On the stability of Hermite spectral methods for the Vlasov-Poisson system and Fokker-Planck equation

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Part I. Vlasov-Poisson system (with M. Bessemoulin-Chatard)

The Vlasov-Poisson system

The Vlasov-Poisson equations of the plasma in dimensionless variables can be written as,

$$\begin{cases} \dfrac{\partial f}{\partial t} + v\,\dfrac{\partial f}{\partial x} + E\,\dfrac{\partial f}{\partial v} \;=\; 0\,, \\[2mm] \dfrac{\partial E}{\partial x} = \rho - \rho_0\,, \\[2mm] f(t=0) = f_0\,, \end{cases} \tag{1}$$

where the density ρ is given by

$$\rho(t,x) \;=\; \int_{\mathbb{R}} f(t,x,v)\,\mathrm{d}\,v\,, \quad t \geq 0,\, x \in \mathbb{T}.$$

To ensure the well-posedness of the Poisson problem we add the compatibility (or normalizing) condition

$$\int_{\mathbb{T}} \rho(t,x)\,\mathrm{d}\,x = \int_{\mathbb{T}}\int_{\mathbb{R}} f(t,x,v)\,\mathrm{d}\,v\,\mathrm{d}\,x \;=\; \mathrm{mes}(\mathbb{T})\,\rho_0, \quad \forall\; t \geq 0,$$

My aim today

To present and study the stability of a class of conservative Spectral method for this Vlasov-Poisson System.

Hermite polynomials and Hermite functions

For a given scaling positive function $t \mapsto \alpha(t)$ which will be determined later, we define the weight as

$$\omega(t,v) \; := \; \sqrt{2\pi}\, \exp\left(\frac{\alpha^2(t)\,|v|^2}{2}\right),$$

and the associated weighted L^2 space

$$L^2(\omega(t)\,\mathrm{d}\,v) := \left\{ g : \mathbb{R} \to \mathbb{R} : \int_{\mathbb{R}} |g(v)|^2\, \omega(t,v)\,\mathrm{d}\,v < +\infty \right\},$$

with $\langle \cdot, \cdot \rangle_{L^2(\omega(t)\,\mathrm{d}\,v)}$ the inner product and $\|\cdot\|_{L^2(\omega(t)\,\mathrm{d}\,v)}$ the corresponding norm. We choose the following basis of normalized scaled time dependent asymmetrically weighted Hermite functions:

$$\Psi_n(t,v) \; = \; \alpha(t)\, H_n\left(\alpha(t) v\right)\, \frac{e^{-(\alpha(t)v)^2/2}}{\sqrt{2\pi}},$$

where α is a scaling function depending on time and H_n are the Hermite polynomials defined by $H_{-1}(\xi) = 0$, $H_0(\xi) = 1$ and for $n \geq 1$, $H_n(\xi)$ has the following recursive relation

$$\sqrt{n}\, H_n(\xi) \; = \; \xi\, H_{n-1}(\xi) - \sqrt{n-1}\, H_{n-2}(\xi)\,, \quad \forall\, n \geq 1\,.$$

The reformulated Vlasov-Poisson system

For any integer $N \geq 1$ and $t \geq 0$, we introduce the space V_N as the subspace of $L^2(\omega(t)\,\mathrm{d}\,v)$ defined by

$$V_N \;:=\; \mathrm{Span}\{\Psi_n(t), \quad 0 \leq n \leq N-1\}.$$

Then we look for an approximate solution f_N of (7) as a finite sum which corresponds to a truncation of a series

$$f_N(t,x,v) = \sum_{n=0}^{N-1} C_n(t,x)\,\Psi_n(t,v)\,, \tag{2}$$

where N is the number of modes and $(C_n)_{0 \leq n \leq N-1}$ are computed using orthogonality property and taking $H_n(\alpha\, v)$ as test function in (7). Therefore, a system of evolution equations is obtained for the modes $(C_n)_{0 \leq n < N}$

$$\begin{cases} \partial_t C_n \,+\, \mathcal{T}_n[C] \,=\, \mathcal{S}_n[C, E_N]\,, \\[2ex] \mathcal{T}_n[C] \,=\, \dfrac{1}{\alpha}\left(\sqrt{n}\,\partial_x C_{n-1} \,+\, \sqrt{n+1}\,\partial_x C_{n+1}\right)\,, \\[2ex] \mathcal{S}_n[C, E_N] \,=\, \dfrac{\alpha'}{\alpha}\,\left(n\,C_n + \sqrt{(n-1)n}\,C_{n-2}\right) \,+\, E_N\,\alpha\,\sqrt{n}\,C_{n-1}\,, \end{cases} \tag{3}$$

