On the hydrodynamic description of aerosols

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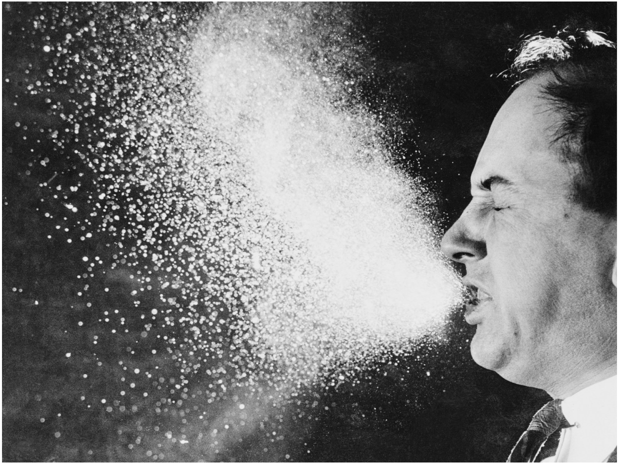
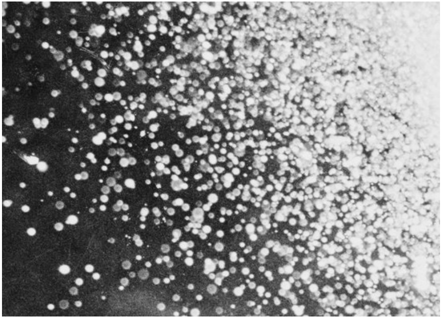
French-Korean Webinar Kinetic and Fluid equations for collective behavior
November 25th 2022

Modelling of aerosols : fluid-kinetic couplings



"*An aerosol is a suspension of fine solid particles or liquid droplets in air or another gas.*" Hinds, Aerosol Technology, 1999.

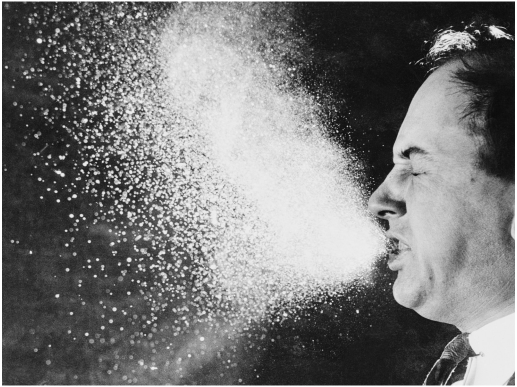
Modelling of aerosols : fluid-kinetic couplings



- **Suspension** : **kinetic** description by a **distribution function** $f(t, x, v)$, giving the continuous distribution of particles with position x and velocity v at time t.
- **Fluid** : viscous, incompressible, homogeneous, described by velocity $u(t, x)$ and pressure $p(t, x)$.

Coupling of equations of **fluid mechanics** with **kinetic equations** [O'Rourke 1981].

Modelling of aerosols : fluid-kinetic couplings



Question : how relevant is it to describe aerosols only by hydrodynamic equations ?

The Vlasov-Navier-Stokes system for $(x, v) \in \mathbb{T}^3 \times \mathbb{R}^3$

$$\begin{cases} \partial_t u + u \cdot \nabla_x u - \Delta_x u + \nabla_x p = -\int_{\mathbb{R}^3} (u - v)\, f \, \mathrm{d}v, \\ \mathrm{div}_x \; u = 0, \\ \partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (f(u - v)) = 0, \\ f\big|_{t=0} = f_0, \quad u\big|_{t=0} = u_0. \end{cases}$$

- **Forcing** in Navier-Stokes : **Brinkman force**
- **Force** in the Vlasov equation : **drag force**
- Moments in velocity

$$\rho_f(t, x) = \int_{\mathbb{R}^3} f(t, x, v)\, \mathrm{d}v, \qquad \text{Density of particles}$$

$$j_f(t, x) = \int_{\mathbb{R}^3} f(t, x, v)\, v \, \mathrm{d}v \qquad \text{Momentum of particles.}$$

