Convergence of discrete Aubry-Mather model in the continuous limit

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Abstract

Let $H(x,p)$ be a Tonelli Hamiltonian defined on $\mathbb{T}^d \times \mathbb{R}^d$. We show how to approximate the solutions of the cell equation $H(x, d_x u) = \bar{H}$ by discrete weak KAM solutions. The cell equation is a degenerate PDE of first order that can be solved using the weak KAM theory developed by Mather and Fathi [Mat91, Mat93, Fat08]. Discrete weak KAM theory is similar to the Frenkel-Kontorova theory developed by Aubry and Le Daeron [ALD83] or Chou and Griffiths [CG86] in dimension one. The theory has then been more thoroughly developed by Gomes [Gom05] and by Garibaldi

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and Thieullen [GT11] in higher dimension. The discrete weak KAM approach approximates the Lagrangian trajectories by one-dimensional chains of atoms. By introducing a family of discrete actions $\{\mathcal{L}_\tau(x,y)\}_{\tau>0}$ related to the Legendre-Fenchel transform of H, we show that the solutions of additive eigenvalue problem $u_\tau(y) + \bar{\mathcal{L}}_\tau = \inf_{x\in\mathbb{R}^d}\{u_\tau(x) + \mathcal{L}_\tau(x,y)\}$ with unknowns $(\bar{\mathcal{L}}_\tau, u_\tau)$ converges in a sense to be defined to a solution of the cell equation with unknowns $(\bar{H}, u)$.
Keywords: discrete weak KAM theory, Frenkel-Kontorova models, Aubry-Mather theory, discounted Lax-Oleinik operator, ergodic cell equation, short-range actions, additive eigenvalue problem

1 Introduction

We consider in this article a periodic time-independent C^2 Hamiltonian $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ with integer $d \geq 1$ satisfying the following assumptions:

(L1) **Positive Definiteness:** $H(x,p)$ is strictly convex with respect to p, i.e., the second partial derivative $\frac{\partial^2 H}{\partial p^2}(x,p)$ is positive definite as a quadratic form uniformly in $x \in \mathbb{T}^d$ and $\|p\| \leq R$, for every $R > 0$;

(L2) **Superliner growth:** $H(x,p)$ is superlinear with respect to p, uniformly in x, that is,
$$\lim_{\|p\|\to+\infty} \inf_{x\in\mathbb{T}^d} \frac{H(x,p)}{\|p\|} = +\infty.$$

We will say that H is a *Tonelli Hamiltonian*. The *PDE cell equation*

$$H(x, P + du(x)) = \bar{H}(P),$$

is a degenerate PDE equation of first order; $P \in \mathbb{R}^d$ is a parameter, $(\bar{H}, u)$ are the two unknowns: $\bar{H}(P)$ is a scalar and $u(x)$ is a continuous periodic function solution in the viscosity sense. The constant $\bar{H}(P)$ is unique and called the *effective energy*. This equation has first been studied by Lions, Papanicolaou and Varadhan [LPV87]. A comprehensive treatment may be found in Crandall, Ishii and Lions [CIL92], Bardi and Capuzzo-Dolcetta [BCD97] or Barles [Bar94]. Some recent overviews may be found in the articles [Ish13, Bar13]. A new approach has been initiated by Fathi [Fat97a, Fat97b, Fat08] by transfering the problem into a Lagrangian setting. Let $L(x,v)$ be the Legendre-Fenchel transform of $H(x,p)$. We call L the *Lagrangian* of the system and we notice that L is again C^2, strictly convex with respect to v, and superlinear. The Legendre-Fenchel transform of $H(x, P+p)$ is the Lagrangian $L(x,v) - Pv$. The constant $\bar{H}(P)$ is also called the *Mañé critical value*; it has been considered for the first time by Mañé [Mn96] and then developed by Contreras and Iturriaga [CI99]. The ergodic definition of the Mañé critical value is given by many equivalent definitions. We choose the following definition

$$-\bar{H}(P) := \lim_{t\to+\infty} \inf_{\gamma:[-t,0]\to\mathbb{R}^d} \Big[\frac{1}{t}\int_{-t}^{0} L(\gamma,\dot\gamma)\,ds - P(\gamma(0)-\gamma(-t))\Big] \tag{1}$$

