Space-Velocity Bridge Is Falling Down
Fractional Mixture Lemma

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Outline

- Boltzmann Equation
- Mixture estimate and its application
- Proof of the Mixture estimate

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Boltzmann Equation: $\partial_t F + \xi \cdot \nabla_x F = Q(F, F)$

t: time; $x \in \mathbb{R}^3$: space; $\xi \in \mathbb{R}^3$: microscopic velocity.

$F(t, x, \xi)$: density distribution function.

- Transport: $\partial_t F + \xi \cdot \nabla_x F$.
- Collision operator:

$$Q(g, h) = \frac{1}{2} \int_{\mathbb{S}^2 \times \mathbb{R}^3} \left[-gh_* - g_* h + g' h'_* + g'_* h' \right] B(|\xi|, \Omega) d\xi_* d\Omega .$$

- ξ, ξ_*: velocity before collision, ξ', ξ'_*: velocity after collision.

$$\xi' = \xi - [(\xi - \xi_*) \cdot \Omega]\Omega , \quad \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \Omega]\Omega.$$

- Very Soft potential: $-3 < \gamma \leq -2$

$$B(|\xi - \xi_*|, \theta) = |\xi_* - \xi|^\gamma b(\theta)$$

- Cutoff assumption: $0 < b(\theta) \leq C|\cos\theta|$

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Around global Maxwellian

- Assumption on collision kernel: $-3 < \gamma \leq -2$ with cutoff
- (Global Maxwellian): $w(\xi) = \frac{1}{(2\pi)^{3/2}} \exp\left(\frac{-|\xi|^2}{2}\right)$.
- Solution around the global Maxwellian $w(\xi)$: $F = w + w^{1/2} f$,

$$\partial_t f + \xi \cdot \nabla_x f = Lf + \Gamma(f, f),$$

$$Lf = -\nu(\xi) f + Kf.$$

- $\nu(\xi) \sim (1 + |\xi|)^\gamma$. $Kf = \int_{\mathbb{R}^3} k(\xi, \eta) f(\eta) d\eta$

$$k(\xi,\eta) = -k_1(\xi,\eta) + k_2(\xi,\eta)$$
$$k_2(\xi,\eta) \leq |\xi-\eta|^\gamma (1+|\xi|+|\eta|)^{\gamma-1} \exp\left\{-\frac{(|\xi|^2-|\eta|^2)^2}{8|\xi-\eta|^2} - \frac{|\xi-\eta|^2}{8}\right\}$$
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provided that $1 \leq q < \frac{3}{-\gamma}$.

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$$|Kg|_{L^q_{\xi,\tau+2-\gamma}} \lesssim |g|_{L^q_{\xi,\tau}}, \; 1 \leq q \leq \infty$$

and

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provided that $1/q + 1/q' = 1$ and $1 \leq q < \frac{3}{-\gamma}$ (that is, $q' > \frac{3}{3+\gamma}$).

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Mechanism

Mechanism: Let $S_{\mathcal{B}}$ be a transport type semigroup and $\mathcal{A}$ be a smoothing integral operator in ξ, then mixing $S_{\mathcal{B}}$ and $\mathcal{A}$ will transfer the ξ regularity coming from $\mathcal{A}$ to the space regularity x.

Review of the Mixture estimate for $-2 < \gamma \leq 1$

$$\begin{cases} \partial_t g + \xi \cdot \nabla_x g + \nu(\xi) g = 0\,, \\ g(0,x,\xi) = g_0(x,\xi)\,. \end{cases} \qquad g = \mathbb{S}^t_\gamma g_0$$

- $\mathbb{S}^t_\gamma$: damped transport operator
- K: smoothing operator in ξ ($\nabla_\xi K$ exists).
- Mixture estimate:

$$\left\| \nabla_x K \mathbb{S}^t_\gamma K h_0 \right\|_{L^2} \lesssim t^{-1} \left\| h_0 \right\|_{L^2}\,.$$

- Mixture Lemma by Liu and Yu: Fourier transform method in x.
- Iterated averaging lemma by Gualdani, Mischler and Mouhot: using $\mathcal{D}_t = t\nabla_x + \nabla_\xi$.
- Generalization (with H.T. Wang, Y-C Lin, M-J Lyu): $\gamma = 1$ (SIMA 14), $-2 < \gamma < 1$ (JSP 18), relativistic Boltzmann (SIMA 20)

