The Fokker-Planck equation as a gradient-flow in the Wasserstein space : estimates in the time-discrete scheme

GRAD. FLOWS

$$\begin{cases} x'(t) = -\nabla F(x(t)) \\ x(0) = x_0 \end{cases} \qquad F : \mathbb{R}^n \to \mathbb{R}$$

TIME-DISCR. FIX $\tau > 0$ $\quad x_0, x_1, x_2, \dots \quad x_k \simeq x(k\tau)$

DEFINED VIA

$$x_{k+1} \in \operatorname{argmin} F(x) + \frac{|x - x_k|^2}{2\tau}$$

$$\nabla F(x) + \frac{x - x_k}{\tau} = 0$$

THIS APPROACH CAN BE EXTENDED TO METRIC SPACES

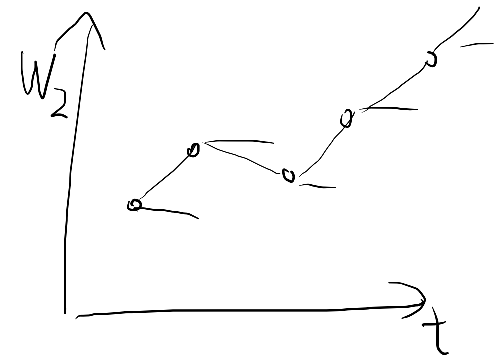
$$\min F(x) + \frac{d^2(x, x_k)}{2\tau}$$

we consider $(X, d) = (\mathcal{P}(\Omega), W_2)$

$t \longmapsto \rho_t \qquad (t,x) \longmapsto \rho(t,x)$

$$\boxed{\rho_{n+1} \in \operatorname{argmin} \; F(\rho) + \frac{W_2^2(\rho, \rho_n)}{2\tau}}$$

$$F : \mathcal{P}(\Omega) \longrightarrow \overline{\mathbb{R}}$$



THIS SCHEME IS CALLED JKO SCHEME

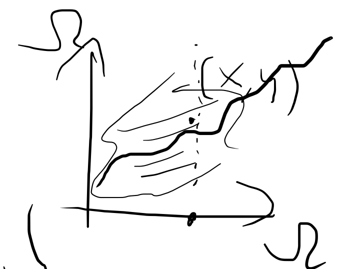
FEW FACTS ABOUT O.T.

given $\mu, \nu \in \mathcal{P}(\Omega)$ $\quad \Omega \subset \mathbb{R}^d$

$$\min \left\{ \int \frac{1}{2} |x - T(x)|^2 \, d\mu \ : \ T_\# \mu = \nu \right\}$$

$$\min \left\{ \int_{\Omega \times \Omega} \frac{1}{2} |x - y|^2 \, d\gamma \quad \gamma \in \Pi(\mu, \nu) \right\}$$

$$\Pi(\mu,\nu) = \left\{ \gamma \in \mathcal{P}(\Omega \times \Omega) : \begin{array}{l} \pi_{x\#}\gamma = \mu \\ \pi_{y\#}\gamma = \nu \end{array} \right\}$$



$$\max \left\{ \int \varphi \, d\mu + \int \psi \, d\nu \ : \ \varphi(x) + \psi(y) \le \frac{1}{2} |x - y|^2 \right\}$$

FACTS $\quad$ if $\mu \ll \mathcal{L}^d$ $\quad \exists! \ \gamma_{OPT}$ $\quad \gamma = (id, T)_\# \mu$ $\qquad T(x) = x - \nabla \varphi(x) = \nabla u$ $\quad$ (BRENIER) $\quad u$ convex

$\exists$ optimal $\varphi \in Lip$

$\frac{|x|^2}{2} - \varphi$ convex

$I - D^2 \varphi \geq 0$

WITH OT WE CAN DEFINE A DISTANCE

$$W_2(\mu,\nu) = \sqrt{\min\left\{\int |x-y|^2 \, d\gamma \; : \; \gamma \in \Pi(\mu,\nu)\right\}}$$

$$\frac{1}{2} W_2^2(\mu,\nu) = \max\left\{\int \varphi \, d\mu + \int \psi \, d\nu \; : \; \varphi \oplus \psi \le \frac{1}{2}|x-y|^2 \right\}$$

IT IS A DISTANCE ON $\mathcal{P}(\Omega)$ (IF Ω IS COMPACT)

WHICH METRIZES $\overset{*}{\rightharpoonup}$

OPTIMALITY CONDITIONS OF

$$\min\; F(\rho) + \frac{W_2^2}{2\tau}(\rho, \rho_n)$$

NOTATION: $\dfrac{\delta F}{\delta \rho}$ = FIRST VARIATION OF F DEFINED VIA

$$\frac{d}{d\varepsilon} F(\rho + \varepsilon \chi) \Big|_{\varepsilon=0} = \int \frac{\delta F}{\delta \rho} \, d\chi$$