The Vlasov-Navier-Stokes system for $(x, v) \in \mathbb{T}^3 \times \mathbb{R}^3$

$$\begin{cases} \partial_t u + u \cdot \nabla_x u - \Delta_x u + \nabla_x p = -\rho_f u + j_f, \\ \mathrm{div}_x \; u = 0, \\ \partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (f(u - v)) = 0, \\ f|_{t=0} = f_0, \quad u|_{t=0} = u_0. \end{cases}$$

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- **Characteristic** curves for Vlasov :

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s} X_{s,t}(x,v) = V_{s,t}(x,v), \quad X_{t,t}(x,v) = x, \\ \frac{\mathrm{d}}{\mathrm{d}s} V_{s,t}(x,v) = u(s, X_{s,t}(x,v)) - V_{s,t}(x,v), \quad V_{t,t}(x,v)) = v. \end{cases}$$

We have

$$\boxed{f(t,x,v) = e^{3t} f_0(X_{0,t}(x,v), V_{0,t}(x,v)).}$$

- **A priori estimates for Vlasov** :

$$f_0 \geq 0 \implies \forall t \geq 0,\ f(t) \geq 0,$$
$$\|f(t)\|_{\mathrm{L}^1(\mathbb{T}^3 \times \mathbb{R}^3)} = \|f_0\|_{\mathrm{L}^1(\mathbb{T}^3 \times \mathbb{R}^3)},$$
$$\|f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} = e^{3t} \|f_0\|_{\mathrm{L}^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}.$$

The Vlasov-Navier-Stokes system : formal properties

- **Energy-dissipation** identity :

$$\mathrm{E}(t) = \frac{1}{2}\int_{\mathbb{T}^3} |u(t,x)|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3} f(t,x,v)|v|^2 \,\mathrm{d}v\,\mathrm{d}x,$$

$$\mathrm{D}(t) = \int_{\mathbb{T}^3} |\nabla u(t,x)|^2 \,\mathrm{d}x + \int_{\mathbb{T}^3\times\mathbb{R}^3} f(t,x,v)|u(t,x)-v|^2 \,\mathrm{d}v\,\mathrm{d}x,$$

we have formally

$$\boxed{\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{E}(t) + \mathrm{D}(t) = 0.}$$

When $f \equiv 0$, this is nothing but the usual energy-dissipation law for incompressible Navier-Stokes.

The Vlasov-Navier-Stokes system : Leray solutions

Theorem (Boudin, Desvillettes, Grandmont, Moussa (2009))

Given $f_0 \geq 0$ *in* $\mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$ *and* $u_0 \in \mathrm{L}^2(\mathbb{T}^3)$ *with* $\mathrm{E}(0) < +\infty$, *there exists a* **global weak solution** $(u(t), f(t))$ *to the Vlasov-Navier-Stokes system that satisfies the energy-dissipation inequality*

$$\mathrm{E}(t) + \int_s^t \mathrm{D}(\tau)\,\mathrm{d}\tau \leq \mathrm{E}(s)$$

for $s = 0$ *and almost all* $s \geq 0$, *and all* $t \geq s$.

We call such solutions **Leray** solutions.

High field limit of the Vlasov-Navier-Stokes system

We shall focus on the following limit $\varepsilon \to 0$ for the Vlasov-Navier-Stokes system :

$$\left\{ \begin{array}{l} \partial_t u_\varepsilon + u_\varepsilon \cdot \nabla_x u_\varepsilon - \Delta_x u_\varepsilon + \nabla_x p_\varepsilon = \frac{1}{\varepsilon}\left(-\rho_{f_\varepsilon} u_\varepsilon + j_{f_\varepsilon}\right), \\ \mathrm{div}_x \; u_\varepsilon = 0, \\ \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \frac{1}{\varepsilon}\nabla_v \cdot (f_\varepsilon(u_\varepsilon - v)) = 0, \\ f_\varepsilon\big|_{t=0} = f_{0,\varepsilon}, \quad u_\varepsilon\big|_{t=0} = u_{0,\varepsilon}. \end{array} \right.$$

This corresponds to the regime where

- The Stokes relaxation time is small compared to the observation time ;
- The particles are small compared to the observation length.

