Semiclassical limit of fermion stars

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White dwarf as a fermion star

White dwarfs are compact stars with high mean density supported by the pressure of degenerate electron gas.

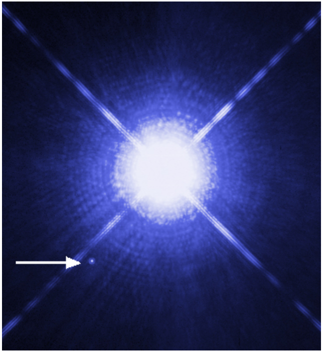

Figure: Image from Wikipedia

See an image of Sirius A and Sirius B. Sirius B, which is faint point of light is a white dwarf.

Fermions and bosons

Bosons may occupy the same quantum states but fermions may not.

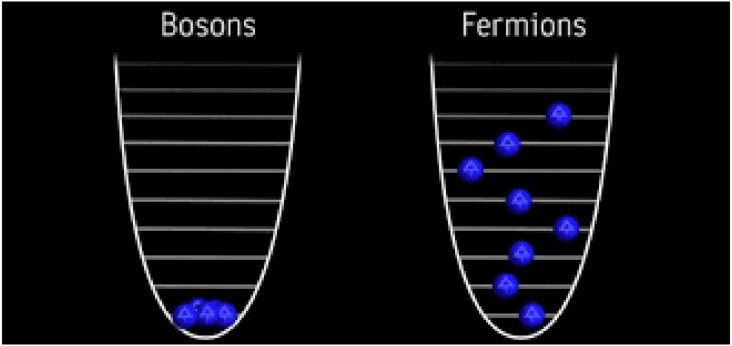

Figure: Image from reddit

Electron degeneracy pressure

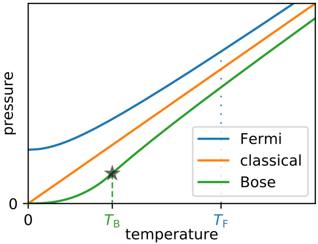

Figure: Image from Wikipedia

Chandrasekhar's theory of white dwarfs

- Equation of state:

$$P(\rho) = Cf(\sqrt[3]{\rho/D}), \quad f(x) = \int_0^x \frac{u^4}{\sqrt{1+u^2}}\,du, \tag{ES}$$

Equation of gravitational hydrostatic equilibrium:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP(\rho)}{dr}\right) = -4\pi G\rho \tag{GHE}$$

- The solution ρ with initial value $\rho(0) = \rho_0$ describes the density of a white dwarf.

Chandrasekhar's theory of white dwarfs

- The value R at which $\rho(r)$ firstly vanishes represents the radius of the white dwarf.
- R is a decreasing function of ρ_0.
- The solution ρ is a decreasing function on the interval $[0, R]$.
- As $\rho_0 \to \infty$, the radius R tends to 0 and the mass of a white star $\int \rho(r) r^2 \, dr$ tends to some positive number M_c. This predicts the gravitational collapse of a star with mass $M > M_c$.
- M_c is called the Chandrasekhar's limit mass.

Semiclassical results by Lieb and Yau (1987 CMP)

- Relativistic Schrödinger Hamiltonian for N particles:

$$H_N = \sum_{i=1}^{N} \sqrt{-\Delta_{x_i} + 1} - 1 - G \sum_{1 \le i < j \le N} \frac{1}{|x_i - x_j|}$$

acting on the Hilbert space

$$\mathcal{H}^{(N)} = (L^2(\mathbb{R}^3) \bigotimes \mathbb{C}^2)^{\wedge N}.$$

- Quantum ground energy level:

$$E_G^Q(N) = \inf \mathrm{spec}\ H_N$$

Then the eigenfunction belonging to $E_G^Q(N)$ represent the wave function of a white dwarf with mass N but its existence is not known.

Semiclassical results by Lieb and Yau (1987 CMP)

Semiclassical result by Lieb-Yau:

Fix $GN^{2/3}$ as a constant τ(fermions). Then there exists a number $\tau_c > 0$ such that if $\tau < \tau_c$

$$\lim_{N\to\infty} \frac{E_G^Q(N)}{E_G^C(N)} = 1,$$

where

$$E_G^C(N) := \inf\{\mathcal{E}_G^C(\rho) \mid \rho \geq 0,\, \rho \in L^{4/3}(\mathbb{R}^3),\, \int \rho = N\} \qquad \text{(SVP)}$$

and

$$\mathcal{E}_G^C(\rho) := \int A(\rho)\, dx - \frac{G}{2} \int \frac{\rho(x)\rho(y)}{|x-y|}\, dxdy,\; A(\rho) = \int_0^\rho \sqrt{\left(\frac{3}{4\pi}u\right)^{\frac{2}{3}} + 1} - 1\, du.$$

Semiclassical results by Lieb and Yau (1987 CMP)

Semiclassical result by Lieb-Yau:

- There exists a number $M_c(G) > 0$ such that

$$\begin{cases} -\infty < E_G^C(N) < 0 & \text{if } N < M_c(G) \\ E_G^C(N) = -\infty & \text{if } N > M_c(G) \end{cases}$$

  E_G^C is a decreasing function in N.