$\dfrac{\delta F}{\delta \rho}$ DEFINED UP TO ADD. CONSTS

EXAMPLES

$F(\rho) = \int V \, d\rho \Rightarrow \dfrac{\delta F}{\delta \rho} = V$

$F(\rho) = \int f(\rho(x)) \, dx \Rightarrow \dfrac{\delta F}{\delta \rho} = f'(\rho)$

$$\frac{\delta F}{\delta \rho} + \frac{1}{\tau} \frac{\delta}{\delta \rho}\left(\frac{1}{2} W_2^2(\cdot, \rho_n)\right) = C$$

$$\frac{\delta}{\delta \rho} \frac{1}{2} W_2^2(\cdot, \nu) = \varphi$$

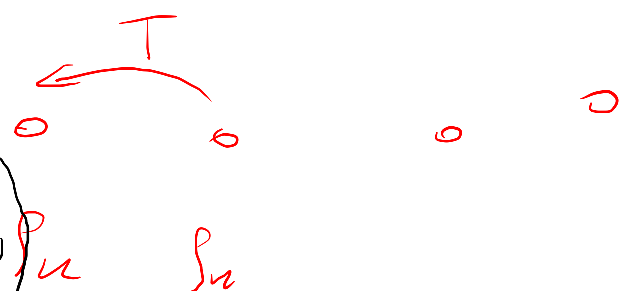
ν fixed

← optimal function (Kant. potential) in this transport from ρ to ν

$$\Rightarrow \nabla \frac{\delta F}{\delta \rho} + \underbrace{\frac{\boxed{\nabla \varphi}}{\tau}}_{=\text{ Discrete } v} = 0$$

$\nabla \varphi = x - T(x)$

$$T_\# \rho_{k+1} = \rho_k$$



The PDE which iteration solves

$$\partial_t \rho + \nabla \cdot (\rho v) = 0$$

$$v = - \nabla \frac{\delta F}{\delta \rho}$$

$$\partial_t \rho + \nabla \cdot (\rho v) = 0$$

$$v = - \nabla \frac{\delta F}{\delta \rho}$$

EXAMPLE $\quad F(\rho) = \int f(\rho(x))\, dx \qquad \frac{\delta F}{\delta \rho} = f'(\rho)$

$$\partial_t \rho - \nabla \cdot (\rho \nabla f'(\rho)) = 0$$

$$\partial_t \rho - \nabla \cdot (\rho f''(\rho) \nabla \rho) = 0$$

IF $f(s) = s \log s \qquad f'(s) = \log s + 1 \qquad f''(s) = \frac{1}{s}$

$\Rightarrow$ HEAT EQUATION

(NOTE THAT HEAT EQ. IS ALSO THE GRAD FLOW OF $F(\rho) = \frac{1}{2}\int |\nabla \rho|^2$
IF I REPLACE W_2 WITH L^2)

Fokker-Planck

$$\partial_t \rho - \Delta\rho - \nabla\cdot(\rho\nabla V) = 0 \qquad V : \Omega \to \mathbb{R}$$

gradient flow in W_2 of $F(\rho) = \int \rho\log\rho + \rho V$

$$\frac{\delta F}{\delta \rho} = \log\rho + V$$

$$\partial_t \rho - \nabla\cdot(\rho\nabla(\log\rho + V)) = 0$$

(Thm by JKO)

ESTIMATES on Fokker Planck via JKO

$$F(\rho) = \int f(\rho) + \rho V \qquad \min F(\rho) + \frac{W_2^2}{2\tau}(\rho, g)$$

$V \in \mathrm{Lip}$ $\qquad$ f convex

$$f'(\rho) + V + \frac{\varphi}{\tau} = c$$

$$T_\# \rho = g \qquad T(x) = x - D\varphi(x) \qquad I - D^2\varphi$$

$$\rho(x) = g(T(x)) \det DT(x)$$

Monge-Ampère eq.

Suppose (for a while) $g \in C^{0,\alpha}$ BDD from below $\quad \varphi \in \mathrm{Lip}$

$\Rightarrow f'(\rho) \in \mathrm{Lip}$

$\Rightarrow \rho \in \mathrm{Lip}$ $\quad$ | $\log \rho \in \mathrm{Lip}$

$\Rightarrow \rho \in \mathrm{Lip}$

$(0 < a \leq \rho \leq b < +\infty)$

Caffarelli's regularity for M.A.