Question : what can we say about the behavior of $(u_\varepsilon, f_\varepsilon)$ when $\varepsilon \to 0$?

Formal limit 1/2

- Scaled energy and dissipation :

$$\mathrm{E}_\varepsilon(t) = \frac{1}{2}\int_{\mathbb{T}^3} |u_\varepsilon(t,x)|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(t,x,v)|v|^2 \,\mathrm{d}v\,\mathrm{d}x,$$

$$\mathrm{D}_\varepsilon(t) = \int_{\mathbb{T}^3} |\nabla u_\varepsilon(t,x)|^2 \,\mathrm{d}x + \frac{1}{\varepsilon}\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(t,x,v)|u_\varepsilon(t,x)-v|^2 \,\mathrm{d}v\,\mathrm{d}x,$$

- Assume $(\rho_{f_\varepsilon}, u_\varepsilon) \rightharpoonup (\rho, u)$. By the energy–dissipation identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}_\varepsilon(t) + \mathrm{D}_\varepsilon(t) = 0,$$

we expect

$$\int_0^{+\infty}\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(t,x,v)|u_\varepsilon(t,x)-v|^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \to_{\varepsilon\to 0} 0,$$

which formally implies

$$\boxed{f_\varepsilon \rightharpoonup_{\varepsilon\to 0} \rho\otimes\delta_{v=u}.}$$

Formal limit 2/2

- Conservation of mass ($\rho_{f_\varepsilon} = \int_{\mathbb{R}^3} f_\varepsilon \mathrm{d}v$) and momentum ($j_{f_\varepsilon} = \int_{\mathbb{R}^3} f_\varepsilon v \mathrm{d}v$) for Vlasov :

$$\begin{cases} \partial_t \rho_{f_\varepsilon} + \mathrm{div}_x\, j_{f_\varepsilon} = 0, \\ \partial_t j_{f_\varepsilon} + \mathrm{div}_x \left(\int_{\mathbb{R}^3} f_\varepsilon v \otimes v \, \mathrm{d}v \right) = -F_\varepsilon, \end{cases}$$

with $F_\varepsilon = \frac{1}{\varepsilon}(-\rho_{f_\varepsilon} u_\varepsilon + j_{f_\varepsilon})$.

- Assuming furthermore that $F_\varepsilon \rightharpoonup_{\varepsilon \to 0} F$ and since $f_\varepsilon \rightharpoonup_{\varepsilon \to 0} \rho \otimes \delta_{v=u}$,

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u) = -F. \end{cases}$$

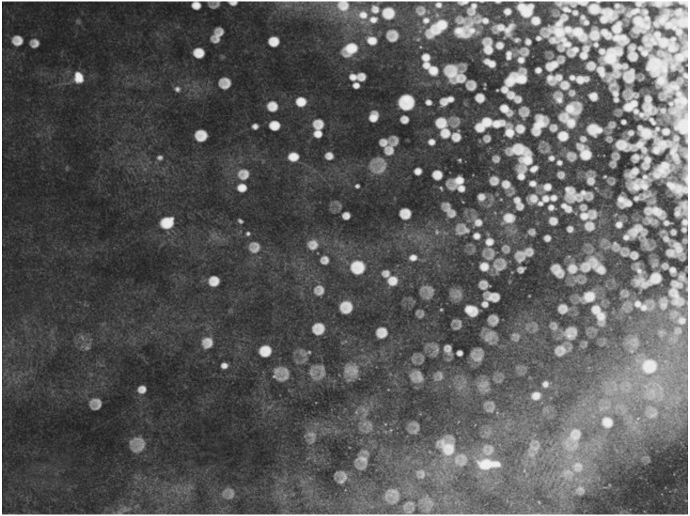
- We can also pass to the limit in the Navier-Stokes equations :

$$\partial_t u + \mathrm{div}_x(u \otimes u) - \Delta_x u + \nabla_x p = F, \qquad \mathrm{div}_x\, u = 0.$$