- If $N < M_c(G)$ the variational problem (SVP) admits a radially symmetric positive minimizer $\rho_{N,G}$ such that it satisfies (ES) and (GHE) (when $G = 1$).

Quantum Mean-field description of fermion stars

Using the Slater determinants $\psi_1(x_1) \wedge \psi_2(x_2) \wedge \cdots \wedge \psi_N(x_N)$, the evolution of unitary group $\{e^{-itH_N}\}$ is approximated by Hartree-Fock equations,

$$i\partial_t \psi_k = \sqrt{-\Delta + m^2}\psi_k - \sum_{l=1}^{N}(\frac{1}{|x|} * |\psi_l|^2)\psi_k + \sum_{l=1}^{N}\psi_l(\frac{1}{|x|} * \{\bar{\psi}_l \psi_k\}), \quad \text{(HF)}$$

$k = 1, \ldots, N.$

Quantum Mean-field description of fermion stars

The energy functional of (HF) is

$$\mathcal{E}(\Psi) = \sum_{k=1}^{N} \langle \psi_k, \sqrt{-\Delta + m^2}\psi_k \rangle - \frac{1}{2} \iint_{\mathbb{R}^6} \frac{\rho_\Psi(x)\rho_\Psi(y) - |\rho_\Psi(x,y)|^2}{|x-y|}\, dxdy,$$

where $\Psi = \{\psi_k\}_{k=1}^N$ are orthonormal and ρ_Ψ denotes the particle density $\sum_{k=1}^N |\psi_k(x)|^2$.

The fermion stars are described as a minimizer of the Hartree-Fock energy.

Operator form of Quantum Mean-field energy

Energy:

$$\mathcal{E}(\gamma) = \mathrm{Tr}\left((\sqrt{1-\Delta} - 1)\gamma\right) - \frac{1}{2}\iint_{\mathbb{R}^6} \frac{\rho_\gamma(x)\rho_\gamma(y) - |\gamma(x,y)|^2}{|x-y|} dxdy,$$

where $\rho_\gamma(x) = \gamma(x,x)$ denotes the density of γ.

Mass:

$$\mathrm{Tr}(\gamma) = m.$$

Variational problem:

$$\tilde{E}_{\mathrm{QM}}(m) = \inf\left\{\mathcal{E}(\gamma) \mid \gamma \in \mathfrak{H}^{\frac{1}{2}},\ 0 \le \gamma \le 1 \text{ and } \mathrm{Tr}(\gamma) = m\right\}. \qquad \text{(QVP)}$$

Chandrasekar's limit mass and existence of fermion stars for quantum mean-field formulation

Let

$$K_{\mathrm{QM}} := \inf_{\gamma \in \mathcal{A}_{\mathrm{QM}} \setminus \{0\}} \frac{\|\gamma\|^{\frac{1}{3}} \left(\mathrm{Tr}\gamma\right)^{\frac{2}{3}} \mathrm{Tr}\left| |\nabla|^{\frac{1}{2}} \gamma |\nabla|^{\frac{1}{2}} \right|}{\|\nabla \Phi_\gamma\|^2_{L^2(\mathbb{R}^3)}},$$

where $\Phi_\gamma = -|\cdot|^{-1} * \rho_\gamma$ and $\mathcal{A}_{\mathrm{QM}} := \{\gamma \in \mathfrak{H}^{\frac{1}{2}} \mid \gamma \geq 0, \|\gamma\| + \mathrm{Tr}(\gamma) < \infty\}$.

Known result (Lenzmann - Lewin):

- If $m^{\frac{2}{3}} > 2K_{\mathrm{QM}}$, then $\tilde{E}_{\mathrm{QM}}(m) = -\infty$.
- If $m^{\frac{2}{3}} < 2K_{\mathrm{QM}}$, then $-\infty < \tilde{E}_{\mathrm{QM}}(m) < 0$ and $\tilde{E}_{\mathrm{QM}}(m)$ is achieved by a minimizer

$$Q_0 = \mathbf{1}_{\{\sqrt{1-\Delta}-1+\Phi_{Q_0}+X_{Q_0}<\mu\}} + \mathcal{R}$$

.

