Vorticity Convergence from Boltzmann to 2D Incompressible Euler equations below Yudovich class

Chanwoo Kim (UW-Madison)

France-Korea IRL webinar in PDE

Hilbert's 6th Problem

N-body Problems $\longrightarrow$ *Boltzmann Equations* $\longrightarrow$ *Fluid PDEs*

"Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes, there merely indicated, which lead from the atomistic view to the laws of motion of continua"- David Hilbert 1900

Classical N-body Problem

- N identical particles, elastic collision
- Liouville equation in punctured phase space, specular reflection BC at the boundary
- Chaos property, Grad-Boltzmann limit

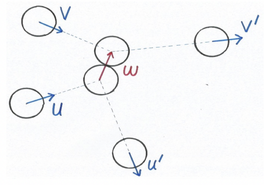
Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = Q(F, F)$$

where

$$Q(F,F)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v-u)\cdot\omega| [F(v')F(u') - F(v)F(u)] \mathrm{d}\omega \mathrm{d}u$$

with $v' = v - [(v-u)\cdot\omega]\omega, \quad u' = u + [(v-u)\cdot\omega]\omega.$



- Equilibrium solutions: Maxwellian

$$M_{R,U,T}(v) = \frac{R}{(2\pi T)^{3/2}} \exp\left\{ -\frac{|v-U|^2}{T} \right\}$$

Scales

- rescale variables (t, x, v) and collision kernel $|(v-u)\cdot\omega|$:

$$\hat{t} = \frac{t}{T},\ \hat{x} = \frac{x}{L},\ \hat{v} = \frac{v}{c},\ |(\hat{v}-\hat{u})\cdot\hat{\omega}| = \frac{1}{\pi r^2 c}|(v-u)\cdot\omega|$$

where the speed of sound $c = \sqrt{\frac{5}{3}\frac{k\Theta}{m}}$.
- Mean free path$\sim \frac{1}{(N/V)\times A} = \frac{1}{(N/L^3)\times\pi r^2}$: average distance between two successive collisions for one gas molecule picked at random
- dimensionless distribution: $\hat{F}(\hat{t},\hat{x},\hat{v}) = \frac{L^3 c^3}{N^3}F(t,x,v)$
- dimensionless Boltzmann equation

$$\frac{L}{cT}\partial_{\hat{t}}\hat{F} + \hat{v}\cdot\nabla_{\hat{x}}\hat{F} = \frac{L}{\text{Mean free path}}\hat{Q}(\hat{F},\hat{F})$$

- $$\boxed{St\partial_{\hat{t}}\hat{F} + \hat{v}\cdot\nabla_{\hat{x}}\hat{F} = \frac{1}{Kn}\hat{Q}(\hat{F},\hat{F})}$$

- Mach number (Ma): $F \sim M_{1,0,1} + O(Ma)$
- $St = \frac{L}{cT} \to c \times St = \frac{L}{T} \geq Ma$: $Ma \lesssim St$
- $St \downarrow 0$ (incompressible) vs $St \sim 1$ (compressible) as $Kn \downarrow 0$
- von Karman relation: $Re\ (Reynolds\ numbers) = \frac{Ma}{Kn}$
- incomp. Navier-Stokes: $St \sim Kn \sim Ma$ (biggest Ma plus $Re \sim 1$)
- incomp. Euler: $St \sim Ma$ and $Kn = \kappa \times Ma$ with $\kappa = O(Ma)$ (biggest Ma plus $Re \uparrow \infty$)
- incomp. Euler limit: Consider solutions $|F_\varepsilon - M_{1,0,1}| = O(\varepsilon)$ solves

  $$\varepsilon \partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\kappa\varepsilon} Q(F_\varepsilon, F_\varepsilon).$$

  Goal: $F_\varepsilon \sim M_{1,\varepsilon u(t,x),1}$ as $\varepsilon \downarrow 0$ and

  $$\begin{aligned} \partial_t u + u \cdot \nabla_x u + \nabla_x p &= 0 \\ \nabla_x \cdot u &= 0 \end{aligned}$$

