Fredholm Property of the Linearized Boltzmann Operator

Mixture of Polyatomic Gases

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Reentry of a shuttle: complexity of the physics

A realistic modeling:

- Mixtures (O_2 and N_2)

- Monoatomic and Polyatomic (N and N_2)

- Different masses (N and N_2)

Polyatomic Gases

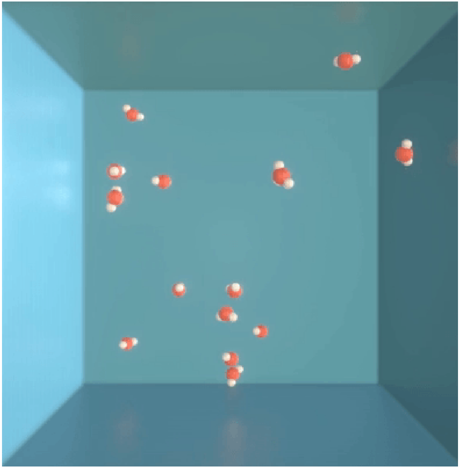

Figure: Polyatomic rarefied gas confined in a box

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- Distribution function

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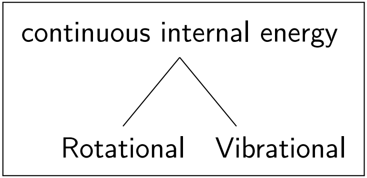
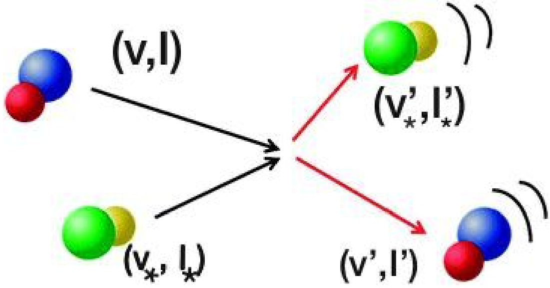


Internal Energy

References

- Continuous internal energy
  1. Borgnakke, Larsen (1975).
  2. Bourgat, Desvillettes, Le Tallec, Perthame (1994)
  3. Desvilettes, Monaco, Salvarani (2005)
  4. Baranger, Bisi, Brull, Desvillettes (2018)
  5. Andries, Le Tallec, Perlat, Perthame (2000)
  6. Brull, Schneider (2009)
- Discrete internal energy
  1. Giovangigli, Multicomponent Flow Modeling (1999)
  2. Bisi, Cáceres (2016)
  3. Bernhoff (2018)
- Undifferentiated
  - Bisi, Borsoni, Groppi (2022)

Colliding Particles



Conservation Equations:

$$m_i v + m_j v_* = m_i v' + m_j v'_*$$
$$\frac{m_i}{2} v^2 + \frac{m_j}{2} {v_*}^2 + I + I_* = \frac{m_i}{2} v'^2 + \frac{m_j}{2} {v'_*}^2 + I'_* + I'$$

The Borgnakke-Larsen Procedure[1]

- **Equivalent Formulation of Conservation Equations**

$$m_i v + m_j v_* = m_i v' + m_j v'_*$$
$$\frac{\mu_{ij}}{2}(v - v_*)^2 + I + I_* = \frac{\mu_{ij}}{2}(v' - v'_*)^2 + I'_* + I' = E$$

- **Partition of total energy by the variable R**

$$\frac{\mu_{ij}}{2}(v' - v'_*)^2 = RE$$
$$I' + I'_* = (1 - R)E$$

- **Partition of internal energy by the variable r**

$$I' = r(1 - R)E$$
$$I'_* = (1 - r)(1 - R)E$$

[1] Borgnakke, Larsen (1975)

Borgnakke-Larsen Model

- **Post collisional velocities**

$$v' = \frac{m_i v + m_j v_*}{m_i + m_j} + \frac{m_j}{m_i + m_j}\sqrt{\frac{2RE}{\mu_{ij}}}\,\sigma$$

$$v'_* = \frac{m_i v + m_j v_*}{m_i + m_j} - \frac{m_i}{m_i + m_j}\sqrt{\frac{2RE}{\mu_{ij}}}\,\sigma,$$

with $\sigma \in S^2$

- $\omega - \sigma$ **notation**

$$\sigma = T_\omega\left[\frac{v - v_*}{|v - v_*|}\right]$$

- $T_\omega(z) = z - 2(z.\omega)\omega$, and $d\omega = \frac{d\sigma}{2\left|\sigma - \frac{v-v_*}{|v-v_*|}\right|}$