- Hence $(1 + \rho, u)$ must satisfy the **inhomogeneous incompressible Navier-Stokes** equations

$$\boxed{\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho u) = 0, \\ \partial_t((1+\rho)u) + \mathrm{div}_x((1+\rho)u \otimes u) - \Delta_x u + \nabla_x p = 0, \\ \mathrm{div}_x\, u = 0. \end{cases}}$$

Formal limit 2/2



Main result

Theorem (DHK, D. Michel (2021+))

Assume $f_{0,\varepsilon} \geq 0 \in \mathrm{L}^1 \cap \mathrm{L}^\infty$ *is decaying fast with respect to* v *and that* $u_{0,\varepsilon}$ *is (a bit) smooth (uniformly in* ε*).*

- *Assume* $\sup_{\varepsilon \in (0,1)} \|f_{0,\varepsilon}\|_{\mathrm{L}^1_v(\mathbb{R}^3;\mathrm{L}^\infty_x(\mathbb{T}^3))} \ll 1$ *and for some* $r > 3$,

  $$\int_{\mathbb{T}^3 \times \mathbb{R}^3} f_{0,\varepsilon} |v - u_{0,\varepsilon}|^r \,\mathrm{d}v \,\mathrm{d}x \lesssim \varepsilon^{r-1},$$

  then there exists $T > 0$ *such that, for all* $t \in [0, T]$,

  $$\boxed{\|u_\varepsilon(t) - u(t)\|_{\mathrm{L}^2(\mathbb{T}^3)} \to_{\varepsilon \to 0} 0, \quad \mathrm{W}_1(f_\varepsilon(t), \rho \otimes \delta_{v=u}) \to_{\varepsilon \to 0} 0,}$$

  where $(1 + \rho, u)$ *satisfies the inhomomogeneous incompressible Navier-Stokes equations.*

- *Assume in addition that* $\sup_{\varepsilon \in (0,1)} \|u_{0,\varepsilon}\|_{\dot{\mathrm{H}}^{1/2}} \ll 1$ *and that we are in a close to equilibrium regime, the convergences hold for all* $T > 0$.

Vlasov-Fokker-Planck-Navier-Stokes

•[Goudon, Jabin, Vasseur 2004] previously studied the same regime for

$$\begin{cases} \partial_t u_\varepsilon + u_\varepsilon \cdot \nabla u_\varepsilon - \Delta u_\varepsilon + \nabla p_\varepsilon = \frac{1}{\varepsilon}\left(-\rho_{f_\varepsilon} u_\varepsilon + j_{f_\varepsilon}\right), \\ \text{div } u_\varepsilon = 0, \\ \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \frac{1}{\varepsilon}\left(\nabla_v \cdot (f_\varepsilon(u_\varepsilon - v)) + \Delta_v f_\varepsilon\right) = 0, \end{cases}$$

also deriving inhomogeneous incompressible Navier-Stokes in the limit $\varepsilon \to 0$.

• **Relative entropy** method ([Brenier 2000] [Golse 2000]) :

$$\boxed{\mathcal{H}_\varepsilon(t) = \int_{\mathbb{T}^3 \times \mathbb{R}^3} \left(f_\varepsilon \log\left(f_\varepsilon / \mathscr{M}_{\rho,u}\right) + \mathscr{M}_{\rho,u} - f_\varepsilon\right) \mathrm{d}v\, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{T}^3} |u_\varepsilon - u|^2\, \mathrm{d}x.}$$

The limit object for VFPNS is the **local Maxwellian**

$$\mathscr{M}_{\rho,u}(t,x,v) = \frac{\rho(t,x)}{(2\pi)^{3/2}} \exp\left(-|v - u(t,x)|^2/2\right),$$

• Key fact :

$$\boxed{\mathcal{H}_\varepsilon(t) \le \mathcal{H}_\varepsilon(0) + \int_0^t \mathcal{H}_\varepsilon(s)\, \mathrm{d}s + \sqrt{\varepsilon}.}$$