Key features of incompressible Euler limit

- "more" Singular limit problem, cf. Navier-Stokes limit (e.g. Golse–Saint-Raymond 2004, Esposito-Guo-K-Marra 2018)
- Need to know Euler solutions, e.g. Lipschitz smoothness (Saint-Raymond 2003)
- "As long as the target solutions are smooth then well-prepared solutions of Boltzmann may have convergence", e.g. , Nishida 1978, Caflisch 1980, De Masi-Esposito-Lebowitz 1989, Guo 2006, etc

2D Incompressible Euler equations

In 2D, $\omega = \nabla^\perp \cdot u = \partial_1 u_2 - \partial_2 u_1$

$$\partial_t \omega + \text{div}\,(\omega u) = 0, \quad u = -\nabla^\perp(-\Delta)^{-1}\omega \tag{1}$$

- Existence and Uniqueness for $\omega \in L^\infty$ by Yudovich (1963) and now-called Yudovich space by Yudovich (1995)

$$\|\omega\|_{Y^\Theta_{\mathrm{ul}}(\Omega)} := \sup_{1\le \mathfrak{p}<\infty} \frac{\|\omega\|_{L^\mathfrak{p}(\Omega)}}{\Theta(\mathfrak{p})}, \quad \Theta(\mathfrak{p}) = \prod_{k=1}^{m} \log_k \mathfrak{p} \tag{2}$$

  for large $\mathfrak{p} > 1$, where $\log_k \mathfrak{p} = \log \circ \cdots \circ \log \mathfrak{p}$ with $\log_0 \mathfrak{p} = 1$.
- Osgood continuity of u implies uniqueness: $|u(t,x) - u(t,y)| \le C(t)\rho(|x-y|)$ with $C(t) \in L^1(0,T)$ and $\int_0^1 \frac{1}{\rho(s)} ds = \infty$
- Existence of $\omega \in L^p$ for $1 < p < \infty$ by DiPerna-Majda (1987)
- Uniqueness for $\omega \in L^p$ for $p < \infty$: Major open problem

Lagrangian solution

- For $u \in L^1_{t,x}$, a regular Lagrangian flow of

$$\frac{d}{ds}X(s;t,x) = u(s, X(s;t,x)), \quad X(s;t,x)|_{s=t} = x. \tag{3}$$

  - $s \in [0,t] \mapsto X(s;t,x)$ is a absolutely continuous integral solution for a.e. x and any t.
  - $X(s;t,x) = x$ for a.e. x.
  - There exists $\mathfrak{C} > 0$ such that $\int \phi(X(s;t,x))\mathrm{d}x \leq \mathfrak{C} \int \phi(x)\mathrm{d}x$ for any measurable function $\phi \geq 0$.
- $\omega \in L^p$ for $p \in [1,\infty]$: Existence and uniqueness of regular Lagrangian flow (DiPerna-Lions (1989))
- Lagrangian solution: $\omega(t,x) = \omega_0(X(0;t,x))$

Main Theorem

We define the Boltzmann (macroscopic) velocity and Boltzmann vorticity:

$$\begin{aligned} u_B^\varepsilon(t,x) &= \frac{1}{\varepsilon}\int_{\mathbb{R}^3} \underline{v}(F^\varepsilon(t,x,v) - M_{1,0,1}(v))\mathrm{d}v, \\ \omega_B^\varepsilon(t,x) &= \nabla^\perp \cdot u_B^\varepsilon(t,x). \end{aligned} \tag{4}$$

Theorem (Joonhyun La-K)

For an arbitrary $T \in (0,\infty)$, suppose $(u_0,\omega_0) \in L^2(\Omega)\times L^{\mathfrak{p}}(\Omega)$ for $\mathfrak{p}\in[1,\infty)$ and (u,ω) be a Lagrangian solution of (E) in $[0,T]$. There exists a family of Boltzmann solutions $F^\varepsilon(t,x,v)$ such that

$$\sup_{t\in[0,T]} \left\| \frac{F^\varepsilon(t) - M_{1,\varepsilon u(t),1}}{\varepsilon\sqrt{M_{1,0,1}}} \right\|_{L^2(\mathbb{T}^2\times\mathbb{R}^3)} \to 0 \tag{5}$$