The Boltzmann Equation (Mixtures)

- Boltzmann equation:

$$\boxed{\partial_t f_i + v.\nabla_x f_i = \sum_{j=1}^{n} Q_{ij}(f_i, f_j), \quad 1 \le i \le n,}$$

$Q_{ij}(f, f)$: quadratic Boltzmann operator

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- Collision Operator:

$$Q_{ij}(f_i, f_j)(v, I) = \int_{\mathbb{R}^3\times\mathbb{R}_+\times S^2\times(0,1)^2} \left(\frac{f_i' f_{j*}'}{I'^{\alpha_i} I_*'^{\alpha_j}} - \frac{f_i f_{j*}}{I^{\alpha_i} I_*^{\alpha_j}} \right) \times \mathcal{B}_{ij} \times$$

$$r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j} I^{\alpha_i} I_*^{\alpha_j}(1-R)R^{1/2}\,\mathrm{d}R\mathrm{d}r\mathrm{d}\omega\mathrm{d}I_*\,\mathrm{d}v_*,$$

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$$r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j} I^{\alpha_i} I_*^{\alpha_j}(1-R)R^{1/2}\, \mathrm{d}R\mathrm{d}r\mathrm{d}\omega\mathrm{d}I_*\, \mathrm{d}v_*,$$

- $f_{j*} = f_j(v_*, I_*)$, $f_i' = f_i(v', I')$, and $f_{j*}' = f_j(v_*', I_*')$

The Boltzmann Equation (Monospecies)

- Boltzmann equation:

$$\boxed{\partial_t f + v.\nabla_x f = Q(f, f)}$$

$Q(f, f)$: quadratic Boltzmann operator

[2] Bourgat, Desvillettes, Le Tallec, Perthame (1994)

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- Collision Operator[2]:

$$Q(f,f)(v,I) = \int_\Delta \left(\frac{f' f'_*}{(I' I'_*)^\alpha} - \frac{f f_*}{(I I_*)^\alpha} \right) \tilde{\mathcal{B}}(v, v_*, I, I_*, r, R, \sigma) \\ \mathrm{d}R\mathrm{d}r\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}v_*,$$

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where $\Delta = \mathbb{R}^3 \times \mathbb{R}_+ \times S^2 \times (0,1)^2$ and

$$\tilde{\mathcal{B}} = (r(1-r))^\alpha (1-R)^{2\alpha}(1-R)R^{1/2} I^\alpha I_*^\alpha \mathcal{B}$$

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The Parameter α

	Translation and rotation		Translation, rotation and vibration
	Linear molecule	Non-linear molecule	
Degrees of freedom	5	6	$3\mathcal{N}$
α	0	$\frac{1}{2}$	$\frac{1}{2}(3\mathcal{N}-5)$

- Micro-reversibility conditions:

$$\mathcal{B}(v, v_*, I, I_*, r, R, \omega) = \mathcal{B}(v_*, v, I_*, I, 1-r, R, \omega)$$
$$\mathcal{B}(v, v_*, I, I_*, r, R, \omega) = \mathcal{B}(v', v'_*, I', I'_*, r', R', \omega),$$

[3] Gamba, Pavić-Čolić: On the Cauchy problem for Boltzmann equation modelling a polyatomic gas

The Collision Cross-Section

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$$\mathcal{B}(v, v_*, I, I_*, r, R, \omega) = \mathcal{B}(v', v'_*, I', I'_*, r', R', \omega),$$

- Lower bound[4]

$$\boxed{C_1 \Phi(r, R) \left| \omega . \frac{(v - v_*)}{|v - v_*|} \right| \left(|v - v_*|^\gamma + I^{\frac{\gamma}{2}} + I_*^{\frac{\gamma}{2}} \right) \leq \mathcal{B}(v, v_*, I, I_*, r, R, \omega)}$$

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- Upper Bound[3]

$$\boxed{\mathcal{B}(v, v_*, I, I_*, r, R, \omega) \leq C_2 \Psi(r, R) \left| \omega . \frac{(v - v_*)}{|v - v_*|} \right| \left(|v - v_*|^\gamma + I^{\frac{\gamma}{2}} + I_*^{\frac{\gamma}{2}} \right)}$$

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Properties of the Cross-section

- Positivity
  - Φ_γ, Ψ, are positive functions such that

$$\Phi_\gamma \leq \Psi$$

- Symmetry
  - $\Phi(r, R) = \Phi(1 - r, R)$
  - $\Psi(r, R) = \Psi(1 - r, R)$
- Boundedness
  - $\Psi^2(r, R)(r(1 - r))^{\min\{2\alpha-1-\gamma,\alpha-1\}} R(1 - R)^{3\alpha-\gamma} \in L^1((0, 1)^2)$.