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Mixture estimate for $-3 < \gamma \leq -2$ (Bridge is falling down)

• Key Point 1: Instead of the derivative estimate of $\nabla_\xi K$, can we still gain some fractional regularity in velocity $(-\Delta_\xi)^{s/2}K$ for appropriate $s > 0$?

$$\nabla_\xi K \Rightarrow (-\Delta_\xi)^{s/2} K?$$

• Key Point 2: Given the fractional derivative estimate for K, is it still possible to transfer the microscopic velocity regularity to macroscopic space regularity in the fractional case by mixture?

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• Mixture estimate: (Mathematische Annalen 2022)
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- Fractional derivative: $(-\Delta_y)^{\frac{s}{2}}$ for $0 < s < 1$

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$$(-\Delta_y)^{\frac{s}{2}} f(y) = \mathrm{p.v.} \int_{\mathbb{R}^3} \frac{f(y+z) - f(y)}{|z|^{3+s}} dz \,,$$

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$$(-\Delta_y)^{\frac{s}{2}} f(y) = \mathrm{p.v.} \int_{\mathbb{R}^3} \frac{f(y+z) - f(y)}{|z|^{3+s}} dz \,,$$

Definition 2:

$$(-\Delta_y)^{\frac{s}{2}} f(y) = \mathcal{F}^{-1}\{|\hat{y}|^s \hat{f}(\hat{y})\} \,,$$

here $\hat{f}(\hat{y}) = \int_{\mathbb{R}^3} e^{iy\cdot\hat{y}} f(y) dy$ is the Fourier transform of $f(y)$ and $\mathcal{F}^{-1}$ be it's corresponding inverse transform.

- The definitions are equivalent on the Lebesgue space L^p $(1 \leq p < \infty)$.

Other important regularization estimate

- Velocity averaging lemma: Golse, Lions, Perthame and Sentis
- A-smoothing Property: Glassey and Strauss
- The L^2-L^∞ approach: Guo

Let $0 < p \leq 2$, $\beta > 3/2$, $\alpha > 0$ sufficiently small, and $j > 0$ sufficiently large. Assume that the initial data ηf_0 satisfies $f_0 \in L^\infty_{\xi,\beta+2j}(e^{\alpha\langle\xi\rangle^p})(L^1_x \cap L^\infty_x)$ where $\eta > 0$ is sufficiently small. Then there is a unique solution f with

$$\begin{aligned} \|f(t)\|_{L^\infty_{\xi,\beta}(e^{\alpha\langle\xi\rangle^p})L^2_x} &\leq \eta C_1(1+t)^{-\frac{3}{4}} \|f_0\|_{L^\infty_{\xi,\beta+2j}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)}, \\ \|f(t)\|_{L^\infty_{\xi,\beta}(e^{\alpha\langle\xi\rangle^p})L^\infty_x} &\leq \eta C_2(1+t)^{-\frac{3}{2}} \|f_0\|_{L^\infty_{\xi,\beta+3j}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)}, \end{aligned}$$

$$\begin{aligned} \|f(t)\|_{L^\infty_{\xi,\beta+2j}(e^{\alpha\langle\xi\rangle^p})L^2_x} &\leq \eta \bar{C}_1 \|f_0\|_{L^\infty_{\xi,\beta+2j}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)}, \\ \|f(t)\|_{L^\infty_{\xi,\beta+2j}(e^{\alpha\langle\xi\rangle^p})L^\infty_x} &\leq \eta \bar{C}_2 \|f_0\|_{L^\infty_{\xi,\beta+2j}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)}, \end{aligned}$$

for some positive constants C_1, C_2, $\bar{C}_1$, $\bar{C}_2$ depending on γ, α, p, β, and j.

Well-posedness and large time behavior

• Based on Liu-Yu's Green function approach: Long wave-short wave decomposition, singular-regular decomposition, bootstrap, Mixture estimate, nonlinear iteration.

• Related works: cutoff Boltzmann, non-smooth initial data

• Torus: large amplitude initial data, enlargement theory. Duan-Huang-Wang-Yang (17), Gualdani-Mischler-Mouhot (17), Cao (22).

• Bounded domain: Guo (10), Liu-Yang (17), Kim-Lee (18).