where the infimum is taken over absolutely continuous paths over $[-t,0]$. (Note that the limit exists since the function $\inf_\gamma \int_{-t}^0$ is super-additive as a function of t). The unknown $u(x)$ is called weak KAM solution by Fathi [Fat97a, Fat97b, Fat08]. It is easy to see that the PDE cell equation is equivalent for Tonelli Hamiltonian to an additive eigenvalue problem for a semi-group of non-linear operators. We call *ergodic cell equation*

$$T^t[u] = u - t\bar{H}(P), \quad \forall t>0, \tag{2}$$

where T^t is the *(backward) Lax-Oleinik semi-group* defined by

$$T^t[u](x) := \inf_{\substack{\gamma:[-t,0]\to\mathbb{R}^d\\ \gamma(0)=x}} \Big[u(\gamma(-t)) + \int_{-t}^{0} L(\gamma,\dot\gamma)\,ds - P(\gamma(0)-\gamma(-t))\Big]. \tag{3}$$

The infimum in T^t is taken over absolutely continuous paths over $[-t,0]$ with terminal point x. For Tonelli Hamiltonian, the infimum is actually attained by a C^2 curve thanks to Tonelli-Weirestrass theorem.

Weak KAM theory is a theory about the structure of the set of continuous paths that minimize the action of the Lagrangian L. A similar theory exists in solid state physics that studies the configurations of a chain of atoms that minimize an interaction energy E. The central model is described by a (generalized) Frenkel-Kontorova model. The configuration of a chain of atoms $\{x_n\}_{n\in\mathbb{Z}}$ is supposed to be infinite and each x_n denotes the position of the n-th atom of the chain in $\mathbb{R}^d$ (not in $\mathbb{T}^d$). The interaction energy $E(x,y)$ is supposed to be short range and periodic: $E(x_n,x_{n+1})$ models the interaction energy between the two nearest atoms and E satisfies the translation-invariance property $E(x+k,y+k)=E(x,y)$ for every $k\in\mathbb{Z}^d$. In general $E(x,y)$ models both the internal interaction between nearest atoms, and the external interaction with the substrate. The *original Frenkel-Kontorova model* [FK38] is given by an interaction in dimension $d=1$,

$$E_P(x,y) = \frac{1}{2}|y-x|^2 + \frac{K}{(2\pi)^2}\Big(1-\cos(2\pi x)\Big) - P(y-x).$$

In solid state physics, it is more appropriate to write the elastic interaction as $\frac{1}{2}|y-x-P|^2$ instead of $\frac{1}{2}|y-x|^2 - P(y-x)$ where P denotes the mean distance between two successive atoms of the chain. In Aubry-Mather theory, the interaction energy is decoupled into two parts

$$E_P(x, y) := E(x, y) - P(y - x), \quad \text{where } P \in \mathbb{R}^d.$$

Here P represents a cohomological term. In the case of Frenkel-Kontorova, the interaction has the form $E(x, y) = W(y - x) + KV(x)$ where W describes the internal energy between two successive atoms, V represents the action of the external periodic media on the chain, and K is a coupling factor.

The main problem in the Frenkel-Kontorova model is to understand the set of configurations that minimize the total energy $\sum_{n\in\mathbb{Z}} E_P(x_n, x_{n+1})$ in a precise sense. Chou and Griffiths [CG86] highlighted first the importance of the two following quantities: $\bar{E}(P)$, the effective energy of the system (or the ground energy in Gibbs theory), and $u(x)$, the effective potential which is a continuous periodic function that calibrates the interaction energy. They showed that $(\bar{E}(P), u)$ can be seen as unknowns of a *discrete additive eigenvalue problem*. We call *discrete backward Lax-Oleinik equation*, the equation

$$u(y) + \bar{E}(P) = \inf_{x\in\mathbb{R}^d} \{u(x) + E_P(x, y)\}. \tag{4}$$

We emphasize the fact that, although u is periodic, we consider $u : \mathbb{R}^d \to \mathbb{R}$ as a function of $\mathbb{R}^d$ and the optimal point $x \in \arg\min\{u(x) + E_P(x, y)\}$ as a point in $\mathbb{R}^d$.