Models

For $\gamma < 2\alpha$, the following models satisfy the required assumptions:

- Model 1

$$\mathcal{B}\left(v, v_*, I, I_*, r, R, \omega\right) = c\left|\omega.\frac{v - v_*}{|v - v_*|}\right|\left(|v - v_*|^{\gamma} + I^{\gamma/2} + I_*^{\gamma/2}\right)$$

- Model 2

$$\mathcal{B}(v, v_*, I, I_*, r, R, \omega) = c\left|\omega.\frac{v - v_*}{|v - v_*|}\right|$$
$$\left(R^{\frac{\gamma}{2}}\,|v - v_*|^{\gamma} + \left(r(1-R)I\right)^{\frac{\gamma}{2}} + \left((1-r)(1-R)I_*\right)^{\frac{\gamma}{2}}\right)$$

Linearization

- Global Maxwellian function: $M(v, I) = \frac{I^{\alpha}}{(2\pi)^{\frac{3}{2}}\Gamma(\alpha+1)} e^{-(\frac{1}{2}v^2+I)}$

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- Define g as $f = M + M^{\frac{1}{2}} g$
- Linearized Boltzmann operator:

$$\mathcal{L}g = M^{-\frac{1}{2}}[Q(M, M^{\frac{1}{2}}g) + Q(M^{\frac{1}{2}}g, M)].$$

Linearized Boltzmann Operator

- Linearized Boltzmann Operator:

$$\mathcal{L}(g) = \left.\begin{aligned} & -\int_\Delta \frac{M^{\frac{1}{2}}}{I^{\frac{\alpha}{2}}}\frac{M_*^{\frac{1}{2}}}{I_*^{\frac{\alpha}{2}}} g(v_*, I_*)\,\tilde{\mathcal{B}}(v, v_*, I, I_*, r, R, \omega)\mathrm{d}r\mathrm{d}R\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}v_* \\ & + \int_\Delta \frac{M_*^{\frac{1}{2}}}{I_*^{\frac{\alpha}{2}}}\frac{M'^{\frac{1}{2}}}{I'^{\frac{\alpha}{2}}} g(v_*', I_*')\tilde{\mathcal{B}}(v, v_*, I, I_*, r, R, \omega)\,\mathrm{d}r\mathrm{d}R\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}v_* \\ & + \int_\Delta \frac{M_*^{\frac{1}{2}}}{I_*^{\frac{\alpha}{2}}}\frac{M_*'^{\frac{1}{2}}}{I_*'^{\frac{\alpha}{2}}} g(v', I')\tilde{\mathcal{B}}(v, v_*, I, I_*, r, R, \omega)\,\mathrm{d}r\mathrm{d}R\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}v_* \end{aligned}\right\} = \mathcal{K}$$

$$-\int_\Delta \frac{M_*}{I_*^{\alpha}} g(v, I)\tilde{\mathcal{B}}(v, v_*, I, I_*, r, R, \omega)\,\mathrm{d}r\mathrm{d}R\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}v_* \quad \Big\} = \nu$$

- Write

$$\mathcal{L}g(v, I) = \underbrace{\mathcal{K}}_{\text{Perturbation operator}} g(v, I) + \underbrace{\nu(v, I)}_{\text{Collision Frequency}} g(v, I)$$

Goal and Plan of Proof

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- Find the kernel form of each $\mathcal{K}_i$, $i = 1, \cdots, 3$
- Prove that the kernel of each $\mathcal{K}_i$ is L^2 integrable