Relative entropy estimates

- **Relative entropy** for Vlasov-Navier-Stokes :

$$\mathscr{H}_\varepsilon(t) = \frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon|v-u|^2\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^3}|u_\varepsilon-u|^2\mathrm{d}x.$$

- There holds

$$\mathscr{H}_\varepsilon(t) + \int_0^t\int_{\mathbb{T}^3}|\nabla_x(u_\varepsilon-u)|^2\mathrm{d}x\,\mathrm{d}s + \frac{1}{\varepsilon}\int_0^t\int_{\mathbb{T}^3\times\mathbb{R}^3}|v-u_\varepsilon|^2 f_\varepsilon\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s$$
$$\leq \mathscr{H}_\varepsilon(0) + \int_0^t\sum_{j=1}^4 I_j\mathrm{d}s.$$

$$I_1 := -\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(v-u)\otimes(v-u):\nabla_x u\,\mathrm{d}v\,\mathrm{d}x,$$

$$I_2 := -\int_{\mathbb{R}^3}(u_\varepsilon-u)\otimes(u_\varepsilon-u):\nabla_x u\,\mathrm{d}x, \quad I_3 := \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(v-u_\varepsilon)\cdot G\,\mathrm{d}v\,\mathrm{d}x,$$

$$I_4 := \int_{\mathbb{T}^3}(\rho_{f_\varepsilon}-\rho)(u_\varepsilon-u)\cdot G\,\mathrm{d}x, \qquad G := \frac{\nabla_x p - \Delta_x u}{1+\rho}.$$

A conditional $\dot{\mathrm{H}}^{-1}$ estimate

The problematic term is I_4 :

$$|I_4| = \left| \int_{\mathbb{T}^3} (\rho_{f_\varepsilon} - \rho)(u_\varepsilon - u) \cdot G \, \mathrm{d}x \right| \lesssim \|\rho_{f_\varepsilon} - \rho\|_{\dot{\mathrm{H}}^{-1}(\mathbb{T}^3)} \|\nabla_x (u_\varepsilon - u)\|_{\mathrm{L}^2(\mathbb{T}^3)}$$

Lemma

Let $T > 0$. Assume there exists $C > 0$ such that

$$\sup_{\varepsilon \in (0,1)} \|\rho_{f_\varepsilon}\|_{\mathrm{L}^\infty(0,T \times \mathbb{T}^3)} \leq C.$$

Then for all $t \in (0, T)$,

$$\|\rho_{f_\varepsilon}(t) - \rho(t)\|_{\dot{\mathrm{H}}^{-1}(\mathbb{T}^3)} \lesssim \|\rho_{f_{\varepsilon,0}} - \rho(0)\|_{\dot{\mathrm{H}}^{-1}(\mathbb{T}^3)} + \int_0^t \|(j_{f,\varepsilon} - \rho_{f,\varepsilon})u\|_{\mathrm{L}^2(\mathbb{T}^3)} \mathrm{d}s + \int_0^t \|u_\varepsilon - u\|_{\mathrm{L}^2(\mathbb{T}^3)} \mathrm{d}s.$$

The key is thus the uniform L^∞ control of ρ_{f_ε}.

Towards a control of ρ_{f_ε}

The conservation laws of the equations seem not enough to ensure such a control. However, by the method of characteristics

$$\rho_{f_\varepsilon}(t,x) = e^{\frac{3t}{\varepsilon}} \int_{\mathbb{R}^3} f_{0,\varepsilon}(X^\varepsilon_{0,t}(x,v), V^\varepsilon_{0,t}(x,v))\,\mathrm{d}v.$$

When $\int_0^T \|\nabla_x u_\varepsilon(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\,\mathrm{d}s \ll 1$, the map $v \mapsto V^\varepsilon_{0,t}(x,v)$ is a $\mathscr{C}^1$ diffeomorphism and

$$|\det \mathrm{D}_v V^\varepsilon_{0,t}(x,v)| \geq e^{\frac{3t}{\varepsilon}}/4.$$

