Rational approximation to real points on quadratic hypersurfaces

Anthony Poëls (joint work with Damien Roy)

France-Korea IRL webinar in Number Theory

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Université Claude Bernard Lyon 1

Introduction

Dirichlet's Theorem (in dimension 1)

For each $\xi \in \mathbb{R}$ and each $X > 1$, there exists $(p, q) \in \mathbb{Z}^2$ such that

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Dirichlet's simultaneous approximation Theorem (in dimension n)

Let $n \geq 2$ be an integer and let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. For each $X > 1$ there is an integer point $\mathbf{x} = (q, p_1, \dots, p_n) \in \mathbb{Z}^{n+1}$ such that

$$1 \leq q \leq X \quad \text{and} \quad \max_{1 \leq i \leq n} |q\xi_i - p_i| \leq X^{-1/n}.$$

Exponents of simultaneous rational approximation

Definition

We define $\hat{\lambda}(\boldsymbol{\xi})$ (resp. $\lambda(\boldsymbol{\xi})$) as the supremum of all $\lambda \in \mathbb{R}$ s.t. for each $X > 1$ large enough (resp. for arb. large X), there is $\mathbf{x} \in \mathbb{Z}^{n+1}$ satisfying

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- (dimension 1) $\hat{\lambda}(\xi) = 1$ and $\lambda(\xi) + 1 =$ irrationality exponent of ξ for each $\xi \in \mathbb{R} \setminus \mathbb{Q}$.

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- We have $\dfrac{1}{n} \leq \hat{\lambda}(\boldsymbol{\xi}) \leq \lambda(\boldsymbol{\xi}) \leq \infty$ for each $\boldsymbol{\xi} \in \mathbb{R}^n$.
- $\lambda(\boldsymbol{\xi}) = \hat{\lambda}(\boldsymbol{\xi}) = 1/n$ for almost every $\boldsymbol{\xi} \in \mathbb{R}^n$ (w.r.t. Lebesgue measure)

Spectra and relation between $\hat{\lambda}$ and λ

LI condition

We denote by $\mathbb{R}^n_{\mathrm{li}}$ the set of $\boldsymbol{\xi} \in \mathbb{R}^n$ such that $1, \xi_1, \dots, \xi_n$ are linearly independent over $\mathbb{Q}$.

Question : Describe the set of values that $\hat{\lambda}$ and λ take when $\boldsymbol{\xi}$ runs through all points of $\mathbb{R}^n_{\mathrm{li}}$?

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$$\hat{\lambda}(\mathbb{R}^n_{\mathrm{li}}) = \left[\frac{1}{n}, 1\right] \quad \text{and} \quad \lambda(\mathbb{R}^n_{\mathrm{li}}) = \left[\frac{1}{n}, +\infty\right].$$

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Question : joint spectrum of $(\hat{\lambda}, \lambda)$? General case conjectured by Schmidt-Summerer (2013) and proved by Marnat-Moshchevitin (2020) :

$$\hat{\lambda}(\boldsymbol{\xi}) + \frac{\hat{\lambda}(\boldsymbol{\xi})^2}{\lambda(\boldsymbol{\xi})} + \cdots + \frac{\hat{\lambda}(\boldsymbol{\xi})^n}{\lambda(\boldsymbol{\xi})^{n-1}} \leq 1 \quad (n \geq 2, \boldsymbol{\xi} \in \mathbb{R}^n_{\mathrm{li}}).$$

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Let $Z \subseteq \mathbb{R}^n$ be such that $Z \cap \mathbb{R}^n_{\mathrm{li}} \neq \emptyset$. We define

$$\hat{\lambda}(Z) := \sup\{\hat{\lambda}(\xi) \mid \xi \in Z \cap \mathbb{R}^n_{\mathrm{li}}\} \in [1/n, 1].$$

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Motivation : related to approximation of ξ by algebraic numbers (resp. algebraic integers) of degree $\leq n$ (resp. $\leq n+1$).

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The (rational) **Witt index** m_q of q is the integer m such that any maximal totally isotropic subspace of $\mathbb{R}^{n+1}$ defined over $\mathbb{Q}$ has dimension $m + \dim\ker(q)$. Recall that $W \subset \mathbb{R}^{n+1}$ is *totally isotropic* iff $q_{|W} = 0$.