The reformulated Vlasov-Poisson system

Meanwhile, we observe that the density ρ_N satisfies

$$\rho_N = \int_{\mathbb{R}} f_N \, \mathrm{d}\, v = C_0 \,,$$

and then the Poisson equation becomes

$$\frac{\partial E_N}{\partial x} = C_0 - \rho_{0,N} \,, \tag{4}$$

Space discretization : discontinuous Galerkin method

Given any $k \in \mathbb{N}$, we define a finite dimensional discrete piecewise polynomial space

$$X_h = \left\{ u \in L^2(\mathbb{T}) : \; u|_{I_j} \in \mathscr{P}_k(I_j), \quad j \in \mathcal{J} \right\} \,,$$

where the local space $\mathscr{P}_k(I)$ consists of polynomials of degree at most k on the interval I.

Hermite spectral form of the Vlasov equation

Spectral methods are commonly used to approximate the solution to the Vlasov-Poisson system [1]

- It starts with the work of Harold Grad in kinetic theory (1949 CPAM);
- J. P. Holloway and J. W. Schumer (1995) (even before Engelmann *et al.* in 1963) applied Hermite functions. Indeed, the product of Hermite polynomials and a Gaussian, seems to be a natural choice for Maxwellian-type velocity profiles.
- More recently, these methods generate a new interest leading to new techniques to improve their efficiency : Le Bourdiec, De Vuyst & Jacquet (2006), Z. Cai, R. Li and Y. Wang (2013, 2018), Camporeale, Delzanno, Bergen and Moulton (2016), Manzini, Delzano, Vencels, Markidis (2016).
- K. Kormann and A. Yurova, stability of Fourier–Hermite method (2021)
- Same spirit as the work of the B. Després : Symmetrization of Vlasov-Poisson equations. SIAM J. Math. Anal. (2014).

[1] M. Shoucri & G. Knorr (1974), A. Klimas & W. Farell (1994), B. Eliasson (2003), J. Holloway & J. Schumer (1998)

Stability estimate in weighted L^2

We set μ_t the measure given as

$$\mathrm{d}\,\mu_t = \omega(t,v)\,\mathrm{d}\,x\,\mathrm{d}\,v$$

where the weight ω is provided before and the following L^2 weighted space given by

$$L^2(\mathrm{d}\,\mu_t) := \left\{ g : \mathbb{T}\times\mathbb{R} \to \mathbb{R} : \iint_{\mathbb{T}\times\mathbb{R}} |g(x,v)|^2\,\mathrm{d}\,\mu_t < +\infty \right\},$$

Proposition

Consider (f,E) a smooth solution of the Vlasov-Poisson system, where f is not necessarily nonnegative. Assuming that the initial data f_0 belongs to $L^2(\mathrm{d}\,\mu_0)$, then there exists $c_0 > 0$ such that solution $f(t)$ satisfies for all $t \geq 0$:

$$\|f(t)\|_{L^2(\mathrm{d}\,\mu_t)} \leq \|f_0\|_{L^2(\mathrm{d}\,\mu_0)}\, e^{t/4\gamma},$$

where α appearing in the definition of the weight ω is given by

$$\alpha(t) = \frac{\alpha_0}{1 + 2\,\alpha_0\,c_0\,\gamma^2\,\|f_0\|^2_{L^2(\mathrm{d}\,\mu_0)}\,(e^{t/2\gamma} - 1)}. \tag{5}$$

Idea of the proof

We compute the time derivative of $\|f(t)\|^2_{L^2(\mathrm{d}\,\mu_t)}$. One has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\,t}\|f\|^2_{L^2(\mathrm{d}\,\mu_t)} = -\iint_{\mathbb{T}\times\mathbb{R}} f\,(v\,\partial_x f\,+\,E\,\partial_v f)\,\mathrm{d}\,\mu_t\,+\,\frac{1}{2}\iint_{\mathbb{T}\times\mathbb{R}} \alpha\,\alpha'\,|v|^2\,f^2\,\mathrm{d}\,\mu_t\,.$$