Kernel of $\mathcal{K}_1$

$$\mathcal{K}_1 g(v, I) = \frac{1}{\Gamma(\alpha+1)(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3\times\mathbb{R}_+} \mathrm{d}I_* \mathrm{d}v_* \; g\left(v_*, I_*\right)$$

$$\left[\int_{S^2\times(0,1)^2} e^{-\frac{1}{4}v_*^2 - \frac{1}{4}v^2 - \frac{1}{2}I_* - \frac{1}{2}I} (r(1-r))^{\alpha} (1-R)^{2\alpha+1} R^{1/2} I^{\frac{\alpha}{2}} I_*^{\frac{\alpha}{2}} \mathcal{B} \, \mathrm{d}r \mathrm{d}R \mathrm{d}\omega \right]$$

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- Kernel of $\mathcal{K}_1$

$$k_1(v,I,v_*,I_*) = \frac{1}{\Gamma(\alpha+1)(2\pi)^{\frac{3}{2}}}$$

$$\int_{S^2\times(0,1)^2} e^{-\frac{1}{4}v_*^2-\frac{1}{4}v^2-\frac{1}{2}I_*-\frac{1}{2}I} (r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2} I^{\frac{\alpha}{2}} I_*^{\frac{\alpha}{2}} \mathcal{B}\, \mathrm{d}r \mathrm{d}R \mathrm{d}\omega$$

- Assumption on $\mathcal{B} \rightsquigarrow k_1$ is L^2 integrable

Kernel of $\mathcal{K}_2$

- Define the change of variable:

$$\mathbf{h} : \mathbb{R}^3 \times \mathbb{R}_+ \longmapsto \mathbf{h}(\mathbb{R}^3 \times \mathbb{R}_+) \subset \mathbb{R}^3 \times \mathbb{R}_+$$

$$(v_*, I_*) \longmapsto (x, y) = \left(\frac{v + v_*}{2} - \sqrt{R(\frac{1}{4}(v - v_*)^2 + I + I_*)}\sigma, \right.$$

$$\left. (1 - R)(1 - r)\left[\frac{1}{4}(v - v_*)^2 + I + I_*\right] \right)$$

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$$(1 - R)(1 - r)\left[\frac{1}{4}(v - v_*)^2 + I + I_*\right] \Bigg)$$

- Jacobian:

$$J = \left| \frac{\partial v_* \partial I_*}{\partial x \partial y} \right| = \frac{8}{(1 - r)(1 - R)}$$

Kernel of $\mathcal{K}_2$

$$\mathcal{K}_2 g(v,I) = \int_\Delta e^{-\frac{I_*}{2}-\frac{1}{2}r(1-R)\left(\frac{(v-v_*)^2}{4}+I+I_*\right)-\frac{1}{4}v_*^2-\frac{1}{4}\left(\frac{v+v_*}{2}+\sqrt{R(\frac{1}{4}(v-v_*)^2+I+I_*)}\,\sigma\right)^2}$$

$$g\left(\frac{v+v_*}{2}-\sqrt{R(\frac{1}{4}(v-v_*)^2+I+I_*)}\sigma,(1-R)(1-r)\left[\frac{1}{4}(v-v_*)^2+I+I_*\right]\right)$$

$$\frac{1}{\Gamma(\alpha+1)(2\pi)^{\frac{3}{2}}}\left|\sigma-\frac{v-v_*}{|v-v_*|}\right|^{-1}\tilde{\mathcal{B}}\;\;I_*^{\prime-\frac{\alpha}{2}}\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma\mathrm{d}I_*\mathrm{d}v_*.$$

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$$g\left(\frac{v+v_*}{2}-\sqrt{R(\frac{1}{4}(v-v_*)^2+I+I_*)}\sigma, (1-R)(1-r)\left[\frac{1}{4}(v-v_*)^2+I+I_*\right]\right)$$
$$\frac{1}{\Gamma(\alpha+1)(2\pi)^{\frac{3}{2}}}\left|\sigma-\frac{v-v_*}{|v-v_*|}\right|^{-1}\tilde{\mathcal{B}}\ I_*'^{-\frac{\alpha}{2}}\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma\mathrm{d}I_*\mathrm{d}v_*.$$