Examples

- $\mathcal{V}_2 = \{(\xi, \xi^2) \mid \xi \in \mathbb{R}\} = Z_q \subset \mathbb{R}^2$ with $q(x_0, x_1, x_2) = x_0 x_2 - x_1^2$ (here $m_q = 1$).

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- More generally : Quadratic hypersurface in $\mathbb{R}^2$ = conic (in that case $m_q \leq 1$).
- Sphere $S^{n-1} \subset \mathbb{R}^n$ with $q(x_0, \dots, x_n) = x_0^2 - (x_1^2 + \dots + x_n^2)$.

Quadratic hypersurface of $\mathbb{R}^n$ ($n \geq 2$)

Theorem (Kleinbock-Moshchevitin, 2019)

Let q be a rational non-degenerate quadratic form on $\mathbb{R}^{n+1}$ such that $Z_q \cap \mathbb{R}^n_{\mathrm{li}} \neq \emptyset$ and $m_q \leq 1$. Then

$$\frac{1}{n} \leq \hat{\lambda}(Z_q) \leq 1/\rho_n,$$

where $\rho_n \in (1,2)$ is the only positive root of $x^n - (x^{n-1} + \cdots + x + 1)$.

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- $1/\rho_2 = 1/\gamma = 0.6180\cdots$
- $1/\rho_3 = 0.5436\cdots$
- $1/\rho_4 = 0.5187\cdots$
- $(\rho_n)_{n\geq 2}$ is increasing and tends to 2 as $n \to \infty$.

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Theorem (P.-Roy, 2021)

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$$\hat{\lambda}(Z_q) = \begin{cases} 1/\rho_n & \text{if } m_q \leq 1, \\ 1 & \text{else.} \end{cases}$$

Moreover, the set $\{\xi \in Z_q \cap \mathbb{R}^n_{\mathrm{li}} \mid \hat{\lambda}(\xi) = \hat{\lambda}(Z_q)\}$ is countably infinite if $m_q \leq 1$, and uncountable otherwise.

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- q can be degenerate.
- Upper-bound $\hat{\lambda}(Z_q) \leq 1/\rho_n$ based on Marnat-Moshchevitin (2020) (relation between $\hat{\lambda}$ and λ).
- $Z_q \cap \mathbb{R}^n_{\mathrm{li}} = \emptyset$ for $q = x_0^2 - x_1^2 = (x_0 - x_1)(x_0 + x_1)$.

Construction - general principles

For any $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n) \in Z_q \cap \mathbb{R}^n_{\mathrm{li}}$ and $\mathbf{x} \in \mathbb{Z}^{n+1}$, we write

$$\|\mathbf{x}\| = \max_{0 \le i \le n} |x_i| \quad \text{and} \quad L_{\boldsymbol{\xi}}(\mathbf{x}) = \max_{1 \le i \le n} |x_0\xi_i - x_i|.$$

Summary of our strategy

Construct by induction a sequence $(\mathbf{x}_i)_{i\ge 0}$ of points in $\mathbb{Z}^{n+1} \setminus \{0\}$ s.t. :

- $(\mathbf{x}_i)_{i\ge 0}$ converges projectively to a point $(1, \boldsymbol{\xi}) \in \mathbb{R}^{n+1}$ and $q(\mathbf{x}_i)/\|\mathbf{x}_i\|$ tends to 0 as $i \to \infty$.

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For any $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n) \in Z_q \cap \mathbb{R}^n_{\mathrm{li}}$ and $\mathbf{x} \in \mathbb{Z}^{n+1}$, we write

$$\|\mathbf{x}\| = \max_{0\le i\le n} |x_i| \quad \text{and} \quad L_{\boldsymbol{\xi}}(\mathbf{x}) = \max_{1\le i\le n} |x_0\xi_i - x_i|.$$

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Construct by induction a sequence $(\mathbf{x}_i)_{i\geq 0}$ of points in $\mathbb{Z}^{n+1} \setminus \{0\}$ s.t. :

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Constructions used in the proof (ideas)

Hypothesis : Witt index $m_q \leq 1$

$q(\mathbf{x}_0) = \cdots = q(\mathbf{x}_i) = 1$. **Induction step** (rigid) :

$$\mathbf{x}_{i+n+1} := b(\mathbf{x}_{i+n}, \mathbf{x}_i)\mathbf{x}_{i+n} - q(\mathbf{x}_{i+n})\mathbf{x}_i \quad (i \geq 0).$$

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Main difficulty : $(n+1)$ consecutive points are linearly independent.

Thank you.