Lemma

There is $\delta > 0$ *such that, if for* $T > 0$

$$\boxed{\sup_{\varepsilon\in(0,1)} \int_0^T \|\nabla_x u_\varepsilon(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\,\mathrm{d}s \leq \delta}$$

then

$$\sup_{\varepsilon\in(0,1)} \|\rho_{f_\varepsilon}\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)} \lesssim 1.$$

Modulated energy

Following [Choi, Kwon 2015] we introduce the **modulated energy**

$$\mathscr{E}_\varepsilon(t) = \frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(t,x,v)\left|v - \frac{\langle j_{f_\varepsilon}(t)\rangle}{\langle \rho_{f_\varepsilon}\rangle}\right|^2 \mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^3} |u_\varepsilon(t,x) - \langle u_\varepsilon(t)\rangle|^2\,\mathrm{d}x + \frac{\langle \rho_{f_\varepsilon}\rangle}{2(1+\langle \rho_{f_\varepsilon}\rangle)}\left|\frac{\langle j_{f_\varepsilon}(t)\rangle}{\langle \rho_{f_\varepsilon}\rangle} - \langle u_\varepsilon(t)\rangle\right|^2.$$

and observe that

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathscr{E}_\varepsilon(t) + \mathrm{D}_\varepsilon(t) = 0,$$

as a consequence of the **energy-dissipation identity** and of the **conservation of global momentum**

$$\frac{\mathrm{d}}{\mathrm{dt}}\langle u_\varepsilon + j_{f_\varepsilon}\rangle(t) = 0.$$

Conditional decay of the modulated energy

Lemma (Choi, Kwon (2015))

Let $T > 0$ *such that*

$$\sup_{\varepsilon \in (0,1)} \|\rho_{f_\varepsilon}\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)} \lesssim 1.$$

Then for all $t \in [0, T]$*, there is* $\lambda, C > 0$ *independent of* ε *such that for all* $\varepsilon \in (0,1)$ *and* $t \in [0, T]$*,*

$$\|u_\varepsilon(t) - \langle u_\varepsilon \rangle(t)\|^2_{\mathrm{L}^2(\mathbb{T}^3)} \lesssim \mathscr{E}_\varepsilon(t) \leq Ce^{-\lambda t}\mathscr{E}_\varepsilon(0).$$

• For $\varepsilon = 1$, [Choi, Kwon 2015] deduced a conditional description of the long time behavior of the VNS system.

• In [HK, Moussa, Moyano 2020] the L^∞ control on ρ_f was proved in a close to equilibrium regime. Our strategy for the high friction limit is actually very much inspired by the latter.

The bootstrap argument

- Introduce

$$T_\varepsilon^\star = \sup \left\{ T > 0,\ \int_0^T \|\nabla_x u_\varepsilon(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \, \mathrm{d}s \leq \delta \right\}.$$

In the close to equilibrium regime, the aim is to show that $T_\varepsilon^\star = +\infty$.

- By **maximal parabolic** estimates for the **Stokes** equation

$$\begin{cases} \partial_t u_\varepsilon - \Delta_x u_\varepsilon + \nabla_x p_\varepsilon = -u_\varepsilon \cdot \nabla_x u_\varepsilon + \frac{1}{\varepsilon}\left(j_{f_\varepsilon} - \rho_{f_\varepsilon} u_\varepsilon\right), \\ \mathrm{div}_x\, u = 0, \end{cases}$$

we get for $r > 3$,

$$\sup_{\varepsilon \in (0,1)} \left(\|\partial_t u_\varepsilon\|_{\mathrm{L}^r(0,T_\varepsilon^\star;\mathrm{L}^r(\mathbb{T}^3))} + \|\mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^r(0,T_\varepsilon^\star;\mathrm{L}^r(\mathbb{T}^3))} \right) \lesssim 1.$$

The bootstrap argument

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$$\boxed{T_\varepsilon^\star = \sup\left\{ T>0,\ \int_0^T \|\nabla_x u_\varepsilon(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\,\mathrm{d}s \leq \delta \right\}.}$$

In the close to equilibrium regime, the aim is to show that $T_\varepsilon^\star = +\infty$.