Then, since

$$\int_{\mathbb{R}} f\,E\,\partial_v f\,\omega\,\mathrm{d}\,v\,=\,-\frac{1}{2}\int_{\mathbb{R}}\alpha^2\,f^2\,E\,v\,\omega\,\mathrm{d}\,v,$$

we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\,t}\|f\|^2_{L^2(\mathrm{d}\,\mu_t)}\,=\,\frac{1}{2}\iint_{\mathbb{T}\times\mathbb{R}} f^2\left(\alpha^2\,E\,v+\alpha\,\alpha'\,|v|^2\right)\mathrm{d}\,\mu_t.$$

hence after Young's inequality and reordering

$$\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}\,t}\|f\|^2_{L^2(\mathrm{d}\,\mu_t)}\,\leq\,\frac{\alpha}{2}\,\left(\frac{\gamma}{2}\,\alpha^3\,\|E\|^2_{L^\infty}\,+\,\alpha'\right)\iint_{\mathbb{T}\times\mathbb{R}} f^2\,|v|^2\,\mathrm{d}\,\mu_t\,+\,\frac{1}{4\gamma}\,\|f\|^2_{L^2(\mathrm{d}\,\mu_t)}.$$

Idea of the proof

In one dimension, we have (in fact the result is better)

$$\|E\|_{L^\infty}^2 \leq \frac{c_0}{\alpha(t)} \|f\|_{L^2(\mathrm{d}\,\mu_t)}^2 \,.$$

Substituting this inequality in the latter estimate, it yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\,t}\|f\|_{L^2(\mathrm{d}\,\mu_t)}^2 \leq \frac{\alpha}{2}\left(\frac{c_0\gamma}{2}\,\alpha^2\,\|f\|_{L^2(\mathrm{d}\,\mu_t)}^2 + \alpha'\right)\iint_{\mathbb{T}\times\mathbb{R}} f^2\,|v|^2\,\mathrm{d}\,\mu_t + \frac{1}{4\gamma}\,\|f\|_{L^2(\mathrm{d}\,\mu_t)}^2 \,.$$

Therefore, choosing α as

$$\alpha(t) = \frac{\alpha_0}{1 + 2\,\alpha_0\,c_0\,\gamma^2\,\|f_0\|_{L^2(\mathrm{d}\,\mu_0)}^2\,(e^{t/2\gamma} - 1)}\,,$$

we get the expected result.

Remark

This approach can be also applied for weighted L^p spaces, with $p > 1$. It provides L^p estimates on the density ρ

$$\|\rho(t)\|_{L^p} \leq C\,\|f(t)\|_{L^p(\mathrm{d}\,\mu_t)}.$$

For the Vlasov-Poisson- system, when $p > N$ it gives a control in L^∞ on the electric field...

Stability for the discrete system

Similarly as in the continuous case, we establish the following stability result for the Hermite/discontinuous Galerkin method

Proposition

For any $t \in [0, T]$, consider the scaling function α provided before and $f_\delta(t)$ the semi-discrete approximate solution given by the truncated series. Assume that $\|f_\delta(0)\|_{L^2(\mathrm{d}\,\mu_0)} < +\infty$. Then, we have

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\|f_\delta(t)\|^2_{L^2(\mathrm{d}\,\mu_t)} := \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\alpha(t)\sum_{n=0}^{N-1}\int_{\mathbb{T}}|C_{\delta,n}|^2\,\mathrm{d}\,x\right)$$
$$\leq -\sum_{n=0}^{N-1}\sum_{j\in\hat{\mathcal{J}}}\nu_n\,[C_{\delta,n}]^2_{j-\frac{1}{2}} + \frac{1}{2\,\gamma}\,\|f_\delta(t)\|^2_{L^2(\mathrm{d}\,\mu_t)},$$

from which we deduce

$$\|f_\delta(t)\|_{L^2(\mathrm{d}\,\mu_t)} \leq \|f_\delta(0)\|_{L^2(\mathrm{d}\,\mu_0)}\,\mathrm{e}^{t/4\gamma}.$$

Our main result is the following.