A particular discrete Lax-Oleinik equation is given by choosing the following *interaction action* $E_P(x, y) = \mathcal{L}_{P,\tau}(x, y)$ where

$$\mathcal{L}_{P,\tau}(x, y) := \tau L\Big(x, \frac{y - x}{\tau}\Big) - P(y - x). \tag{5}$$

(Notice that the physical dimension of H or L is an *energy* ML^2T^{-2}, while the discrete action $\mathcal{L}_\tau(x, y)$ has the dimension *time* $\times$ *energy*).

Our first objective is to show that, for any solution u_τ of the discrete eigenvalue problem (4) with $E_P = \mathcal{L}_{P,\tau}$, there exists a sub-sequence $\tau_i \to 0$ such that u_{τ_i} converges to a solution of the ergodic cell equation (2). We notice that without loss of generality we may assume $P = 0$. Our first main result is the following:

Theorem 1. *Let $H(x, p) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a C^2, time-independent, periodic in x, Tonelli Hamiltonian (satisfying the hypotheses (L1) and (L2)). Let $L(x, v)$ be the Lagrangian associated to H, and $\mathcal{L}_\tau(x, y)$ be the one-parameter family of discrete actions*

$$\mathcal{L}_\tau(x, y) := \tau L\Big(x, \frac{y - x}{\tau}\Big), \quad \textit{for every } \tau \in (0, 1]. \tag{6}$$

We call discrete (backward) Lax-Oleinik equation, the equation

$$u_\tau(y) + \bar{\mathcal{L}}_\tau = \min_{x\in\mathbb{R}^d} \{u_\tau(x) + \mathcal{L}_\tau(x, y)\}, \quad \forall y \in \mathbb{R}^d, \tag{7}$$

where the two unknowns are $\bar{\mathcal{L}}_\tau$ which is a scalar, and $u_\tau : \mathbb{R}^d \to \mathbb{R}$ which is a C^0 periodic function. Then the following conclusions holds.

i. *Equation* (7) *admits a solution* $(\bar{\mathcal{L}}_\tau, u_\tau)$. *The constant $\bar{\mathcal{L}}_\tau$ is unique and is called the effective action of $\mathcal{L}_\tau(x, y)$. The function u_τ may not be unique. We call discrete weak KAM solution any u_τ solution of* (7).

ii. *There exists a constant $C > 0$ such that*

$$\Big|\frac{\bar{\mathcal{L}}_\tau}{\tau} + \bar{H}(0)\Big| \le C\tau, \quad \forall \tau \in (0, 1].$$

iii. *There exist constants $C, R > 0$ such that, for every $\tau \in (0, 1]$ and for every discrete weak KAM solution u_τ of* (7),

   (a) *Lip*$(u_\tau) \le C$, *in particular* $\|u_\tau\|_\infty \le C$ *if* $\min(u_\tau) = 0$,

   (b) $\forall y \in \mathbb{R}^d$, *if* $x \in \arg\min_{x\in\mathbb{R}^d} \{u_\tau(x) + \mathcal{L}_\tau(x, y)\}$ *then* $\|y - x\| \le \tau R$.

iv. *There exist a subsequence $\tau_i \to 0$ and discrete weak KAM solutions u_{τ_i} such that $u_{\tau_i} \to u$ uniformly. Moreover every such u is a solution of the ergodic cell equation (a weak KAM solution in Fathi's terminology) or equivalently a viscosity solution of the PDE cell equation*

$$T^t[u] = u - t\bar{H}(0),\ \forall t > 0 \iff H(x, du(x)) = \bar{H}(0).$$

The convergence of the solutions of the discrete Lax-Oleinik equation to the solutions of the ergodic cell equation has been addressed by Gomes [Gom05] and Camilli, Cappuzzo-Dolcetta, Gomes [CCDG08], but their proofs require a particular form of the Lagrangian: there exists a (uniform) constant $R > 0$ such that for every x, h, v, w

$$L(x + h, v + w) - 2L(x, v) + L(x - h, v - w) \le R(|h|^2 + |w|^2)$$

that we do not assume. Several other numerical schemes have been studied for computing the effective Hamiltonian, see [GO04], [Ror06], [FR10] but the properties *(i)–(iv)* are not stated explicitly, see also [BFZ12] for a mechanical Lagrangian of the form $L(t, x, v) = W(v) + V(t, x)$.