- $\mathcal{K}_2$ becomes

$$\mathcal{K}_2 g = \frac{1}{\Gamma(\alpha+1)(2\pi)^{\frac{3}{2}}}\int_{(0,1)^2\times S^2}\int_{\mathbf{h}(\mathbb{R}^3\times\mathbb{R}_+)} g(x,y)J\mathcal{B}\left|\sigma-\frac{v-x-\sqrt{Ray}\sigma}{|v-x-\sqrt{Ray}\sigma|}\right|^{-1}$$
$$e^{-\frac{ay-I-(x-v+\sqrt{Ray}\sigma)^2}{2}-\frac{r}{2(1-r)}y-\frac{1}{4}(2x+2\sqrt{Ray}\sigma-v)^2-\frac{1}{4}(x+2\sqrt{Ray}\sigma)^2}$$
$$(r(1-r))^{\alpha}(1-R)^{2\alpha+1}R^{1/2}I^{\frac{\alpha}{2}}I_*^{\alpha}\mathrm{d}y\mathrm{d}x\mathrm{d}\sigma\mathrm{d}r\mathrm{d}R$$

Kernel of $\mathcal{K}_2$

- Extracting the kernel

$$\mathcal{K}_2 g(v,I) = \frac{1}{\Gamma(\alpha+1)(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3\times\mathbb{R}_+} \Bigg[\int_{H_{x,y}^{v,I}} y^{-1} J\,\mathcal{B} \left| \sigma - \frac{v-x-\sqrt{Ray}\sigma}{|v-x-\sqrt{Ray}\sigma|} \right|^{-1} e^{-\frac{ay-I-(x-v+\sqrt{Ray}\sigma)^2}{2} - \frac{r}{2(1-r)}y - \frac{1}{4}(2x+2\sqrt{Ray}\sigma-v)^2 - \frac{1}{4}(x+2\sqrt{Ray}\sigma)^2} (r(1-r))^{\alpha}(1-R)^{2\alpha+1} R^{1/2} I^{\frac{\alpha}{2}} I_*^{\alpha} \mathrm{d}\sigma \mathrm{d}r \mathrm{d}R \Bigg] g(x,y) \mathrm{d}y \mathrm{d}x$$

- Kernel of $\mathcal{K}_2$ in $L^2(\mathbb{R}^3\times\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}_+)$?

L^2 integrability of kernel

$$\|k_2\|_{L^2}^2 = \frac{1}{\Gamma(\alpha+1)(2\pi)^{\frac{3}{2}}} \int_{(\mathbb{R}^3\times\mathbb{R}_+)^2} \left[\int_{H_{x,y}^{v,I}} y^{-1} J\mathcal{B} \left| \sigma - \frac{v - x - \sqrt{Ray}\sigma}{|v - x - \sqrt{Ray}\sigma|} \right|^{-1} \right.$$

$$e^{-\frac{ay-I}{2} - \frac{1}{2}(x-v+\sqrt{Ray}\sigma)^2 - \frac{r}{2(1-r)}y - \frac{1}{4}(2x+2\sqrt{Ray}\sigma-v)^2 - \frac{1}{4}(x+2\sqrt{Ray}\sigma)^2}$$

$$\left. (r(1-r))^{\alpha}(1-R)^{2\alpha+1}R^{1/2}I^{\frac{\alpha}{2}}I_*^{\alpha}\mathrm{d}\sigma\mathrm{d}r\mathrm{d}R \right]^{2} \mathrm{d}y\mathrm{d}x\mathrm{d}I\mathrm{d}v$$

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$$e^{-\frac{ay-I}{2}-\frac{1}{2}(x-v+\sqrt{Ray}\sigma)^2-\frac{r}{2(1-r)}y-\frac{1}{4}(2x+2\sqrt{Ray}\sigma-v)^2-\frac{1}{4}(x+2\sqrt{Ray}\sigma)^2}$$

$$(r(1-r))^{\alpha}(1-R)^{2\alpha+1}R^{1/2}I^{\frac{\alpha}{2}}I_*^{\alpha}\mathrm{d}\sigma\mathrm{d}r\mathrm{d}R\Bigg]^{2} \mathrm{d}y\mathrm{d}x\mathrm{d}I\mathrm{d}v$$

Apply Cauchy-Schwarz:

$$\|k_2\|_{L^2}^2 \leq \frac{1}{\Gamma(\alpha+1)(2\pi)^{\frac{3}{2}}} \int_{(\mathbb{R}^3\times\mathbb{R}_+)^2} \int_{H_{x,y}^{v,I}} y^{-2} J\mathcal{B}^2 \left| \sigma - \frac{v-x-\sqrt{Ray}\sigma}{|v-x-\sqrt{Ray}\sigma|} \right|^{-2}$$

$$e^{-ay-I-(x-v+\sqrt{Ray}\sigma)^2-\frac{r}{(1-r)}y-\frac{1}{2}(2x+2\sqrt{Ray}\sigma-v)^2-\frac{1}{2}(x+2\sqrt{Ray}\sigma)^2}$$

$$(r(1-r))^{2\alpha}(1-R)^{4\alpha+2}RI^{\alpha}I_*^{2\alpha}\mathrm{d}\sigma\mathrm{d}r\mathrm{d}R\mathrm{d}y\mathrm{d}x\mathrm{d}I\mathrm{d}v$$