Theorem

For any $t \in [0, T]$*, consider the scaling function* α *provided before and let* $f(t,.) \in H^m(\mathrm{d}\,\mu_t)$ *be the solution of the Vlasov-Poisson system* (7) *where* $m \geq k+1$ *and* f_δ *be the approximation defined by Hermite/DG. Then there exists a constant* $\mathcal{C} > 0$*, independent of* $\delta = (h, 1/N)$ *but depending on the scaling function* $\alpha(t)$ *such that*

$$\|f(t) - f_\delta(t)\|_{L^2(\mathrm{d}\,\mu_t)} \leq \mathcal{C}\left[\frac{1}{N^{(m-1)/2}} + h^{k+1/2}\right]. \tag{6}$$

- This result shows spectral accuracy in velocity and classical order of convergence of the discontinuous Galerkin method for the space discretization.
- Of course $\mathcal{C} > 0$ denpends on the scaling parameter α;

Key estimate

Let us define the Fokker-Planck operator $\mathcal{F}$ as

$$\mathcal{F}[g](v) = -\partial_v \left(\omega^{-1}(t)\, \partial_v \left(g\, \omega(t)\right)\right) ,$$

where $\omega(t)$ is the weight defined before. Hence, the Hermite function Ψ_n is the n-th eigenfunction of the following singular Liouville problem:

$$-\mathcal{F}[g](v) + \lambda\, g(v) = 0, \qquad v \in \mathbb{R},$$

with corresponding eigenvalues $\lambda_n = \alpha^2(t)\, n$.

Proposition

Let $r \geq 0$. For any $g \in H^r(\omega(t)\, \mathrm{d}\, v)$, it holds for all $N \geq 0$

$$\| g - \mathcal{P}_{V_N} g \|_{L^2(\omega(t)\, \mathrm{d}\, v)} \leq \frac{\mathcal{C}}{(\alpha^2(t)\, N)^{r/2}} \|g\|_{H^r(\omega(t)\, \mathrm{d}\, v)} ,$$

with $\mathcal{C} > 0$ independent of N and t.

Filtering

It is a common procedure to reduce the effects of the Gibbs phenomenon.

The filter will consist in multiplying some spectral coefficients by a scaling factor σ in order to reduce the amplitude of high frequencies, for any $N_H \geq 4$,

$$\widetilde{C}_n = C_n \, \sigma\left(\frac{n}{N_H}\right).$$

Here, we simply apply a filter, called Hou-Li's filter[2] for $\beta = 36$,

$$\sigma(s) = \begin{cases} 1\,, & \text{if } 0 \leq |s| \leq 2/3\,, \\ \exp(-\beta\,|s|^{\beta})\,, & \text{if } \ |s| > 2/3\,. \end{cases}$$

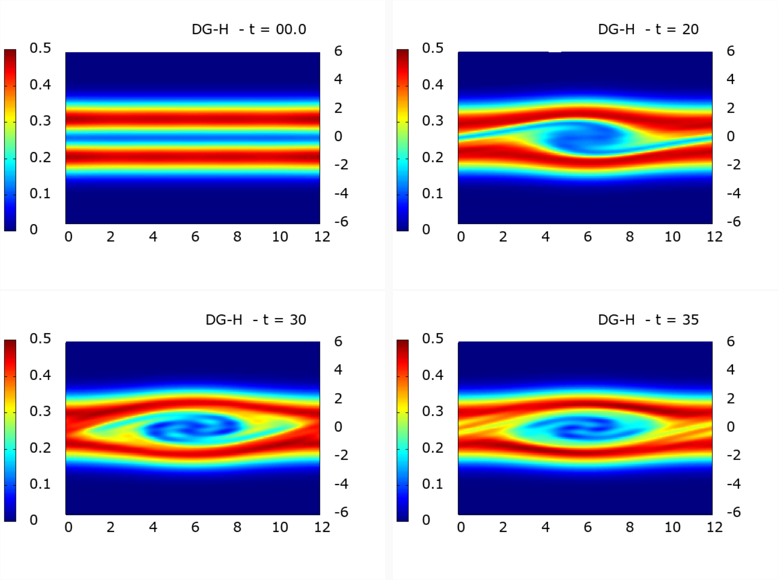
Remark

- Observe that the filter is applied only when $N_H \geq 4$, hence the filtering process does not modify the coefficients $(C_k)_{0 \leq k \leq 2}$

- It is possible to adapt the number of modes N_H along the simulations

[2] Th. Y. Hou and R. Li, (2007)

