Asymptotic tracking of a point cloud moving on Riemannian manifolds

Junhyeok Byeon

French-Korean IRL in Mathematics

giugi2486@snu.ac.kr

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Object. Design a suitable **(?)** *interacting particle* system that can guarantee asymptotic tracking to the given point cloud.

Figure: Pyeongchang Olympics Drone Art

Backbone model

Cucker-Smale model.

$$\begin{cases} \dot{x}_i = v_i, \quad t > 0, \quad i \in \{1, 2, \cdots, N\} =: [N], \\ \dot{v}_i = \dfrac{\kappa}{N} \displaystyle\sum_{j \in [N]} \phi(\|x_i - x_j\|)\,(v_j - v_i). \end{cases}$$

Definition

1. The agents $\{x_i\}_{i\in[N]}$ exhibit *asymptotic flocking* if

$$\sup_{t\geq 0} \max_{i,j\in[N]} \|x_i - x_j\| < \infty, \quad \lim_{t\to\infty} \max_{i,j\in[N]} \|v_i - v_j\| = 0.$$

2. The agents avoid collision if

$$\min_{\substack{i,j\in[N] \\ i\neq j}} \|x_i(t) - x_j(t)\| > 0, \quad \forall t > 0.$$

Previous results[1]

Model. For $\phi_f = r^{-\alpha}, \quad \alpha > 0$ and external contrial signal $\{u_i\}$,

$$\begin{cases} \dot{x}_i = v_i, \quad t > 0, \quad i \in \{1, 2, \cdots, N\} =: [N], \\ \dot{v}_i = \dfrac{\kappa_f}{N} \displaystyle\sum_{j \in [N]} \phi_f(\|x_i - x_j\|)\,(v_j - v_i) + M u_i, \\ u_1 = -\phi(\|x_1 - x_2 - z_1\|^2)(x_1 - x_2 - z_1), \\ u_j = \phi(\|x_{j-1} - x_j - z_{j-1}\|^2)(x_{j-1} - x_j - z_{j-1}) \\ \qquad -\phi(\|x_j - x_{j+1} - z_j\|^2)(x_j - x_{j+1} - z_j), \quad j \in [N-1], \\ u_N = \phi(\|x_{N-1} - x_N - z_{N-1}\|^2)(x_{N-1} - x_N - z_{N-1}). \end{cases}$$

Result. Collision avoidance, asymptotic flocking, and formation control:

$$\exists \lim_{t \to \infty} x_i(t) = x_i^* \quad \text{where} \quad x_i^* = x_{i-1}^* - z_{i-1}, \quad i \in [N].$$

[1] Choi, Y.-P., Kalsie, D., Peszek, J. and Peters, A.: *A collisionless singular Cucker–Smale model with decentralized formation control.* SIAM J. Appl. Dyn. Syst. **18** (2019), 1954–1981.

Previous results[2]

Model. For well-structured ψ_f, ψ_c and ψ_r,

$$\begin{cases} \dot{x}_i = v_i, \quad t > 0, \quad i \in [N], \\ \dot{v}_i = \frac{\kappa_f}{N} \sum_{j \in [N]} \phi_f(\|x_i - x_j\|)\,(v_j - v_i) - \frac{\kappa_c}{N} \sum_{j=1, j \neq i}^{N} \phi_c(\|x_i - x_j\|^2)(x_j - x_i) \\ \qquad + \frac{\kappa_r}{N} \sum_{j \in [N]} \phi_r(\|x_j - x_i - (x_j^* - x_i^*)\|^2)\Big(x_j - x_i - (x_j^* - x_i^*)\Big). \end{cases}$$

Result. Collision avoidance, asymptotic flocking, and asymptotic tracking of relative distances:

$$\lim_{t \to \infty} \|(x_i(t) - x_j(t)) - (x_i^* - x_j^*)\| = 0, \quad i, j \in [N].$$

[2] Dong, J.-G.: *Avoiding collisions and pattern formation in flocks.* SIAM J. Appl. Math. **81** (2021), 2111–2129.