Moreover the Boltzmann vorticity converges to ω:

$$\omega_B^\varepsilon \to \omega. \tag{6}$$

Theorem (Joonhyun La-K)

Choose an arbitrary $T \in (0, \infty)$. *Suppose* $\omega_0 \in Y_{\mathrm{ul}}^{\Theta}(\mathbb{T}^2)$, *and let* (u, ω) *be a unique solution of (E). Then there exists a family of Boltzmann solutions* $F^\varepsilon(t, x, v)$ *such that*

$$\sup_{t \in [0,T]} \left\| \frac{F^\varepsilon(t) - M_{1,\varepsilon u(t),1}}{\varepsilon \sqrt{M_{1,0,1}}} \right\|_{L^2(\mathbb{T}^2 \times \mathbb{R}^3)} \to 0. \tag{7}$$

Moreover, with an explicit rate,

$$\begin{aligned} \sup_{0 \le t \le T} \|u_B^\varepsilon(t,\cdot) - u(t,\cdot)\|_{L^2(\mathbb{T}^2)} &\lesssim \mathrm{Rate}(\beta), \\ \sup_{0 \le t \le T} \|\omega_B^\varepsilon(t,\cdot) - \omega(t,\cdot)\|_{L^{\mathfrak{p}}(\mathbb{T}^2)} &\lesssim \mathrm{Rate}_\omega(\beta). \end{aligned} \tag{8}$$

Highlights

- Selection Principle when $\omega \in L^{\mathfrak{p}}$ with $\mathfrak{p} < \infty$.
- Vorticity Convergence: capture the singularity (e.g. Vortex patch), beyond the relative Entropy method
- Convergence rate when $\omega \in Y^{\Theta}_{\mathrm{ul}}(\mathbb{T}^2)$
- The proof does not rely on inviscid limit of Navier-Stokes, cf. Jang-K 2021

Difficulties

- Global Maxwellian vs Local Maxwellian: Hilbert expansion around the global Maxwellian does not work: $M_{1,0,1} + \varepsilon f_1 \sqrt{M_{1,0,1}} + \cdots$ produces an $e^{t/\kappa}$ growth for the remainder.
- Without using inviscid limit of Navier-Stokes, we cannot go beyond the Lipschitz regularity of u.
- velocity mixing is weak (e.g. velocity average/elliptic regularity).
- Commutator makes higher derivatives estimate worse.

Key Ideas

1. Set $\mu = M_{1,\varepsilon u^\beta,1}$ (global Maxwellian is not working!) where $u^\beta = \varphi^\beta * u$

$$F^\varepsilon = \mu + \varepsilon^2 f_2\sqrt{\mu} + \varepsilon f_R\sqrt{\mu} \tag{9}$$

Then

$$\partial_t f_R + \frac{v}{\varepsilon}\cdot\nabla_x f_R + \left(\frac{(\partial_t + \frac{v}{\varepsilon}\cdot\nabla_x)\sqrt{\mu}}{\sqrt{\mu}}\right) f_R + \frac{1}{\varepsilon^2\kappa}Lf \tag{10}$$

$$= \frac{1}{\varepsilon\kappa}\Gamma(f_R, f_R) - \{\underbrace{\partial_t u + u\cdot\nabla_x u + \nabla_x p}_{=0(Euler)} - \kappa\Delta u\} + \cdots \tag{11}$$

2. At best an Energy $f_R \sim \kappa$; Dissipation $(\mathbf{I}-\mathbf{P})f_R \sim \varepsilon\kappa^{3/2}$; elliptic regularity/velocity average $\mathbf{P}f_R \sim \kappa^{1/2}$

$$v \cdot \nabla_x f_R \sim \frac{1}{\varepsilon\kappa} L(\mathbf{I}-\mathbf{P})f_R \sim \kappa^{1/2} \tag{12}$$