- By interpolation, for $\alpha \in (0,1)$,

$$\|\nabla_x u_\varepsilon\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \lesssim \|u_\varepsilon - \langle u_\varepsilon\rangle\|^{\alpha}_{\mathrm{L}^2(\mathbb{T}^3)} \|\mathrm{D}^2_x u_\varepsilon\|^{1-\alpha}_{\mathrm{L}^r(\mathbb{T}^3)}$$

and we can use the (conditional) exponential decay of the modulated energy :

$$\boxed{\int_0^{T_\varepsilon^\star} \|\nabla_x u_\varepsilon(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\,\mathrm{d}s \lesssim \mathscr{E}_\varepsilon(0)^{\alpha/2} \leq \delta/2,}$$

for $\mathscr{E}_\varepsilon(0) \ll 1$ (= close to equilibrium), that is

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} f_{0,\varepsilon}\left| v - \frac{\langle j_{f_{0,\varepsilon}}\rangle}{\langle \rho_{f_{0,\varepsilon}}\rangle}\right|^2 \mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{T}^3} |u_{0,\varepsilon} - \langle u_{0,\varepsilon}\rangle|^2\,\mathrm{d}x + \left|\frac{\langle j_{f_{0,\varepsilon}}\rangle}{\langle \rho_{f_{0,\varepsilon}}\rangle} - \langle u_{0,\varepsilon}\rangle\right|^2 \ll 1.$$

Higher dissipation functionals

• In the maximal parabolic estimates, we need to desingularize the Brinkman force $F_\varepsilon = \frac{1}{\varepsilon}\left(j_{f_\varepsilon} - \rho_{f_\varepsilon} u_\varepsilon\right)$.
• **Higher dissipation functional** introduced in [HK, 2022] for the large time behavior of VNS on $\mathbb{R}^3 \times \mathbb{R}^3$:

$$\boxed{\mathrm{D}_\varepsilon^{(r)} = \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon \frac{|v-u_\varepsilon|^r}{\varepsilon^r}\,\mathrm{d}v\,\mathrm{d}x,}$$

which is useful, since

$$\|F_\varepsilon\|_{\mathrm{L}^r((0,T)\times\mathbb{T}^3)} \leq \|\rho_{f_\varepsilon}\|_{\mathrm{L}^\infty(0,T)\times\mathbb{T}^3)}^{1-\frac{1}{r}} \left(\int_0^T \mathrm{D}_\varepsilon^{(r)}\,\mathrm{d}t\right)^{\frac{1}{r}}.$$

We prove that for $\eta \ll 1$,

$$\int_0^T \mathrm{D}_\varepsilon^{(r)}\mathrm{d}t + \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(t)\frac{|v-u_\varepsilon(t)|^r}{\varepsilon^{r-1}}\mathrm{d}v\,\mathrm{d}x$$
$$\lesssim \|f_{0,\varepsilon}\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}\|\partial_t u_\varepsilon\|_{\mathrm{L}^r((0,T)\times\mathbb{T}^3)}^r$$
$$+\eta\|\mathrm{D}_x^2 u_\varepsilon\|_{\mathrm{L}^r((0,T)\times\mathbb{T}^3)}^r + \int_{\mathbb{T}^3\times\mathbb{R}^3} f_{0,\varepsilon}\frac{|v-u_{0,\varepsilon}|^r}{\varepsilon^{r-1}}\mathrm{d}v\,\mathrm{d}x + 1.$$

Higher dissipation functionals

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Some open questions

- Remove the smallness assumptions ?
- Remove the smoothness assumptions : derivation from Leray solutions of Vlasov-Navier-Stokes to Leray solutions of inhomogeneous Navier-Stokes ?
- Inclusion of the effect of **gravity** : see inertialess limits by [Höfer 2018] for Vlasov-Stokes and [Ertzbischoff 2022] for Vlasov-Navier-Stokes on the half-space with absorption condition on the boundary.

Thanks for listening !