Previous results[3]

Model. For a fixed common target x^* on $\mathcal{M}$,

$$\begin{cases} \dot{x}_i = v_i, \quad t > 0, \quad i \in [N], \\ \nabla_{v_i} v_i = \frac{\kappa_f}{N} \sum_{j \in [N]} \phi_f(d_{ij}) \left(P_{ij} v_j - v_i \right) - \gamma_i v_i + \kappa_{r,i} \log_{x_i} x^*, \end{cases}$$

Result. Asymptotic flocking, and asymptotic tracking, and rendezvous control:

$$\lim_{t \to \infty} d(x_i, x^*) = 0 \quad \text{and} \quad \lim_{t \to \infty} v_i = 0, \quad i \in [N].$$

[3] Li, X., Wu, Y. and Zhu, J.: *Rendzvous control design for generalized Cucker–Smale model on Riemannian manifolds*. IEEE Trans. Automat. Control. DOI: 10.1109/TAC.2022.3190974

The previous models contain one of the following issues:

- A priori condition (e.g. $\min_{i,j\in[N],i\neq j} \liminf_{t\to\infty} \|x_i(t) - x_j(t)\| > 0$),
- fixed target configuration,
- collision avoidance,
- spatial constraint (i.e. Manifold).

Object. Design an *interacting particle* system on a manifold that guarantee

- collision avoidance,
- asymptotic flocking,
- and asymptotic tracking to the given *moving* point cloud.

The Model. (On $\mathcal{M}$)

$$\begin{cases} \dot{x}_i = v_i, \quad t > 0, \quad i \in [N], \\ \nabla_{\dot{x}_i} v_i = \dfrac{\kappa_f}{N} \displaystyle\sum_{j=1}^{N} \phi_f(d_{ij})\,(P_{ij} v_j - v_i) - \gamma_i v_i - \dfrac{\kappa_c}{N} \displaystyle\sum_{j \neq i} \phi_c(d_{ij}^2) \log_{x_i} x_j \\ \qquad\qquad - \Big(\kappa_{r,1} g(v_i, \log_{x_i} x_i^*) - \kappa_{r,2} \phi_r(d_{ii^*}^2) \Big) \log_{x_i} x_i^*. \end{cases}$$

The Model. (On $\mathbb{R}^d$)

$$\begin{cases} \dot{x}_i = v_i, \quad t > 0, \quad i \in [N], \\ \dot{v}_i = \dfrac{\kappa_f}{N} \displaystyle\sum_{j=1}^{N} \phi_f(\|x_i - x_j\|)\,(v_j - v_i) - \gamma_i v_i - \dfrac{\kappa_c}{N} \displaystyle\sum_{j \neq i} \phi_c(\|x_i - x_j\|^2)(x_j - x_i) \\ \qquad\qquad - \Big(\kappa_{r,1} \langle v_i, x_i^* - x_i \rangle - \kappa_{r,2} \phi_r(\|x_i - x_i^*\|^2) \Big)(x_i^* - x_i), \end{cases} \tag{TCS}$$

where each κ's and γ are positive, and ϕ's are nonnegative and locally Lipschitz. We further assume that ϕ_r is positive.

Remark on the model.

$$\begin{cases} \dot{x}_i = v_i, \quad t > 0, \quad i \in [N], \\ \dot{v}_i = \underbrace{\frac{\kappa_f}{N}\sum_{j=1}^{N}\phi_f(\|x_i - x_j\|)(v_j - v_i)}_{\text{flocking force}} \overbrace{-\gamma_i v_i}^{\text{friction}} \underbrace{-\frac{\kappa_c}{N}\sum_{j\neq i}\phi_c(\|x_i - x_j\|^2)(x_j - x_i)}_{\text{collision avoidance force}} \\ \qquad \underbrace{-\Big(\kappa_{r,1}\langle v_i, x_i^* - x_i\rangle - \kappa_{r,2}\phi_r(\|x_i - x_i^*\|^2)\Big)(x_i^* - x_i)}_{\text{relaxation force}}. \end{cases}$$

- The red terms are motivated from the previous studies in the slide.
- The blue terms are based on the inter-particle bonding feedback control.[4]

[4] Ahn, H., Byeon, J., Ha, S.-Y. and Yoon, J. (2021). Emergent dynamics of second-order nonlinear consensus models with bonding feedback controls. arXiv preprint arXiv:2112.14875.