Note that the discrete Lax-Oleinik equation possesses a second form: the *discrete forward Lax-Oleinik equation*,

$$u_\tau(x) - \bar{\mathcal{L}}_\tau = \max_{y\in\mathbb{R}^d} \{u_\tau(y) - \mathcal{L}_\tau(x, y)\}.$$

Theorem 1 is also valid for the forward Lax-Oleinik equation with the same effective action $\bar{\mathcal{L}}_\tau$ and possibly a different u_τ. From now on we only study the backward problem.

Our second objective is to show that, by introducing a discounted factor δ in the discrete Lax-Oleinik equation (7), we do not need to take a sub-sequence in time to obtain the convergence of the discrete solution to a solution of the PDE cell equation. Our result is twofold: for fixed $\tau > 0$, by letting $\delta \to 0$, $u_{\tau,\delta} - \frac{\bar{\mathcal{L}}_\tau}{\tau\delta}$ converges to a particular discrete weak KAM solutions u^*_τ; for $\tau, \delta \to 0$ satisfying the constraint $\tau = o(\delta)$, $u_{\tau,\delta} - \frac{\bar{\mathcal{L}}_\tau}{\tau\delta}$ converges to the weak KAM solution u^* that is described in [DFIZ14].

We recall that the *discounted cell equation* is given by the PDE

$$\delta u_\delta(x) + H(x, d_x u_\delta(x)) = 0, \tag{8}$$

where u_δ is understood in the viscosity sense. It is known (see [LPV87, CIL92, Bar94, BCD97]) that the equation (8) admits a unique periodic solution u_δ given by the integral formula,

$$u_\delta(x) = \inf\left\{\int_{-\infty}^{0} e^{s\delta} L(\gamma(s), \dot\gamma(s))\, ds \ : \ \gamma : (-\infty, 0] \to \mathbb{R}^d,\ \gamma(0) = x\right\}, \tag{9}$$

where the infimum is taken over the set of absolutely continuous paths ending at x. The authors of [DFIZ14] showed that a correct normalization of u_δ converges to some solution of the cell equation, namely

$$\lim_{\delta\to 0}\left(u_\delta + \frac{\bar{H}(0)}{\delta}\right) = u^* \quad \text{(exists in the } C^0 \text{ topology).} \tag{10}$$

The limit u^* is a particular weak KAM solution that we call the *asymptotically-discounted weak KAM solution*. We obtain a similar statement in the discrete case. Our second main result is the following:

Theorem 2. *We assume that $H(x, p)$ satisfies the same hypotheses as in theorem 1. We call discounted discrete Lax-Oleinik equation*

$$u(y) = \min_{x\in\mathbb{R}^d} \{(1 - \tau\delta)u(x) + \mathcal{L}_\tau(x, y)\}, \quad \forall y \in \mathbb{R}^d, \tag{11}$$

where u is a C^0 periodic unknown function.

i. *Equation* (11) *admits a unique solution $u_{\tau,\delta}$ called discounted discrete weak KAM solutions that can be written as,*

$$u_{\tau,\delta}(x) := \inf_{\substack{\{x_{-k}\}_{k=0}^{+\infty} \\ x_0 = x}} \sum_{k=0}^{\infty} (1 - \tau\delta)^k \mathcal{L}_\tau(x_{-(k+1)}, x_{-k}).$$

(The infimum is taken over all backward configurations $x_{-k} \in \mathbb{R}^d$ starting at $x_0 = x$). Moreover there exist constants $R, C > 0$ such that, for every $\tau, \delta \in (0,1]$, $Lip(u_{\tau,\delta}) \le C$, and for every $y \in \mathbb{R}^d$

$$x \in \arg\min_{x\in\mathbb{R}^d} \{(1-\tau\delta)u_{\tau,\delta}(x) + \mathcal{L}_\tau(x,y)\} \quad \Rightarrow \quad \|y-x\| \le \tau R.$$

ii. For every $\tau \in (0,1]$ fixed, as $\delta \to 0$, $u_{\tau,\delta} - \frac{\bar{\mathcal{L}}_\tau}{\tau\delta}$ converges uniformly in C^0 topology to a Lipschitz periodic function u^*_τ. The limit u^*_τ is a particular discrete weak KAM solution that admits the two characterizations

$$\begin{aligned} u^*_\tau(y) &= \sup\{w(y) \,:\, w \text{ is a discrete weak KAM solution s.t.} \\ &\qquad\qquad \textstyle\int w\,d\mu \le 0,\ \forall\mu \text{ projected } \tau\text{-minimizing measure}\} \\ &= \inf\{\textstyle\int \Phi_\tau(x,y)\,d\mu(x) \,:\, \mu \text{ is a projected } \tau\text{-minimizing measure}\}. \end{aligned}$$