• Whole space: Liu-Yu (Green function for hard sphere, 04), Ukai-Yang (Large time behavior for hard potential, 06), Duan-Huang-Wang-Yang (Global existence for large amplitude initial data 17)

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Outline

- Boltzmann Equation
- Mixture estimate and its application
- Proof of the Mixture estimate

- Fractional derivative of ν and K (Singular integral definition):
- For any $t > 0$ and $0 < s < 3+\gamma$, we have

$$\left|(-\Delta_\xi)^{\frac{s}{2}} e^{-\nu(\xi)t}\right| \lesssim \langle\xi\rangle^{-s}.$$

- If $0 < s < 3+\gamma$

$$\left|(-\Delta_\xi)^{\frac{s}{2}} k(\xi,\eta)\right| \lesssim |\xi-\eta|^{\gamma-s}(1+|\xi|+|\eta|)^{\gamma+1}e^{-\frac{|\xi-\eta|^2}{C}} + (1+|\xi-\eta|)^{-3-s}(1+|\xi|+|\eta|)^{\gamma-1} + (1+|\eta|)^{\gamma-1}(1+|\xi|+|\eta|)^{-3-s}.$$

- To control the singularity (for $|\xi-\eta|$ small)
- To maintain the decay estimates of K (for $|\xi|$ or $|\eta|$ large)

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$-3 < \gamma \leq -2$ and $0 < s < 3+\gamma$

- $$\int_{\mathbb{R}^3} \left|(-\Delta_\xi)^{\frac{s}{2}} k(\xi,\eta)\right|^q d\eta \lesssim \langle \xi \rangle^{q(\gamma+1)},$$
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Proof of the Mixture estimate $(-\Delta_x)^{\frac{s}{2}} K\mathbb{S}_\gamma^t K h_0$

Step1: Free transport equation Let

$$h(t,x,\xi) = \mathbb{S}^t h_0 = h_0(x-\xi t,\xi)\,,$$

if we take the Fourier transform in both x and ξ variables and let $\hat{x}$ and $\hat{\xi}$ be the Fourier variables of x and ξ respectively, then we have

$$\hat{h}(t,\hat{x},\hat{\xi}) = \hat{h}_0(\hat{x},\hat{\xi}+t\hat{x})\,.$$

Note that

$$|\hat{x}|^s\hat{h}(t,\hat{x},\hat{\xi}) = t^{-s}|\hat{\xi}|^s\hat{h}(t,\hat{x},\hat{\xi}) + t^{-s}\left(|t\hat{x}|^s - |\hat{\xi}|^s\right)\hat{h}(t,\hat{x},\hat{\xi})\,,$$

one has (Fourier transform definition)

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Remark: Similar idea in F. Bouchut (JMPA 2002).

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Step2: Estimate of $K\mathbb{S}^t_\gamma$ Let $0 < s < 3+\gamma$, if $\frac{1}{p} = \frac{1}{2}(3 - \frac{2}{q})$ with $1 < q < \frac{3}{-\gamma+s}$,

$$\left\| (-\Delta_x)^{\frac{s}{2}} K\mathbb{S}^t_\gamma h_0 \right\|_{L^2} \lesssim t^{-s} \left\| \mathbb{S}^t h_0 \right\|_{L^p_\xi L^2_x} + t^{-s} \| (-\Delta_\xi)^{\frac{s}{2}} h_0 \|_{L^2} .$$

Proof. By Step 1:

$$(-\Delta_x)^{\frac{s}{2}} K\mathbb{S}^t_\gamma h_0 = t^{-s} K e^{-\nu(\xi)t} (-\Delta_\xi)^{\frac{s}{2}} \mathbb{S}^t h_0 + t^{-s} K e^{-\nu(\xi)t} \mathcal{F}^{-1} \left\{ \left(|t\hat{x}|^s - |\hat{\xi}|^s \right) \hat{h}(t,\hat{x},\hat{\xi}) \right\} .$$