Theorem (Informal statement)

[a] *Consider the model (TCS) on a complete, connected, and smooth Riemannian manifold. Suppose that*

- *the trajectory of a point cloud has finite length,*
- *the initial spacing between agents is sufficiently large,*
- *relaxation kernel ϕ_r is sufficiently strong near the origin and decays sufficiently fast as it approaches infinity,*
- *collision avoidance kernel ϕ_c is both sufficiently strong and concentrated,*
- *the relaxation force dominates the collision avoidance force in a suitable sense.*

Here, the terms 'sufficient' and 'suitable' are determined by the initial conditions and system parameters. Under these conditions, (TCS) exhibits

- $\inf_{t\in\mathbb{R}_+} \min_{i,j\in[N], i\neq j} d(x_i(t), x_j(t))) > 0$,
- $\lim_{t\to\infty} \max_{i\in[N]} \|v_i(t)\| = 0$,
- $\lim_{t\to\infty} \max_{i\in[N]} d(x_i(t), x_i^*(t)) = 0$.

[a] Ahn, H., Byeon, J., Ha, S.-Y. and Yoon, J. (2023). Asymptotic tracking of a point cloud moving on Riemannian manifolds. To appear in SICON.

Remark. A rigorous statement of a framework on $\mathbb{R}^d$ is as follows.

- (Trajectory of target configuration):

$$\max_{i\in[N]} \int_0^\infty \|v_i^*(s)\| ds =: \overline{\ell^*} < \infty, \qquad \sup_{t\in\mathbb{R}_+} \max_{i,j\in[N]} d(x_i^*(t), x_j^*(t)) =: \overline{d^*} < \infty.$$

- (Kernel functions, initial data, and system parameters): for some nonnegative constant $\underline{r}$ and $\overline{r}$ satisfying $0 < \underline{r} \le \overline{r}$,

$$\begin{cases} \sup_{s\in\mathbb{R}_+} (s\phi_r(s^2)) = \overline{\Phi_r} < \infty, \quad \min_{i,j\in[N], i\neq j} d_{ij}(0) > \sqrt{\underline{r}} \ge 0, \\ \mathcal{E}(0) + \kappa_{r,2} N\overline{\ell^*} \cdot \overline{\Phi_r} < \min\left\{ \frac{\kappa_{r,2}}{2} \int_0^\infty \phi_r(s)ds,\ \frac{\kappa_c}{2N} \int_{\underline{r}}^\infty \phi_c(s)ds \right\}, \\ U := \sup_{s\in\mathbb{R}_+} \left\{ \int_0^{s^2} \phi_r(s)ds = \frac{2}{\kappa_{r,2}} \left(\mathcal{E}(0) + \kappa_{r,2} N\overline{\ell^*} \cdot \overline{\Phi_r} \right) \right\} \ge \max_{i\in[N]} d_{ii^*}(0), \\ L := \inf_{s\in\mathbb{R}_+} \left\{ \int_{s^2}^\infty \phi_c(s)ds = \frac{2N}{\kappa_c} \left(\mathcal{E}(0) + \kappa_{r,2} N\overline{\ell^*} \cdot \overline{\Phi_r} \right) \right\} > \sqrt{\underline{r}}, \\ \mathrm{supp}(\phi_c) \subset [\underline{r}, \overline{r}]. \end{cases}$$

- System parameters satisfy

$$2\kappa_{r,2} \min_{\bar{L}\le s\le U} \phi_r(s^2)s^2 - \kappa_c U(N-1) \max_{L\le s\le \sqrt{\overline{r}}} (\phi_c(s^2)s) > 0,$$

$$\bar{L} := \inf_{t\in\mathbb{R}_+} \min_{i,j\in[N], i\neq j} d_{i^*j^*}(t) - U - \sqrt{\overline{r}} > 0.$$

Idea of Proof

Observation.