Kinetic description of fermion stars?

Can we suggest a kinetic theory for fermion stars standing between the relativistic mean-field quantum theory and the Chandrasekhar theory?

Quantum Mean-field Theory

$\underbrace{\rightarrow}_{\text{semiclassical limit}}$ Kinetic Theory

$\underbrace{\rightarrow}_{\text{reducing to density functional}}$ Chandrasekhar Theory

Relativistic gravitational Vlasov-Poisson energy

For a distribution function $f(t,x,v)$, we define the relativistic gravitational Vlasov-Poisson energy

$$H(f) = \iint_{\mathbb{R}^6} (\sqrt{1+|v|^2} - 1) f(t,x,v)\, dxdv - \frac{1}{2} \iint_{\mathbb{R}^6} \frac{\rho_f(t,x)\rho_f(t,y)}{|x-y|}\, dxdy,$$

where $\rho_f = \int f\, dv$.

The total mass is given by

$$M(f) = \int_{\mathbb{R}^3} \rho_f(t,x)\, dx = \iint_{\mathbb{R}^6} f(t,x,v)\, dvdx$$

Kinetic variational problem

The kinetic variational problem is defined by

$$\tilde{E}_{\mathrm{CM}}(m) = \min_{f \in \mathcal{A}} H(f) \tag{KVP}$$

where

$$\mathcal{A} = \left\{ f \in L^1(\mathbb{R}^6) \mid M(f) = m,\, 0 \le f \le 1,\, \mathrm{supp}(f) \text{ is bdd} \right\}.$$

The point-wise constraint $0 \le f \le 1$ inherits the quantum feature of fermions.

Chandrasekar's limit mass and existence of fermion stars for kinetic formulation

Theorem (J. Jang and S.)

Let

$$K_{\mathrm{CM}} := \inf_{f \in \mathcal{E} \setminus \{0\}} \frac{\||v|f\|_{L^1} \|f\|_{L^1}^{\frac{2}{3}} \|f\|_{L^\infty}^{\frac{1}{3}}}{\|\nabla \Phi_f\|_{L^2}^2},$$

where $\Phi_f = -|\cdot|^{-1} * \rho_f$.

- *If* $m^{\frac{2}{3}} > 2K_{\mathrm{CM}}$, *then* $\tilde{E}_{\mathrm{CM}}(m) = -\infty$.
- *If* $m^{\frac{2}{3}} < 2K_{\mathrm{CM}}$, *then* $-\infty < \tilde{E}_{\mathrm{CM}}(m) < 0$ *and* $\tilde{E}_{\mathrm{CM}}(m)$ *is achieved by a minimizer*

$$f_0 = \mathbf{1}_{\{\sqrt{1+|p|^2} - 1 + \Phi_{f_0} \leq \mu\}}$$

.

Recovery of the Chandrasekhar theory

Theorem (J. Jang and S.)

Let f_0 be a minimizer of (KVP). Then $\rho_0 := \rho_{f_0}$ is a minimizer of (SVP). Consequently, ρ_0 satisfies (ES) and (GHE).

Semiclassical limit of fermion stars

Quantum energy with $\hbar$:

$$\mathcal{E}^{\hbar}(\gamma) = \mathrm{Tr}^{\hbar}\left((\sqrt{1-\hbar^2\Delta}-1)\gamma\right) - \frac{1}{2}\iint_{\mathbb{R}^6} \frac{\rho_\gamma^{\hbar}(x)\rho_\gamma^{\hbar}(y) - |\gamma^{\hbar}(x,y)|^2}{|x-y|} dxdy$$

where

$$\mathrm{Tr}^{\hbar} = (2\pi\hbar)^3\mathrm{Tr}, \quad \gamma^{\hbar}(x,y) = (2\pi\hbar)^3\gamma(x,y), \quad \rho_\gamma^{\hbar}(x) = \gamma^{\hbar}(x,x).$$

Variational problem with $\hbar$:

$$\tilde{E}_{\mathrm{QM}}^{\hbar}(m) = \inf\left\{\mathcal{E}^{\hbar}(\gamma) \mid \gamma \in \mathfrak{H}^{\frac{1}{2}},\ 0 \le \gamma \le 1 \text{ and } \mathrm{Tr}^{\hbar}(\gamma) = m\right\}. \quad (\hbar\mathrm{QVP})$$