Therefore

$$\begin{aligned} &\left\| (-\Delta_x)^{\frac{s}{2}} K\mathbb{S}^t_\gamma h_0 \right\|_{L^2} \\ &\leq t^{-s} \left\| K e^{-\nu(\xi)t} (-\Delta_\xi)^{\frac{s}{2}} \mathbb{S}^t h_0 \right\|_{L^2} \\ &\quad + t^{-s} \left\| K e^{-\nu(\xi)t} \mathcal{F}^{-1} \left\{ \left(|t\hat{x}|^s - |\hat{\xi}|^s \right) \hat{h}(t,\hat{x},\hat{\xi}) \right\} \right\|_{L^2} \\ &=: t^{-s} (T_1 + T_2) . \end{aligned}$$

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\end{aligned}$$

Proof of the Mixture estimate $(-\Delta_x)^{\frac{s}{2}} K\mathbb{S}^t_\gamma K h_0$

- Estimate of T_2: ($f(z) = |z|^s$ is Holder continuous of order s)

$$T_2 \leq \left\| |t\hat{x} + \hat{\xi}|^s \hat{h}_0(\hat{x}, t\hat{x} + \hat{\xi}) \right\|_{L^2_{\hat{x}} L^2_{\hat{\xi}}} \leq \left\| (-\Delta_\xi)^{\frac{s}{2}} h_0 \right\|_{L^2} .$$

- Estimate of $T_1 = \left\| K e^{-\nu(\xi)t} (-\Delta_\xi)^{\frac{s}{2}} \mathbb{S}^t h_0 \right\|_{L^2}$: Note that

$$K e^{-\nu(\xi)t} (-\Delta_\xi)^{\frac{s}{2}} h = \int_{\mathbb{R}^3} (-\Delta_\eta)^{\frac{s}{2}} \left[k(\xi,\eta) e^{-\nu(\eta)t} \right] h(t,x,\eta) d\eta .$$

We need the the kernel estimate:

$$\int_{\mathbb{R}^3} \left| (-\Delta_\eta)^{\frac{s}{2}} \left[k(\xi,\eta) e^{-\nu(\eta)t} \right] \right|^r d\eta .$$

Fractional Leibniz rule: Let $1 < r < \infty$, $1 < p_1, p_2, q_1, q_2 < \infty$ satisfy $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. Given $0 < s < 1$, we have

$$\left| (-\Delta)^{\frac{s}{2}} (fg) \right|_{L^r} \lesssim \left| (-\Delta)^{\frac{s}{2}} f \right|_{L^{p_1}} |g|_{L^{q_1}} + |f|_{L^{p_2}} \left| (-\Delta)^{\frac{s}{2}} g \right|_{L^{q_2}} .$$

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Fractional Leibniz rule: Let $1 < r < \infty$, $1 < p_1, p_2, q_1, q_2 < \infty$ satisfy $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. Given $0 < s < 1$, we have

$$\left| (-\Delta)^{\frac{s}{2}} (fg) \right|_{L^r} \lesssim \left| (-\Delta)^{\frac{s}{2}} f \right|_{L^{p_1}} |g|_{L^{q_1}} + |f|_{L^{p_2}} \left| (-\Delta)^{\frac{s}{2}} g \right|_{L^{q_2}} .$$

- Estimate of T_2: ($f(z) = |z|^s$ is Holder continuous of order s)

$$T_2 \leq \left\| |t\hat{x} + \hat{\xi}|^s \hat{h}_0(\hat{x}, t\hat{x} + \hat{\xi}) \right\|_{L^2_{\hat{x}} L^2_{\hat{\xi}}} \leq \left\| (-\Delta_\xi)^{\frac{s}{2}} h_0 \right\|_{L^2} .$$

- Estimate of $T_1 = \left\| K e^{-\nu(\xi)t} (-\Delta_\xi)^{\frac{s}{2}} \mathbb{S}^t h_0 \right\|_{L^2}$: Note that

$$K e^{-\nu(\xi)t} (-\Delta_\xi)^{\frac{s}{2}} h = \int_{\mathbb{R}^3} (-\Delta_\eta)^{\frac{s}{2}} \left[k(\xi,\eta) e^{-\nu(\eta)t} \right] h(t,x,\eta) d\eta .$$

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Step3: Estimate of $K\mathbb{S}^t_\gamma K$

Proof. In Step 2:

$$\left\|(-\Delta_x)^{\frac{s}{2}} K\mathbb{S}^t_\gamma h_0\right\|_{L^2} \lesssim t^{-s}\left\|\mathbb{S}^t h_0\right\|_{L^p_\xi L^2_x} + t^{-s}\|(-\Delta_\xi)^{\frac{s}{2}} h_0\|_{L^2}.$$