Two stream instability



Two stream instability

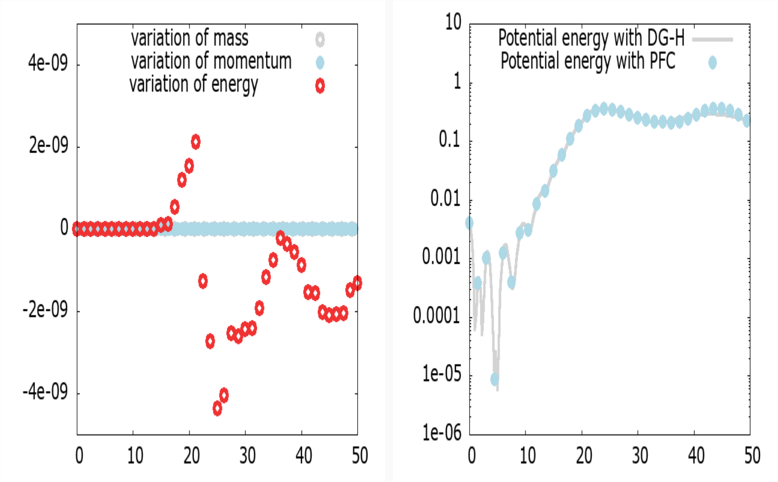


Figure 1: Two stream instability: (a) deviation of mass, momentum and energy, (b) time evolution of the electric field in L^2 norm in logarithmic value with DG-H: $N_x \times N_H = 64 \times 128$ and the reference solution is from the PFC scheme with $N_x \times N_v = 256 \times 1024$.

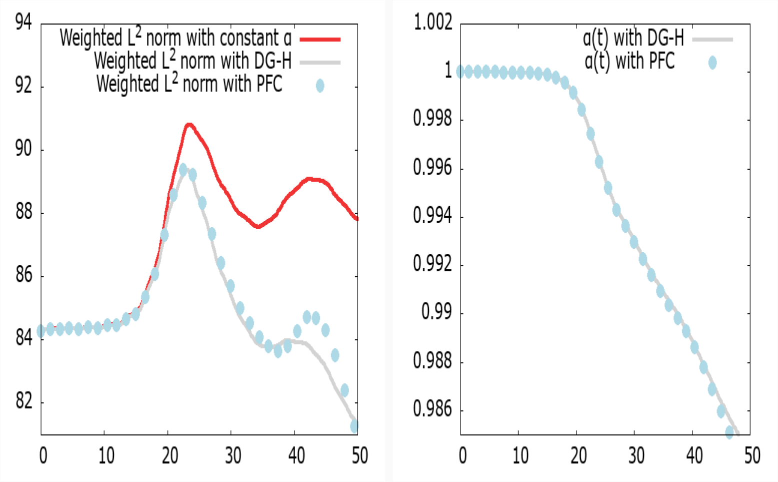


Figure 2: Two stream instability: (a) time evolution of the weighted L^2 norm of f, (b) time evolution of the scaling function α for DG-H with $N_x \times N_H = 64 \times 128$ and the reference solution is from the PFC scheme with $N_x \times N_v = 256 \times 1024$.

Part II : Discrete hypocoercive estimates for the Vlasov-Fokker-Planck equation (with A. Blaustein)

We consider the one dimensional Vlasov-Fokker-Planck equation with periodic boundary conditions in space, it reads

$$\partial_t f + \frac{1}{\varepsilon}\left(v\,\partial_x f + E\,\partial_v f\right) = \frac{1}{\tau(\varepsilon)}\,\partial_v\left(v\,f + \partial_v f\right)\,, \tag{7}$$

where the electric field derives from a potential Φ such that $E = -\partial_x \Phi$, with the following regularity assumption

$$\Phi \in W^{2,\infty}(\mathbb{T})\,.$$

The distribution function f relaxes towards the stationary solution to the Vlasov-Fokker-Planck equation $\rho_\infty\,\mathcal{M}$, where the Maxwellian $\mathcal{M}$ is given by

$$\mathcal{M}(v) = \frac{1}{\sqrt{2\pi\,T_0}}\,\exp\left(-\frac{|v|^2}{2\,T_0}\right)\,,$$