$g, \rho \in C^{0,\alpha}$ BDD below $\Rightarrow \varphi \in C^{2,\alpha}$

Ω convex, smooth

L^∞ Bounds on ρ $\qquad V=0$

$x_0 \in \operatorname{argmax} \rho \Rightarrow \max f'(\rho)$

$\Rightarrow \min \varphi \Rightarrow$ (IF $x_0 \notin \partial\Omega$)

$\nabla\varphi = 0$

$D^2\varphi \geq 0$

$$\rho(x_0) = g(\underset{\substack{\shortparallel \\ x_0}}{T(x_0)}) \det(\underset{\substack{\geq \\ 1}}{\underbrace{I - D^2\varphi(x_0)}})$$

$$\Rightarrow \underset{\substack{\shortparallel \\ \|\rho\|_{L^\infty}}}{\rho(x_0)} \leq g(x_0) \leq \|g\|_{L^\infty}$$

If V not 0 Then $x_0 \in \operatorname{argmax} f'(\rho) + V \Rightarrow \varphi$ min

$$\rho(x_0) = g(x_0)\,\det\big(\underset{\substack{\wedge\!| \\ 1}}{I - D^2\varphi}\big)$$

$$\Rightarrow f'(\rho(x_0)) \leq f'(g(x_0))$$

$$(f'(\rho)+V)(x_0) \leq (f'(g)+V)(x_0) \leq \|f'(g)+V\|$$

$$k \longmapsto \|f'(\rho_k)+V\|_{L^\infty} \searrow$$

DIFFICULT POINT: $x_0 \in \partial\Omega$

IF $\min \varphi$ ATTAINED on $\partial\Omega$

$T(x_0) = x_0 - \nabla\varphi(x_0) \in \Omega \Rightarrow \nabla\varphi(x_0)\cdot n \geq 0$

$\Rightarrow$ NOT POSSIBLE TO HAVE $\min \varphi$ on $\partial\Omega$

WHAT ABOUT $\min \rho$

VERY SAME ARGUMENT, BUT WE HAVE TO EXCLUDE

$$\operatorname{argmax} \varphi \in \partial\Omega$$

IF ρ, g SMOOTH T SMOOTH T IS A DIFFEO : $\Omega \to \Omega$

$$T(\partial\Omega) = \partial\Omega$$

$$T(x_0) = x_0 - \nabla\varphi(x_0) \in \partial\Omega$$

BUT $T = \nabla u$ WAS MONOTONE

L^∞ bounds on $\nabla(\log \rho + V)$ for F-P

$$\nabla\left(\log \rho + V\right) + \frac{\nabla\varphi}{L} = c$$

$$\log \rho = \log \rho \circ T + \log\left(\det\left(I - D^2\varphi\right)\right)$$

$$\frac{\partial}{\partial x_i} \quad (\log \rho)_i = (\log \rho)_j \, T^j_i + B^{jk} \varphi_{jki} \quad \leftarrow \sum_{jk}$$

$$B = \left(I - D^2\varphi\right)^{-1}$$

$$\max \frac{|D\varphi|^2}{2} \Rightarrow D^2\varphi D\varphi = 0$$

$$\varphi_{ijk}\varphi_j + \underbrace{\varphi_{ij}\varphi_{ik}}^{(D^2\varphi)^2} \leq 0$$

$$(\log \rho)_i = (\log g^{(1)})_j \, T^j_i + B^{jk} \varphi_{jki}$$

Thus φ_i

$$\nabla\varphi \cdot \nabla \log \rho = \nabla \log g \cdot (I - D^2\varphi)\nabla\varphi$$

$$- \underbrace{B^{jk}\varphi_{jki}\varphi_i}_{\leq 0}$$

$$\nabla\varphi \cdot \nabla \ln \rho \geq \nabla \log g^{(1)} \cdot \nabla\varphi$$

$V = 0$

$$\nabla \ln g = -\frac{\nabla\varphi}{\tau} \qquad -\frac{|\nabla\varphi|^2}{\tau} \geq \nabla \ln g \cdot \nabla\varphi$$

$$|\nabla \log \rho| = \frac{|\nabla\varphi|^{\cancel{2}}}{\tau} \leq |\nabla \log \rho| + \cancel{\nabla\varphi}|$$

$$\vee$$

$$\|\nabla \log \rho\|_{L^\infty} \qquad K \longmapsto \|\nabla \log \rho_K\|_{L^\infty} \searrow$$

(IF $V = 0$ IF NOT MOD ∇V
(MOD THE $D^2 V \geq \lambda I$)

OF COURSE ONE NEEDS TO EXCLUDE THE
CASE $x_0 \in \partial\Omega$

<u>PROP</u> IF φ, g SMOOTH φ KANT POT $\Omega = B(0,R)$
$\max |\nabla\varphi|^2$ IS NEVER ATTAINED ON $\partial\Omega$

$$|x - \nabla\varphi(x)|^2 = R^2 \quad \text{FOR} \quad |x| = R \qquad \Rightarrow \quad x \cdot \nabla\varphi = \frac{1}{2}|\nabla\varphi|^2$$

Thm

- If $a \leq \log \rho_0 + V \leq b \Rightarrow \forall k$

$$a \leq \log \rho_k + V \leq b$$

(if Ω is convex, any V)

- If $\mathrm{Lip}(\log \rho_0 + V) \leq M$

If Ω is convex $D^2 V \geq \lambda I$

$$\Rightarrow \quad \mathrm{Lip}(\log \rho_k + V) \leq M (1 - \lambda \tau)^k$$

$$\mathrm{Lip}(\log \rho_t + V) \leq \mathrm{Lip}(\log \rho_0 + V)\, e^{-\lambda t}$$