- Consider a damped harmonic oscillator $m\ddot{x} = -\mu\dot{x} - kx$. Then

$$E = \frac{1}{2}m(\dot{x})^2 + \frac{1}{2}kx^2 \quad \Rightarrow \quad \frac{d}{dt}E = -\mu(\dot{x})^2 \leq 0.$$

  is non-negative, and decreasing in time.

- Therefore, as $t \to \infty$, we expect

$$E \to E^\infty \geq 0.$$

  That is, the energy converge to the state whose the **derivative is zero**:

$$\begin{aligned} \dot{E} = -\mu(\dot{x})^2 \equiv 0 \quad &\Rightarrow \quad \dot{x} \equiv 0 \\ &\Rightarrow \quad \ddot{x} \equiv 0 \quad \Rightarrow \quad x = 0. \quad (\because m\ddot{x} + \mu\dot{x} = kx) \end{aligned}$$

- This actually happens as $t \to \infty$!

Definition

Let $\varphi_\cdot(x) : t \mapsto \mathcal{M}$ be an orbit generated by the vector field $F(x) \in \mathcal{X}(\mathcal{M})$:

$$\begin{cases} \dfrac{\partial \varphi_t(x)}{\partial t} = F(\varphi_t(x)), \quad t > 0, \\ \varphi_t(x)\Big|_{t=0+} = x. \end{cases}$$

The ω-limit set of x, denoted by $\omega(x)$, is the collection limit points of the orbit:

$$\omega(x) := \left\{ y \in \mathcal{M} : \ \exists \ \{t_n\}_{n\geq 1} \quad \text{such that} \quad \lim_{n\to\infty} t_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \varphi_{t_n}(x) = y \right\}.$$

Proposition (LaSalle's invariance principle)

[a]Suppose that the vector field F on $\mathcal{M}$ and a nonempty open set Ω satisfy the following conditions:

1. F is locally Lipschitz continuous vector field on $\mathcal{M}$, and let $\{\varphi_t\}$ be the flow generated by F on $\mathcal{M}$.
2. Let $\mathcal{L} : \Omega \to \mathbb{R}$ be a continuously differentiable functional such that its orbital derivative is nonpositive:

$$\dot{\mathcal{L}}(y) = \nabla \mathcal{L} \cdot F(y) \leq 0 \quad \text{for all } y \in \Omega.$$

3. $\omega(x)$ is bounded and contained in Ω.

Then, the following assertions hold.

1. $\mathbb{R}_+ \subset \mathcal{I}_x :=$ maximal existence interval of $\varphi_x(t)$ for x.
2. As $t \to \infty$, the orbit $\phi_t(x)$ approaches the largest invariant set contained in $\{\dot{\mathcal{L}}(y) = 0\}$.

[a] LaSalle, J. P. and Rath, R. J.: *Eventual stability*. IFAC Proceedings Volumes **1** (1963), 556–560.