Semiclassical limit of fermion stars

Invariance of K_{QM}:

$$K_{\mathrm{QM}} = \inf_{\gamma \in \mathcal{A}_{\mathrm{QM}} \setminus \{0\}} \frac{\|\gamma\|^{\frac{1}{3}} \left(\mathrm{Tr}^{\hbar}\gamma\right)^{\frac{2}{3}} \mathrm{Tr}^{\hbar} \left| |\hbar\nabla|^{\frac{1}{2}} \gamma |\hbar\nabla|^{\frac{1}{2}} \right|}{\|\nabla \Phi_{\gamma}^{\hbar}\|_{L^2(\mathbb{R}^3)}^2},$$

where $\Phi_{\gamma}^{\hbar} = -|\cdot|^{-1} * \rho_{\gamma}^{\hbar}$ and $\mathcal{A}_{\mathrm{QM}} := \{\gamma \in \mathfrak{H}^{\frac{1}{2}} \mid \gamma \geq 0, \|\gamma\| + \mathrm{Tr}(\gamma) < \infty\}$.

Theorem (Y. Hong, S. Jin, S.)

- *(Quantum limit mass $\leq$ Classical limit mass)* $K_{\mathrm{QM}} \leq K_{\mathrm{CM}}$
- *(Semiclassical limit) Let $Q_{\hbar}$ be a family of minimizers of ($\hbar$QVP) and f_0 be a minimizer of (KVP). Then as $\hbar \to 0$,*

$$\rho_{Q_{\hbar}} \rightharpoonup \rho_{f_0} \text{ weakly in } L^q \quad \forall q > 1$$
$$\||\nabla|^{-1}\rho_{Q_{\hbar}} - |\nabla|^{-1}\rho_{f_0}\|_{L^2 \cap L^\infty} \to 0.$$

Main ingredients for proof: Key inequalties

Lemma (Kinetic interpolation inequality)

If $0 \leq f \leq 1$, *then*

$$\|\rho_f\|_{L^{4/3}(\mathbb{R}^3)}^{4/3} \lesssim \||p|f\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)}.$$

Proof: The density function ρ_f satisfies the following trivial inequality

$$\rho_f = \int_{|p|\leq R} + \int_{|p|\geq R} f(\cdot,p)dp \lesssim R^3 + \frac{1}{R}\||p|f(\cdot,p)\|_{L^1_p(\mathbb{R}^3)}.$$

Optimizing the right hand side, we obtain $(\rho_f)^{4/3} \lesssim \||p|f(\cdot,p)\|_{L^1_p(\mathbb{R}^3)}$.
Then, integrating, we obtain the desired inequality.

Main ingredients for proof: Key inequalties

As a quantum analogue of the kinetic interpolation inequality, we have the Lieb-Thirring inequality.

Lemma (Lieb-Thirring inequality)

Let $\hbar \in (0,1]$, $s \geq 0$ *and* $\alpha \in [0, \frac{3}{2})$. *If* γ *is a compact self-adjoint operator on* $L^2(\mathbb{R}^3)$ *and* $0 \leq |\hbar\nabla|^{\alpha}\gamma|\hbar\nabla|^{\alpha} \leq 1$, *then*

$$\|\rho_\gamma^\hbar\|_{L^{\frac{3+2s-2\alpha}{3-2\alpha}}(\mathbb{R}^3)}^{\frac{3+2s-2\alpha}{3-2\alpha}} \lesssim \mathrm{Tr}^\hbar(|\hbar\nabla|^s\gamma|\hbar\nabla|^s),$$

where the implicit constant is independent of $\hbar$.

Main ingredients for proof: Relativistic Weyl's law

For each $\hbar > 0$, we denote the negative eigenvalues of the operator

$$\sqrt{1 - \hbar^2 \Delta} - 1 + \Phi_\hbar + X_\hbar$$

in non-decreasing order (counting multiplicities) by

$$\mu_1^\hbar < \mu_2^\hbar \leq \mu_3^\hbar \leq \cdots < 0.$$

Assumption: For a family $\{\Phi_\hbar\}_{\hbar \in (0,1]}$ of potentials and a family $\{X_\hbar\}_{\hbar \in (0,1]}$ of self-adjoint operators on $L^2(\mathbb{R}^3)$, the following hold.

1. $\Phi_\hbar$ is non-positive;
2. $\|\Phi_\hbar\|_{C^1(\mathbb{R}^3)}$ is bounded uniformly in $\hbar \in (0, 1]$;
3. There exists $\mu < 0$ such that $\|(\Phi_\hbar - \frac{\mu}{2})_-\|_{L^{3/2}(\mathbb{R}^3)}$ is bounded uniformly in $\hbar \in (0, 1]$.
4. $\|X_\hbar\| = O(\sqrt{\hbar})$.