L^2 integrability of kernel

Move backwards by h^{-1}

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$$\|k_2\|_{L^2}^2 \leq c \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{(0,1)^2 \times S^2} J(r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2} I^{\frac{\alpha}{2}} I_*^\alpha$$

$$e^{-r(1-R)\left(\frac{(v-v_*)^2}{4}+I+I_*\right)-\frac{1}{4}\left(\frac{v+v_*}{2}-\sqrt{R(\frac{1}{4}(v-v_*)^2+I+I_*)}\,\sigma\right)^2}$$

$$e^{-I_*-\frac{1}{2}v_*^2} \mathcal{B}^2(v, v_*, I, I_*, r, R, \sigma) \mathrm{d}\sigma \mathrm{d}r \mathrm{d}R \mathrm{d}I_* \mathrm{d}v_* \mathrm{d}I \mathrm{d}v$$

Assumptions on $\mathcal{B}$+ The following changes of variables

- $I \mapsto E = I + I_* + \frac{1}{4}|v - v_*|^2$ where $dI = dE$,
- $\tilde{V} \mapsto \frac{v}{2} + \frac{v_*}{2} + \sqrt{RE}\sigma$ where $d\tilde{V} = \frac{1}{2^3} dv_*$

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Finally imply

$$\|k_2\|_{L^2}^2 \leq c \int_{(0,1)^2} \Psi^2(r,R)r^{2\alpha-1-\gamma}(1-r)^{\alpha-1}R(1-R)^{3\alpha-\gamma}\mathrm{d}r\mathrm{d}R < \infty.$$

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- Use the identity

$$\Psi(r,R) = \Psi(1-r,R)$$

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- Spectrum of $-\mathcal{L}$ lies in $[0, \infty)$
- $-\mathcal{L}$ has a spectral gap

Comparison to the Mixtures?

- The pre-post collisional relations:
  - Post-collisional velocities

$$v' = \frac{m_i v + m_j v_*}{m_i + m_j} + \frac{m_j}{m_i + m_j}\sqrt{\frac{2RE}{\mu_{ij}}}\; T_\omega\left[\frac{v - v_*}{|v - v_*|}\right]$$

$$v'_* = \frac{m_i v + m_j v_*}{m_i + m_j} - \frac{m_i}{m_i + m_j}\sqrt{\frac{2RE}{\mu_{ij}}}\; T_\omega\left[\frac{v - v_*}{|v - v_*|}\right],$$

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$$I' = r(1 - R)E$$

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- The change of variable map $\mathbf{h} : (v_*, I_*) \mapsto (x, y)$ has Jacobian:

$$J_{ij} = \left(\frac{m_i + m_j}{m_j}\right)^3 \frac{1}{(1-r)(1-R)}$$

State of Art

- Single monoatomic gas
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- Mixture of monoatomic gases
  - Compactness of $\mathcal{K}$: Boudin, Grec, Pavic, Salvarani (2014)
  - Spectral gap estimate Mouhot, Daus, Jungel, Zamponi (2015)
  - Stability of spectral gap Bondesan, Briant, Boudin, Grec (2018)
- Single polyatomic gas
  - Compactness of $\mathcal{K}$: Bernhoff (2022)
  - Compactness of $\mathcal{K}$: Borsoni, Boudin, Salvarani(2022)(resonant model)
- Mixture of polyatomic gases
  - Compactness of $\mathcal{K}$: Brull, Shahine, Thieullen (2022): Continuous Internal Energy
  - Compactness of $\mathcal{K}$: Bernhoff (2022)Discrete Internal Energy

Perspectives

- Perspectives
  - Establish the same result for a mixture of monoatomic and polyatomic gases
  - Spectral gap estimates of the linearized Boltzmann operator
  - L^p compactness of $\mathcal{K}$

And Finally

Thanks for your ATTENTION!