We define the energy functionals

$$
\begin{aligned}
\mathcal{E}_k(t) &:= \text{ Kinetic energy } = \frac{1}{2}\sum_{i=1}^{N}\|v_i(t)\|, \\
\mathcal{E}_p(t) &:= \text{ Potential energy } = \frac{\kappa_c}{4N}\sum_{j=1, i\neq j}^{N}\int_{d_{ij}(t)^2}^{\infty}\phi_c(s)ds, \\
\mathcal{E}_{r,1}(t) &:= \kappa_{r,2}\sum_{i=1}^{N}\int_0^t \phi_r(d_{ii^*}(s)^2)\langle x_i(s) - x_i^*(s), v_i^*(s)\rangle ds, \\
\mathcal{E}_{r,2}(t) &:= \frac{\kappa_{r,2}}{2}\sum_{i=1}^{N}\int_0^{d_{ii^*}(t)^2}\phi_r(s)ds, \\
\mathcal{E}_r(t) &:= \text{ Relaxation energy } = \mathcal{E}_{r,1}(t) + \mathcal{E}_{r,2}(t), \\
\mathcal{E}(t) &:= \text{ Total energy } = \mathcal{E}_k(t) + \mathcal{E}_p(t) + \mathcal{E}_r(t).
\end{aligned}
$$

Proposition (Energy estimate)

Let $\{(x_i, v_i)\}$ be a local solution in $[0, \tau)$. Then total energy $\mathcal{E}$ monotonically decreases on $t \in [0, \tau)$:

$$\mathcal{E}(t) + \int_0^t \Lambda(s)ds = \mathcal{E}(0), \quad t \in (0, \tau),$$

where Λ is a nonnegative energy production rate defined by

$$\Lambda := \frac{\kappa_f}{2N} \sum_{i,j=1}^{N} \phi_f(d_{ij}) \|v_j - v_i\|^2 + \sum_{i=1}^{N} \gamma_i \|v_i\|^2 + \kappa_{r,1} \sum_{i=1}^{N} \Big| \langle v_i, x_i^* - x_i \rangle \Big|^2 \geq 0.$$

Proof of Theorem

Step 1. (Collision avoidance and uniform boundedness of distances)

For $\tau \in (0, \infty]$, let $\{(x_i, v_i)\}_{i=1}^N$ be a local solution to (TCS) on $t \in [0, \tau)$. By energy dissipation,

$$(i)\ 0 \leq \sqrt{\underline{r}} < L \leq \inf_{0 \leq t < \tau} \min_{i,j \in [N], i \neq j} d(x_i(t), x_j(t)).$$

$$(ii)\ \sup_{0 \leq t < \tau} \max_{i \in [N]} d(x_i(t), x_i^*(t)) \leq U.$$

$$(iii)\ \sup_{0 \leq t < \tau} \max_{i,j \in [N]} d(x_i(t), x_j(t)) \leq 2U + \overline{d^*}.$$

Remark. The above facts prevents the blow-up of kernels, and hence $\tau = \infty$.

Step 2. (Uniform boundedness of state configuration): Again from the energy dissipation,

$$\begin{aligned}
\max_{i\in[N]} \|v_i(t)\|^2 &\le 2\mathcal{E}(0) - 2\mathcal{E}_{r,1} \\
&\le 2\mathcal{E}(0) + 2\kappa_{r,2}\sum_{i=1}^{N}\int_0^t \phi_r(d_{ii^*}^2(s))d_{ii^*}(s)\|v_i^*(s)\|ds \\
&\le 2\mathcal{E}(0) + 2\overline{\Phi_r}\kappa_{r,2}\sum_{i=1}^{N}\int_0^t \|v_i^*(s)\|ds \\
&\le 2\mathcal{E}(0) + 2\kappa_{r,2}N\overline{\ell^*}\cdot\overline{\Phi_r}.
\end{aligned}$$

Remark.

- This enable us to apply the invariance principle (refer to the next slide).
- The integrability of each target's velocity is necessary because we do not have quantitative information for the decay of d_{ii^*}.

Step 3. (Zero convergence of speeds): We set

$$z_i := (x_i, v_i) \in T\mathcal{M}, \quad i \in [N], \qquad Z(0) := (z_1(0), \cdots, z_N(0)).$$

- From the boundedness of state configuration, $\omega(Z(0))$ is nonempty.
- Thus we can apply the LaSalle invariance principle with $\mathcal{L} = \mathcal{E}$.
- Recall the energy production rate:

$$-\dot{\mathcal{E}} = \sum_{i=1}^{N} \gamma_i \|v_i\|^2 + \text{ Some nonnegative terms },$$

which yields

$$\lim_{t\to\infty} \|v_i(t)\|^2 = 0, \quad i \in [N].$$

Step 4. (Refining a lower bound for d_{ij}).

