3)$$

would appear in a fourth moment computation. I am currently working on this.