Main ingredients for proof: Relativistic Weyl's law

Given the energy level $E < 0$, we denote the number of eigenvalues $< E$ by

$$N^\hbar(E) = N^\hbar(E; \Phi_\hbar, X_\hbar) = \mathrm{Tr}\big(\mathbf{1}_{\{\sqrt{1-\hbar^2\Delta}-1+\Phi_\hbar+X_\hbar<E\}}\big),$$

and define the associated sum

$$S^\hbar(E) = S^\hbar(E; \Phi_\hbar, X_\hbar) = \sum_{\mu_j^\hbar < E} (\mu_j^\hbar - E)$$

Lemma (Relativistic Weyl's law)

Suppose that $\{\Phi_\hbar\}_{\hbar\in(0,1]}$ and $\{X_\hbar\}_{\hbar\in(0,1]}$ satisfy Assumption with some $\mu < 0$. Then, for $\mu_\hbar \le \mu$, we have

$$(2\pi\hbar)^3 N^\hbar(\mu_\hbar) = \left|\left\{(q,p) \in \mathbb{R}^3 \times \mathbb{R}^3 : \sqrt{1+|p|^2} - 1 + \Phi_\hbar(q) \le \mu_\hbar\right\}\right| + O(\hbar^{1/4}).$$

$$(2\pi\hbar)^3 S^\hbar(\mu_\hbar) = \iint_{\sqrt{1-|p|^2}-1+\Phi_\hbar(q)<\mu_\hbar} \big(\sqrt{1-|p|^2} - 1 + \Phi_\hbar(q) - \mu_\hbar\big) dq dp + O(\sqrt{\hbar}),$$

Idea of proof: Upper energy estimate

- Let $f_0 = \mathbf{1}_{\{\sqrt{1+|p|^2}-1+\Phi_{f_0}\le\mu\}}$ be a minimizer of (KVP). Define

$$\gamma_\hbar = \mathbf{1}_{\{\sqrt{1-\hbar^2\Delta}-1+\Phi_{f_0}<\tilde{\mu}_\hbar\}} + \tilde{\mathcal{R}}_\hbar,$$

  where $0 \le \tilde{\mathcal{R}}_\hbar \le 1$ is a self-adjoint operator on the eigenspace of $\sqrt{1-\hbar^2\Delta} - 1 + \Phi$ associated to a negative eigenvalue $\tilde{\mu}_\hbar$.
- By using Relativistic Weyl's law, $\tilde{\mu}_\hbar$ and $\tilde{\mathcal{R}}_\hbar$ can be chosen so that $\mathrm{Tr}^\hbar(\gamma_\hbar) = m$.
- Using Relativistic Weyl's law,

$$\tilde{E}^\hbar_{\mathrm{QM}}(m) \le \mathcal{E}^\hbar(\gamma_\hbar) = H(Q) + o(1) = \tilde{E}_{\mathrm{CM}}(m) + o(1),$$

  which shows

$$\limsup_{\hbar\to 0} \tilde{E}^\hbar_{\mathrm{QM}}(m) \le \tilde{E}_{\mathrm{CM}}(m)$$

Idea of proof: Lower energy estimate

- To obtain the lower energy estimate

$$\liminf_{\hbar \to 0} \tilde{E}^{\hbar}_{\mathrm{QM}}(m) \geq \tilde{E}_{\mathrm{CM}}(m),$$

  we do the similar work with the auxiliary kinetic distribution function

$$f_\hbar = \mathbf{1}_{\{\sqrt{1+|p|^2}-1+\Phi_{Q_\hbar}(q)<\tilde{\mu}'_\hbar\}},$$

  where $\tilde{\mu}'_\hbar$ is chosen to be $M(f_\hbar) = m$ and $\Phi_{Q_\hbar}$ is the potential of a minimizer $Q_\hbar$ of ($\hbar$QVP).

- Thus we have

$$\lim_{\hbar \to 0} \tilde{E}^{\hbar}_{\mathrm{QM}}(m) = \tilde{E}_{\mathrm{CM}}(m),$$

Convergence and Regularity

- The convergence of the ground state energy shows $\{f_\hbar\}$ is a minimizing sequence for (KVP). Then it is possble to shows that as $\hbar \to 0$

$$\Phi_{Q_\hbar} \to \Phi_{f_0} \quad \text{in } \dot{H}^1,$$

which is equivalent to

$$\||\nabla|^{-1}\rho_{Q_\hbar} - |\nabla|^{-1}\rho_{f_0}\|_{L^2}.$$

- Further regularity estimate comes from iteratively applying the Lieb-Thirring inequality.

Thank you for your attention!