(The notion of τ-minimizing measures is explained in definition 13, and the Mañé potential Φ_τ is defined in definition 20). The limit u^*_τ is called the asymptotically-discounted discrete weak KAM solution.

iii. There exists a constant $C > 0$ such that for every $\tau, \delta \in (0,1]$,

$$\|u_{\tau,\delta} - u_\delta\|_\infty \le C\frac{\tau}{\delta} \quad \text{and} \quad \Big\| \Big(u_{\tau,\delta} - \frac{\bar{\mathcal{L}}_\tau}{\tau\delta}\Big) - \Big(u_\delta + \frac{\bar{H}(0)}{\delta}\Big) \Big\|_\infty \le C\frac{\tau}{\delta}.$$

In particular, if $\tau \to 0$, $\delta \to 0$, and $\frac{\tau}{\delta} \to 0$, then $u_{\tau,\delta} - \frac{\bar{\mathcal{L}}_\tau}{\tau\delta}$ converges uniformly in C^0 norm to the asymptotically discounted weak KAM solution u^* defined by equation (10).

Theorem 1 is proved in section 5 and theorem 2 is proved in section 6. Both proofs use a new *a priori bound* valid for more general actions called short-range actions (see propositions 4, 5 and 19). We also show that the infimum in the representation formula (9) is attained by a $C^{1,1}$ path. The last estimate item (iii) improves similar estimates in [Ror06, FR10, BFZ12].

2 General discrete Aubry-Mather model

In order to understand better the difference between the discrete Lax-Oleinik equation (4), where $E_P(x,y) = \mathcal{L}_{P,\tau}(x,y)$, and the ergodic cell equation (2), it will be helpful to introduce an intermediate one-parameter family of actions. Define for every $\tau > 0$, $x, y \in \mathbb{R}^d$, and $P \in \mathbb{R}^d$,

$$\mathcal{E}_{P,\tau}(x,y) := \inf_{\substack{\gamma\in C^{ac}([0,\tau],\mathbb{R}^d) \\ \gamma(0)=x,\ \gamma(\tau)=y}} \left\{ \int_0^\tau L(\gamma(t),\dot\gamma(t))\,dt - P(y-x) \right\}.$$

Note that the infimum in the definition of $\mathcal{E}_{P,\tau}$ is reached by some C^2 curve due to Tonelli–Weierstrass theorem. Such a curve is called a minimizer. Again there is no loss of generality by assuming $P = 0$. We call *minimal action*

$$\mathcal{E}_\tau(x, y) := \inf_{\substack{\gamma\in C^{ac}([0,\tau],\mathbb{R}^d) \\ \gamma(0)=x,\ \gamma(\tau)=y}} \int_0^\tau L(\gamma(t), \dot\gamma(t))\, dt. \tag{12}$$

The action $\mathcal{E}_\tau(x, y)$ is called by Mather [Mat93] in section 4, the *variational principle associated to L*, and by Fathi [Fat08] in section 4.4 the *minimal action*. We also stress that $\mathcal{E}_\tau(x, y)$ has been defined over the universal covering space $\mathbb{R}^d$ and not over $\mathbb{T}^d$.

We have just defined two families of actions: the discrete action $\mathcal{L}_\tau(x, y)$ and the minimal action $\mathcal{E}_\tau(x, y)$. We are thus led to consider more general one-parameter families of actions $\{E_\tau(x, y)\}_{\tau\in(0,1]}$ which cover the two previous examples. We focus on the fact that $\|y - x\|$ and τ should have the same order of magnitude as $\tau \to 0$: we call this property *short-range*. We list in what follows the only properties on $E_\tau(x, y)$ we are going to use.