whereas the density ρ_∞ is determined by

$$\rho_\infty = c_0\,\exp\left(-\frac{\Phi}{T_0}\right),$$

where the constant c_0 is fixed by the conservation of mass

About the diffusive limit

In the case where $\tau(\varepsilon) \sim \tau_0\,\varepsilon^2$, for some $\tau_0 > 0$. The spatial density converges to a time dependent ρ whose dynamics are driven by a drift-diffusion equation depending on the force field E.
Indeed, performing the change of variable

$$x \to x + \tau_0\,\varepsilon\,v$$

and integrating with respect to v, we deduce that the quantity

$$\pi\,(t,x) = \int_{\mathbb{R}} f\,(t, x - \tau_0\,\varepsilon\,v, v)\,\mathrm{d}\,v\,,$$

solves the following equation

$$\partial_t\,\pi + \tau_0\,\partial_x \left(\int_{\mathbb{R}} E\,f\,(t, x - \tau_0\,\varepsilon\,v, v)\,\mathrm{d}\,v - \partial_x\,\pi \right) = 0\,.$$

According to its definition, π verifies: $\rho \sim \pi$ in the limit $\varepsilon \to 0$. Therefore, we may formally replace π with ρ and ε with 0 in the latter equation. This yields

$$f(t,x,v) \underset{\varepsilon \to 0}{\longrightarrow} \rho_{\tau_0}(t,x)\,\mathcal{M}(v)\,,$$

where ρ_{τ_0} solves

$$\partial_t \rho_{\tau_0} + \tau_0\,\partial_x\,(E\,\rho_{\tau_0} - \partial_x\,\rho_{\tau_0}) = 0\,.$$

We again consider the decomposition of f n the Hermite basis

$$f\,(t,x,v) \,=\, \sum_{k\in\mathbb{N}} C_k\,(t,x)\,\Psi_k(v)\,.$$

The natural functional framework here is the L^2 space with weight ρ_∞^{-1}. Unfortunately, it is not very well adapted to the space discretization, hence we set

$$D_k \,:=\, \frac{C_k}{\sqrt{\rho_\infty}}$$

and get that

$$\begin{cases} \partial_t D_k \,+\, \dfrac{1}{\varepsilon}\left(\sqrt{k}\,\mathcal{A}\,D_{k-1} \,-\, \sqrt{k+1}\,\mathcal{A}^\star D_{k+1}\right) \,=\, -\dfrac{k}{\tau(\varepsilon)}\,D_k\,, \\[2ex] D_k(t=0) = D_k^{0,\varepsilon}\,, \end{cases} \tag{8}$$

where operators $\mathcal{A}$ and $\mathcal{A}^\star$ are given by

$$\begin{cases} \mathcal{A}\,u \,=\, +\partial_x u \,-\, \dfrac{E}{2}\,u\,, \\[2ex] \mathcal{A}^\star\,u \,=\, -\partial_x u \,-\, \dfrac{E}{2}\,u\,. \end{cases}$$

Basic properties on $\mathcal{A}$

1. First, $\mathcal{A}^\star$ is its dual operator in $L^2(\mathbb{T})$, indeed for all $u,\, v \in H^1(\mathbb{T})$ it holds
$$\langle \mathcal{A}^\star u,\, v\rangle = \langle \mathcal{A} v,\, u\rangle\,;$$
2. we have $D_{\infty,0}$ lies in the kernel of $\mathcal{A}$, indeed
$$\mathcal{A}\, D_{\infty,0} \;=\; 0\,;$$
3. we also point out that since $\mathcal{A} + \mathcal{A}^\star \;=\; \partial_x \Phi$, it holds
$$\|\, (\mathcal{A} + \mathcal{A}^\star)\, u\|_{L^2} \;\leq\; \|\Phi\|_{W^{1,\infty}} \|u\|_{L^2}\,;$$
4. the operators $\mathcal{A}$ and $\mathcal{A}^\star$ do not commute and we have
$$[\mathcal{A},\, \mathcal{A}^\star] \;=\; \mathcal{A}\, \mathcal{A}^\star - \mathcal{A}^\star\, \mathcal{A} \;=\; \partial_{xx}\Phi\,,$$
which yields
$$\|\, [\mathcal{A},\, \mathcal{A}^\star]\, u\|_{L^2} \;\leq\; \|\Phi\|_{W^{2,\infty}}\, \|u\|_{L^2}\,.$$