Claim. There exists a constant $\tau \in (0, \infty)$ satisfying

$$\inf_{\tau \le t < \infty} \min_{i,j \in [N], i \neq j} d_{ij}(t) > \sqrt{r},$$

Claim proof.

1. If not, there exists a time sequence $\{t_n\}_{n=1}^{\infty}$ and set of index tuples $\mathcal{N}$ satisfying
   1. $t_k < t_{k+1}$ for all $k \in \mathbb{N}$ with $\lim_{k \to \infty} t_k = \infty$.
   2. $(i, j) \in \mathcal{N} \Longleftrightarrow d_{ij}(t_n) \le \sqrt{r}, \quad \forall n \in \mathbb{N}$.

   From the Pigeonhole principle, $\mathcal{N} - \{(1,1), (2,2), \cdots, (N,N)\}$ is nonempty.
2. From the invariance principle and governing equation of (TCS), configuration approach to the state such that

$$\begin{aligned} v_i \equiv 0 \quad &\Rightarrow \quad \dot{v}_i \equiv 0 \\ &\Rightarrow \quad 0 \equiv \sum_{i=1}^{N} \langle x_I^* - x_i, \dot{v}_i \rangle \\ &\qquad = \kappa_{r,2} \sum_{i=1}^{N} \phi_r(d_{ii^*}^2) d_{ii^*}^2 + \frac{\kappa_c}{N} \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \phi_c(d_{ij}^2) \langle x_j - x_i, x_i^* - x_i \rangle =: \mathcal{F}. \end{aligned}$$

1. Then for $t = t_n$, the support of ϕ_c and definition of $\mathcal{N}$ gives

$$\begin{aligned}\mathcal{F} =& \kappa_{r,2} \sum_{i=1}^{N} \phi_r(d_{ii^*}^2) d_{ii^*}^2 + \frac{\kappa_c}{N} \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \phi_c(d_{ij}^2) \langle x_j - x_i, x_i^* - x_i \rangle \\ &\geq \kappa_{r,2} \sum_{i=1}^{N} \phi_r(d_{ii^*}^2) d_{ii^*}^2 - \frac{\kappa_c U}{N} \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \phi_c(d_{ij}^2) d_{ij} \\ &= \kappa_{r,2} \sum_{i=1}^{N} \phi_r(d_{ii^*}^2) d_{ii^*}^2 - \frac{\kappa_c U}{N} \sum_{(i,j)\in\mathcal{N}, i\neq j} \phi_c(d_{ij}^2) d_{ij} \\ &\geq 2\kappa_{r,2} \min_{\bar{L}\leq s\leq U} \phi_r(s^2) s^2 - \kappa_c U(N-1) \max_{L\leq s\leq\sqrt{r}} (s\phi_c(s^2)) > 0.\end{aligned}$$

In the last inequality, we used $|\mathcal{N} - \{(1,1), \cdots, (N,N)\}| \leq N(N-1)$.

2. However, this yields a contradiction:

$$0 < \inf_{n\in\mathbb{N}} \mathcal{F}(t_n) \leq \mathcal{F}(t_n) \xrightarrow[n\to\infty]{\text{Invariance principle}} 0.$$

Step 5. (Asymptotic tracking)

1. From the result of **step 4**, we have

$$\inf_{\tau \le t < \infty} \min_{i,j \in [N], i \neq j} d_{ij}(t) > \sqrt{r} \quad \Rightarrow \quad \phi_c((d_{ij}(t))^2) \equiv 0, \quad t \ge \tau,$$

   for some $\tau > 0$.