Main hypotheses 3. We call *family of short-range actions*, a one-parameter family of functions $E_\tau(x, y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ indexed by $\tau \in (0, 1]$ satisfying:

(H1) $E_\tau(x, y)$ is **continuous** in (x, y);

(H2) $E_\tau(x, y)$ is **translational periodic**:

$$E_\tau(x + k, y + k) = E_\tau(x, y), \quad \forall k \in \mathbb{Z}^d \quad \text{and} \quad \forall x, y \in \mathbb{R}^d;$$

(H3) $E_\tau(x, y)$ is **uniformly coercive**:

$$\lim_{R\to+\infty} \inf_{\tau\in(0,1]} \inf_{\|x-y\|\geq\tau R} \tfrac{1}{\tau} E_\tau(x, y) = +\infty;$$

(H4) $E_\tau(x, y)$ is **uniformly bounded**: for every $R > 0$

$$\inf_{\tau\in(0,1]} \inf_{x,y\in\mathbb{R}^d} \tfrac{1}{\tau} E_\tau(x, y) > -\infty, \qquad \sup_{\tau\in(0,1]} \sup_{\|y-x\|\leq\tau R} \tfrac{1}{\tau} E_\tau(x, y) < +\infty.$$

We will also reinforce the continuity and coerciveness properties for the main step of the proof by asking

(H5) $E_\tau(x, y)$ is **uniformly superlinear**:

$$\lim_{R\to+\infty} \inf_{\tau\in(0,1]} \inf_{\|x-y\|\geq\tau R} \frac{E_\tau(x, y)}{\|x - y\|} = +\infty;$$

(Note that superlinearity implies coerciveness.)

(H6) $E_\tau(x,y)$ is **uniformly Lipschitz**: for every $R > 0$, there exists a constant $C(R) > 0$ such that, for every $\tau \in (0,1]$ and for every $x,y,z \in \mathbb{R}^d$,

– if $\|y-x\| \leq \tau R$ and $\|z-x\| \leq \tau R$ then

$$|E_\tau(x,z) - E_\tau(x,y)| \leq C(R)\|z-y\|,$$

– if $\|z-x\| \leq \tau R$ and $\|z-y\| \leq \tau R$ then

$$|E_\tau(x,z) - E_\tau(y,z)| \leq C(R)\|y-x\|.$$

The proof of theorem 1 will follow mainly from two results. The first one gives an a priori bound on discrete weak KAM solutions for every short-range actions $E_\tau(x,y)$; the second one shows that the two actions $\mathcal{L}_\tau(x,y)$ and $\mathcal{E}_\tau(x,y)$ are particular cases of short-range actions which are comparable in the sense that $\mathcal{L}_\tau(x,y) - \mathcal{E}_\tau(x,y) = O(\tau^2)$ uniformly on $\|y-x\| = O(\tau)$.

We call *discrete Lax-Oleinik operator*,

$$T_\tau[u](y) := \min_{x \in \mathbb{R}^d} \{u(x) + E_\tau(x,y)\}, \quad \forall y \in \mathbb{R}^d,\ \forall \tau \in (0,1], \tag{13}$$

for every continuous periodic function $u \in C^0(\mathbb{R}^d)$.

We call *discrete weak KAM solution*, any periodic continuous function u solution of the additive eigenvalue problem,

$$T_\tau[u] = u + \bar{E}_\tau, \tag{14}$$

where $\bar{E}_\tau$ is a scalar which can be understood as an additive eigenvalue. We ask the reader to notice the two notations: the continuous time operator T^t defined in (3) and the discrete time operator T_τ defined in (13).

Proposition 4 (A priori compactness for short-range actions)**.** *We consider a family of short-range actions $\{E_\tau(x,y)\}_{\tau > \in (0,1]}$ satisfying the hypotheses (H1)–(H6).*

i. There exist constants $C, R > 0$ such that, if $\tau \in (0,1]$ and u_τ is a discrete weak KAM solution of (14)*, then*

(a) u_τ is Lipschitz and $\mathrm{Lip}(u_\tau) \leq C$,

(b) $\forall y \in \mathbb{R}^d,\ x \in \arg\min_{x \in \mathbb{R}^d} \{u_\tau(x) + E_\tau(x,y)\} \ \Rightarrow\ \|y-x\| \leq \tau R.$

ii. For every Lipschitz periodic function u, $\lim_{\tau \to 0} T_\tau[u] = u$ *uniformly. More precisely, for every constant $L > 0$, there exist constants $R_L, C_L > 0$ such that, if u is any Lipschitz function satisfying $Lip(u) \leq C_L$, and if $\tau \in (0,1]$, then* *(a)* $\forall y \in \mathbb{R}^d$, $x \in \arg\min_{x\in\mathbb{R}^d} \{u(x) + E_\tau(x, y)\} \Rightarrow \|y - x\| \le \tau R_L$,

(b) $\| T_\tau[u] - u \|_\infty \le \tau C_L$.