We define the following H^1 norm, defined for all $D = (D_k)_{k\in\mathbb{N}}$ as follows

$$\|\mathcal{B}\, D\|_{L^2}^2 = \sum_{k\in\mathbb{N}} \|\mathcal{B}_k\, D_k\|_{L^2(\mathbb{T})}^2 \,,$$

where the family of differential operator $\mathcal{B} = (\mathcal{B}_k)_{k\geq 0}$ is defined as follows

$$\mathcal{B}_k = \begin{cases} \mathcal{A}\,, \text{ if } k = 0\,, \\ \mathcal{A}^\star\,, \text{ else}\,. \end{cases}$$

Theorem

(i) under the condition $\|D(0)\|_{L^2} < +\infty$, *it holds*

$$\|D(t) - D_\infty\|_{L^2} \leq C \exp\left(-\frac{\tau(\varepsilon)}{\varepsilon^2}\,\kappa\, t\right);$$

(ii) under the condition $\|\mathcal{B}\, D(0)\|_{L^2} + \|D(0)\|_{L^2} < +\infty$, *it holds*

$$\|\mathcal{B}D(t)\|_{L^2} \leq C \exp\left(-\frac{\tau(\varepsilon)}{\varepsilon^2}\,\kappa\, t\right);$$

Comments on these results

- The main difficulty here consists in proving the convergence of the first coefficient D_0 in the Hermite decomposition of f towards the equilibrium $\sqrt{\rho_\infty}$. We adapt hypocoercivity methods developed in Dolbeault-Schmeiser & Mouhot (TAMS 2015) to the framework of Hermite decomposition
- we introduce modified entropy functionnals in order to recover dissipation and thus a convergence rate on D_0 :

$$\mathcal{H}_0[D|D_\infty] = \frac{1}{2}\|D(t) - D_\infty\|_{L^2}^2 + \alpha_0 \left\langle \frac{\tau(\varepsilon)}{\varepsilon}\,\mathcal{A}^\star D_1, u^\varepsilon \right\rangle ,$$

where u^ε is the particular solution to equation

$$\begin{cases} \mathcal{A}^\star \mathcal{A}\, u \;=\; D_0 \;-\; D_{\infty,0}\,, \\[2ex] \displaystyle\int_{\mathbb{T}} u\,\sqrt{\rho_\infty}\;\mathrm{d}\,x \;=\; 0\,. \end{cases}$$

We denote by $\mathcal{E}_1$ the relative entropy $\|f - \rho\,\mathcal{M}\|$:

$$\mathcal{E}_1(t) \;=\; \frac{1}{2}\sum_{k\geq 1}\|D_k(t)\|^2_{L^2}\,.$$

Then we have the following theorem

Theorem

Suppose that $\tau(\varepsilon) = \tau_0\,\varepsilon^2$. For all positive ε, consider $D = (D_k)_{k\in\mathbb{N}}$ the solution to (8) with an initial datum $D(0)$ such that

$$\|D(0)\|^2_{H^1} \;:=\; \|\mathcal{B}D(0)\|^2_{L^2} \,+\, \|D(0)\|^2_{L^2} \;<\; +\infty\,.$$

and consider $D_{\tau_0} = (D_{\tau_0,k})_{k\in\mathbb{N}}$ given by limit drift-diffusion equation,

$$\mathcal{E}_1(t) \;\leq\; \mathcal{E}_1(0)\,e^{-t/(2\tau_0\varepsilon^2)} \;+\; C\,\varepsilon^2\,\|D(0) - D_\infty\|_{H^1}\;e^{-\tau_0\,\kappa\,t}\,.$$

On the other hand, it holds

$$\|(D_0 - D_{\tau_0,0})(t)\|_{H^{-1}} \;\leq\; C\left(\|(D_0 - D_{\tau_0,0})(0)\|_{H^{-1}} \,+\, \varepsilon\right)e^{-\tau_0\,\kappa\,t}$$

Conclusion & Perspectives

- We propose a Hermite decomposition of the Vlasov-Poisson system and the linear Vlasov-Fokker-Planck equation : this approach allows us to construct accurate and stable numerical approximation.
- This framework is well suited for discrete hypocoercive estimates : long time behavior and diffusive limit
- Perspectives : apply this latter strategy for the Vlasov-Poisson-Fokker-Planck system