But nonlinear estimate:

$$\int \frac{1}{\varepsilon}\Gamma(\frac{f_R}{\kappa},\frac{f_R}{\kappa})\frac{f_R}{\kappa} \sim \|\kappa^{1/2}\Gamma(\frac{f_R}{\kappa},\frac{f_R}{\kappa})\|_{L^2}\|\varepsilon^{-1}\kappa^{\frac{-3}{2}}(\mathbf{I}-\mathbf{P})f_R\|_{L^2}$$
$$\lesssim \kappa^{-1/2}\|\mathbf{P}f_R\|_{L^\infty}\sqrt{\mathcal{E}}\sqrt{\mathcal{D}}$$

3. Higher order estimate. Commutator $[[\mathbf{P}, \nabla_x]]$: $\nabla_x \sim \kappa^{-1/2}$, Energy and Dissipation

$$\mathcal{E}(t) = \sum_{s\leq 2} \|\kappa^{-1+\frac{s}{2}}\partial^s f_R\|^2_{L^\infty_t L^2_{x,v}} \tag{13}$$

$$\mathcal{D}(t) = \sum_{s\leq 2} \|\varepsilon^{-1}\kappa^{-\frac{3}{2}+\frac{s}{2}}\nu^{1/2}(\mathbf{I}-\mathbf{P})\partial^s f_R\|^2_{L^\infty_t L^2_{x,v}} \tag{14}$$

Recall the Agmon: $\|\mathbf{P}f_R\|_{L^\infty_x} \lesssim \|\mathbf{P}f_R\|^{1/2}_{L^2_x}\|\nabla^2_x\mathbf{P}f_R\|^{1/2}_{L^2_x} \sim \kappa^{1/2}1^{1/2}$

5. Cancel the diffusion term:

$$F^\varepsilon = \mu + \varepsilon^2 p^\beta \mu - \varepsilon^2 \kappa(\nabla_x u^\beta) : A\sqrt{\mu} + \varepsilon\kappa \tilde{u}^\beta \cdot (v - \varepsilon u^\beta)\mu + \varepsilon^2 \kappa \tilde{p}^\beta \mu + \varepsilon f_R \sqrt{\mu}, \tag{15}$$

- A pressure $-\Delta p^\beta = \mathrm{div}(\mathrm{div}(u^\beta \otimes u^\beta))$ with $\int_\Omega p^\beta = 0$.

$$\begin{aligned} (\partial_t + u^\beta \cdot \nabla_x)u^\beta + \nabla_x p^\beta = 0, (t,x) \in (0,T) \times \mathbb{T}^2 \\ \nabla_x \cdot u^\beta = 0, (t,x) \in (0,T) \times \mathbb{T}^2 \\ u^\beta(x,0) = u_0^\beta(x), x \in \mathbb{T}^2 \end{aligned} \tag{16}$$

- $LA = v \otimes v - \frac{|v|^2}{3} id$
- Also, we will consider the following auxiliary equation.

$$\begin{aligned} (\partial_t + u^\beta \cdot \nabla_x)\tilde{u}^\beta + \tilde{u}^\beta \cdot \nabla_x u^\beta + \nabla_x \tilde{p}^\beta - \eta_0 \Delta_x u^\beta = 0, \\ \nabla_x \cdot \tilde{u}^\beta = 0, (t,x) \in (0,T) \times \mathbb{T}^2 \\ \tilde{u}^\beta(0,x) = \tilde{u}_0(x), x \in \mathbb{T}^2. \end{aligned} \tag{17}$$

The key reason to introduce correctors $\varepsilon\kappa\tilde{\boldsymbol{u}}^{\beta} \cdot (v - \varepsilon \boldsymbol{u}^{\beta})\mu$ and $\varepsilon^2\kappa\tilde{\boldsymbol{p}}^{\beta}\mu$ is to get rid of hydrodynamic terms of order $\varepsilon^2\kappa$ (diffusion): as a payback, we obtained terms of order $\varepsilon\kappa$, which is larger, but all of them are non-hydrodynamic, so they are small in our scale:

$$\text{cancel} \;\; \varepsilon^2\kappa \in \mathcal{N} \text{ and get } \varepsilon\kappa \in \mathcal{N}^{\perp} \tag{18}$$

6. Stability of regular Lagrangian flow when $\omega \in L^{\mathfrak{p}}$ for $\mathfrak{p} \geq 1$ (Crippa-De Lellis 2008)

Thank You!