Proposition 5 (Comparison estimate)**.** *Let $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a C^2, periodic, time-independent, Tonelli Hamiltonian and L be the associated Lagrangian. Then*

i. the two families of actions $\{\mathcal{L}_\tau(x, y)\}_{\tau>0}$ and $\{\mathcal{E}_\tau(x, y)\}_{\tau>0}$, which is defined in equations (6) *and* (12) *respectively, satisfy the hypotheses (H1)–(H6);*

ii. for every $R > 0$, there exist a constant $C(R) > 0$ such that, if $\tau \in (0, 1]$, $x, y \in \mathbb{R}^d$ satisfy $\|y - x\| \le \tau R$, then

$$|\mathcal{E}_\tau(x, y) - \mathcal{L}_\tau(x, y)| \le \tau^2 C(R).$$

Proposition 4 will be proved in section 3 by using an equivalent definition of the effective action. Proposition 5 is proved in section 4. The proof of item (i) of this proposition for the discrete action $\mathcal{L}_\tau(x, y)$ is trivial. For the minimal action $\mathcal{E}_\tau(x, y)$, the proof of (i) and the comparison estimate (ii) will be proved as a consequence of an a priori compactness estimate for Tonelli minimizers (lemma 14).

We recall that, for Tonelli Hamiltonian, there exists a C^2 minimizer of either

$$\mathcal{E}_\tau(x, y) = \inf_{\gamma(0)=x,\ \gamma(\tau)=y} \int_0^\tau L(\gamma, \dot\gamma)\, ds \quad \text{or} \quad T^t[0](y) = \inf_{x\in\mathbb{R}^d} \mathcal{E}_t(x, y),$$

where the infimum is taken over absolutely continuous paths $\gamma : [0, \tau] \to \mathbb{R}^d$ and T^t has been defined in (3). We say that a path $\gamma : [0, \tau] \to \mathbb{R}^d$ is a characteristic if it is obtained as the projection of an orbit of the Hamiltonian flow or the Euler-Lagrange flow.

3 Short-range actions

This section is mainly devoted to the proof of proposition 4 and more generally to results that are valid for every family of short-range actions. We begin by recalling some properties of the discrete Lax-Oleinik operator (13). The main point is that discrete weak KAM solutions exist and that the additive eigenvalue $\bar{E}_\tau$ is unique. We also show different ways to compute $\bar{E}_\tau$. The following estimates are obvious.

Proposition 6. *For any $u, v \in C^0(\mathbb{T}^d)$, and $c \in \mathbb{R}$, for any subset $\mathcal{U}$ of nonnegative continuous periodic functions, we have*

i. if $u \le v$, then $T_\tau[u] \le T_\tau[v]$;

ii. $\|T_\tau[u] - T_\tau[v]\|_\infty \le \|u - v\|_\infty$;

iii. $T_\tau[u+c] = T_\tau[u] + c$*;*

iv. $T_\tau[\inf_{u\in\mathcal{U}} u] = \inf_{u\in\mathcal{U}} T_\tau[u]$.

The following theorem, though fundamental, is easy to establish. Different proofs may be found as for instance in [Nus91], [Gom05] and [GT11]. We will nevertheless give a different proof using the discounted Lax-Oleinik operator defined in section 6.

Theorem 7 (Lax-Oleinik equation for short-range actions)**.** *Let* $\{E_\tau(x,y)\}_{\tau\in(0,1]}$ *be a family of short-range actions satsifying the hypotheses (H1)–(H4). Then for every* $\tau \in (0,1]$ *there exists a unique scalar* $\bar{E}_\tau$*, called effective action, such that*

$$T_\tau[u] = u + \bar{E}_\tau$$

admits a continuous periodic solution u.

Solutions of this equation are called discrete weak KAM solutions.