2. By the invariance principle once again, the governing equation yields

$$\lim_{t \to \infty} \kappa_{r,2} \phi_r(\|x_i - x_i^*\|^2)(x_i^* - x_i) = 0.$$

3. Since $\phi_r(s) > 0$ for $s > 0$ and $\|x_i^* - x_i\|$ is uniformly bounded by $U < \infty$,

$$\lim_{t \to \infty} \|x_i^* - x_i\| = 0.$$

□

Technical Remark

Proposition (Informal)

Consider the flocking model of the form

$$\begin{cases} \dot{x}_i = v_i, \quad t > 0, \quad i \in \{1, 2, \cdots, N\} =: [N], \\ \dot{v}_i = \frac{\kappa}{N} \sum_{j \in [N]} \phi(\|x_i - x_j\|)\,(v_j - v_i) + f_i. \end{cases}$$

If $|\max_i \sup_{t \geq 0} f_i| < \infty$ and $\int_0^{\varepsilon} \phi(r)dr = \infty$, then we expect the collision avoidance.

1. However, collision avoidance is not sufficient. We need:

$$\inf_{t \geq 0} \min_{i,j,i \neq j} \|x_i(t) - x_j(t)\| > 0.$$

2. If not, one may have

$$\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0,$$

and the $\omega-$limit set is not properly defined.

Alternative model

Question. What if

$$\int_0^\infty \|v_i^*(s)\| ds = \infty?$$

Answer. We propose the following alternative model:

$$\begin{cases} \dot{x}_i = v_i, \quad \dot{x}_i^* = v_i^*, \quad t > 0, \quad i \in [N], \\ \nabla_{\dot{x}_i}(\nabla_{\dot{x}_i} \log_{x_i} x_i^*) = \dfrac{\kappa_f}{N} \displaystyle\sum_{j=1}^N \phi_f(d_{ij}) \left(P_{ij} \nabla_{\dot{x}_j} \log_{x_j} x_j^* - \nabla_{\dot{x}_i} \log_{x_i} x_i^* \right) - \gamma_i \nabla_{\dot{x}_i} \log_{x_i} x_i^* \\ \qquad\qquad - \left(\kappa_{r,1} g_{x_i}(\nabla_{\dot{x}_i} \log_{x_i} x_i^*, \log_{x_i} x_i^*) + \kappa_{r,2} \phi_r(d_{ii^*}^2) \right) \log_{x_i} x_i^*, \\ (x_i, v_i)\Big|_{t=0+} = (x_i^0, v_i^0) \in T\mathcal{M}, \end{cases} \tag{TCS2}$$

Motivation. Consider (TCS) on $\mathbb{R}^d$ with $\phi_c \equiv 0$ under *rendezvous problem*:

$$x_i^*(t) = 0, \quad t \geq 0, \quad i \in [N].$$

In this setting, (TCS) becomes

$$\begin{cases} \dot{x}_i = v_i, \quad t > 0, \quad i \in [N], \\ \dot{v}_i = \dfrac{\kappa_f}{N} \displaystyle\sum_{j=1}^{N} \phi_f(\|x_i - x_j\|)(v_j - v_i) - \gamma_i v_i - \kappa_{r,1}\langle v_i, x_i \rangle x_i - \kappa_{r,2}\phi_r(\|x_i\|^2)x_i. \end{cases} \tag{TCS'}$$

Then for new trajectory $\{y_i^*(t)\}$, under $\phi_f \equiv 1$ we have

$$(\text{TCS'})|_{x_i \leftarrow y_i^* - x_i} \equiv (\text{TCS2})|_{x_i^* \leftarrow y_i^*,\ \mathcal{M}=\mathbb{R}^d}$$

In other word, (TCS2) is a constructed in the spirit that

$$\text{Asymptotic tracking of } x_i \text{ toward } x_i^* \quad \Leftrightarrow \quad \text{Asymptotic rendezvous of } x_i^* - x_i$$