Therefore

$$\begin{aligned}\left\|(-\Delta_x)^{\frac{s}{2}} K\mathbb{S}^t_\gamma Kh_0\right\|_{L^2} &\lesssim t^{-s}\left\|\mathbb{S}^t Kh_0\right\|_{L^p_\xi L^2_x} + t^{-s}\|(-\Delta_\xi)^{\frac{s}{2}} Kh_0\|_{L^2}\\ &\lesssim t^{-s}\|Kh_0\|_{L^p_\xi L^2_x} + t^{-s}\|h_0\|_{L^2}\\ &\lesssim t^{-s}\left\|\langle\xi\rangle^{\gamma-2} h_0\right\|_{L^p_\xi L^2_x} + t^{-s}\|h_0\|_{L^2}\\ &\lesssim t^{-s}\|h_0\|_{L^2} + t^{-s}\|h_0\|_{L^2}.\end{aligned}$$

Proof of the Mixture estimate $(-\Delta_x)^{\frac{s}{2}} K\mathbb{S}^t_\gamma K h_0$

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$$\left\|(-\Delta_x)^{\frac{s}{2}} K\mathbb{S}^t_\gamma h_0\right\|_{L^2} \lesssim t^{-s} \left\|\mathbb{S}^t h_0\right\|_{L^p_\xi L^2_x} + t^{-s}\| (-\Delta_\xi)^{\frac{s}{2}} h_0\|_{L^2} .$$

Therefore

$$\begin{aligned}\left\|(-\Delta_x)^{\frac{s}{2}} K\mathbb{S}^t_\gamma K h_0\right\|_{L^2} &\lesssim t^{-s} \left\|\mathbb{S}^t K h_0\right\|_{L^p_\xi L^2_x} + t^{-s}\| (-\Delta_\xi)^{\frac{s}{2}} K h_0\|_{L^2} \\ &\lesssim t^{-s} \left\|K h_0\right\|_{L^p_\xi L^2_x} + t^{-s}\| h_0\|_{L^2} \\ &\lesssim t^{-s} \left\|\langle\xi\rangle^{\gamma-2} h_0\right\|_{L^p_\xi L^2_x} + t^{-s}\| h_0\|_{L^2} \\ &\lesssim t^{-s} \left\|h_0\right\|_{L^2} + t^{-s}\| h_0\|_{L^2} .\end{aligned}$$

Step3: Estimate of $K\mathbb{S}^t_\gamma K$

Proof. In Step 2:

$$\left\|(-\Delta_x)^{\frac{s}{2}} K\mathbb{S}^t_\gamma h_0\right\|_{L^2} \lesssim t^{-s}\left\|\mathbb{S}^t h_0\right\|_{L^p_\xi L^2_x} + t^{-s}\|(-\Delta_\xi)^{\frac{s}{2}} h_0\|_{L^2}.$$

Therefore

$$\begin{aligned}\left\|(-\Delta_x)^{\frac{s}{2}} K\mathbb{S}^t_\gamma K h_0\right\|_{L^2} &\lesssim t^{-s}\left\|\mathbb{S}^t K h_0\right\|_{L^p_\xi L^2_x} + t^{-s}\|(-\Delta_\xi)^{\frac{s}{2}} K h_0\|_{L^2}\\ &\lesssim t^{-s}\|K h_0\|_{L^p_\xi L^2_x} + t^{-s}\|h_0\|_{L^2}\\ &\lesssim t^{-s}\left\|\langle\xi\rangle^{\gamma-2} h_0\right\|_{L^p_\xi L^2_x} + t^{-s}\|h_0\|_{L^2}\\ &\lesssim t^{-s}\|h_0\|_{L^2} + t^{-s}\|h_0\|_{L^2}.\end{aligned}$$

Conclusion

• We obtain the pointwise velocity fractional derivative estimate of the kernel function $k(\xi, \eta)$.

• We obtain the L^2 spatial fractional derivative estimate of $K\mathbb{S}^t_\gamma K$.

• We obtain the well-posedness of the cut-off Boltzmann equation for very soft potential with non-smooth initial perturbation.

• Other applications ?

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THANK YOU