Theorem (Informal statement)

[a] *Consider the model (TCS2) on a complete, connected, and smooth Riemannian manifold. Suppose the same conditions as in the previous theorem except*

- $\phi_c \equiv 0$,
- *the trajectory of a point cloud can have infinity length,*
- *and the spacing between target points are sufficiently large in any time.*

Then (TCS2) exhibits

- $\inf_{t \in \mathbb{R}_+} \min_{i,j \in [N], i \neq j} d(x_i(t), x_j(t))) > \underline{r} \geq 0$,
- $\lim_{t \to \infty} \max_{i \in [N]} \|v_i(t)\| = 0$,
- $\lim_{t \to \infty} \max_{i \in [N]} d(x_i(t), x_i^*(t)) = 0$.

[a] Ahn, H., Byeon, J., Ha, S.-Y. and Yoon, J. (2023). Asymptotic tracking of a point cloud moving on Riemannian manifolds. To appear in SICON.

Sketch of proof.

1. Almost every step except collision avoidance is analogous to the previous proof.
2. For the collision avoidance, we observe that $d(x_i, x_i^*)$ and $d(x_i^*, x_j^*)$ are bounded.

Numerical simulations. Under $\mathcal{M} = \mathbb{S}^2$, we set

$$N = 5, \quad \Delta t = 10^{-2}, \quad \kappa_f = \kappa_c = 10^{-3}, \quad \kappa_{r,1} = 0, \quad \kappa_{r,2} = 10^{-2}, \quad \gamma = 10^{-1},$$

$$\phi_f(s) = 1, \quad \phi_r(s) = \frac{1}{\sqrt{s+1}}, \quad \phi_c(s) = \begin{cases} -10s(s-\pi), & s \in (0,\pi), \\ 0, & \text{otherwise.} \end{cases}$$

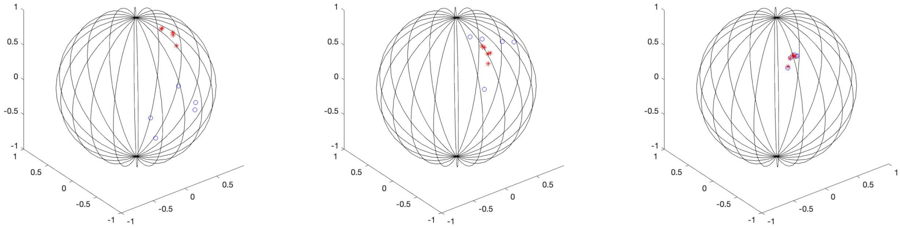

Figure: Asymptotic tracking simulation on $\mathbb{S}^2$ ($t = 0, 40, 80$)

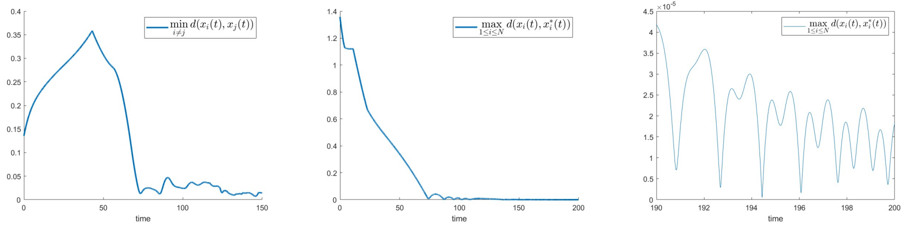

Figure: Collision avoidance and asymptotic tracking

Remark.

- Our method(invariance principle) does not yield quantitative information for the decay estimate.
- The distances between agents are target are not decreasing.
- This is a realization of the fact that individual energy is not necessarily decreasing.
- Since the total energy decreases, increments of some energy is compensated by decrement of other energies.

Ongoing works

Preprints.

1. Synchronization tracking
2. Target switching
3. Finite time tracking
4. The mean-field formulation

The End