Proof. The proof of the existence of u is done in section 6, we just prove the uniqueness of the effective action. Suppose there exist $u, v \in C^0(\mathbb{T}^d)$ and $\lambda, \mu \in \mathbb{R}$ such that

$$T_\tau[u] = u + \lambda \quad \text{and} \quad T_\tau[v] = v + \mu.$$

Therefore, there exist c and y_* such that

$$u(y) \le v(y) + c, \ \ \forall\, y \in \mathbb{T}^d, \quad \text{and} \quad u(y_*) = v(y_*) + c.$$

Consequently, for any $y \in \mathbb{T}^d$, we have

$$u(y) + \lambda = T_\tau[u](y) \le T[v](y) + c = v(y) + c + \mu.$$

Taking $y = y_*$ implies $\lambda \le \mu$. The opposite inequality holds similarly. □

The additive eigenvalue $\bar{E}_\tau$ may be seen as a discrete analogue of Mañé critical value $-c[0]$. The discrete weak KAM solutions play the role of the discrete viscosity solutions. The following proposition recall two ways of computing directly the effective action.

Proposition 8. *Under the hypotheses (H1)–(H4), the effective action can be computed either by*

$$\begin{aligned} \bar{E}_\tau &= \sup_{u\in C^0(\mathbb{T}^d)} \inf_{x,y\,\in\,\mathbb{R}^d} \{E_\tau(x,y) - [u(y)-u(x)]\}, \\ &= \sup_{v\in\mathcal{B}(\mathbb{R}^d)} \inf_{x,y\,\in\,\mathbb{R}^d} \{E_\tau(x,y) - [v(y)-v(x)]\}, \end{aligned} \tag{15}$$

(where $\mathcal{B}(\mathbb{R}^d)$ denotes the space of bounded functions not necessarily periodic) or as a mean action per site

$$\bar{E}_\tau = \lim_{k\to+\infty} \inf_{z_0,\dots,z_k\in\mathbb{R}^d} \frac{1}{k}\sum_{i=0}^{k-1} E_\tau(z_i, z_{i+1}). \tag{16}$$

Property (15) is called the *discrete sup-inf formula* and is analogue to the sup-inf formula introduced by [CIPP98] for continuous-time Tonelli Hamiltonian systems. Equation (16) shows that the effective action is similar to the notion of ground state for one-dimensional chains of atoms $\{z_i\}_{i\in\mathbb{Z}^d}$ as in [CG86].

Note that if we define $\bar{E}_\tau(k) := \inf_{z_0,\dots,z_k}\sum_{i=0}^{k-1} E_\tau(z_i, z_{i+1})$, then $\bar{E}_\tau(k)$ is super additive (i.e., $E(k+l) \geq E(k) + E(l)$), which implies in particular that the limit $\lim_{k\to+\infty}\frac{1}{k}\bar{E}_\tau(k) = \sup_{k\geq1}\frac{1}{k}\bar{E}_\tau(k)$ exists.

The second equality of property (15) is new and we give a proof.

Proof of proposition 8. Part 1. Let be

$$\lambda := \sup_{u\in C^0(\mathbb{T}^d)} \inf_{x,y\in\mathbb{R}^d} \{E_\tau(x,y) - [u(y) - u(x)]\}.$$

Let u be a discrete weak KAM solution of (14). Then

$$u(y) - u(x) + \bar{E}_\tau \leq E_\tau(x,y), \qquad \forall\, x,y\in\mathbb{R}^d.$$

In particular, on obtains $\lambda \geq \bar{E}_\tau$. Conversely, let v be a bounded function and $c = \sup_{z\in\mathbb{R}^d}\{v(z) - u(z)\}$. On the one hand $v(z) \leq u(z) + c$ for every $z\in\mathbb{R}^d$, and by the monotonicity of T_τ, we obtain

$$T_\tau[v](z) \leq T_\tau[u](z) + c, \qquad \forall\, z\in\mathbb{R}^d.$$

On the other hand, for every $\epsilon > 0$, one can choose $y_\epsilon \in \mathbb{R}^d$ such that $v(y_\epsilon) > u(y_\epsilon) + c - \epsilon$. Consequently,

$$\begin{aligned} T_\tau[v](y_\epsilon) - v(y_\epsilon) &< \big(T_\tau[u](y_\epsilon) + c\big) - \big(u(y_\epsilon) + c - \epsilon\big) \\ &= T_\tau[u](y_\epsilon) - u(y_\epsilon) + \epsilon = \bar{E}_\tau + \epsilon. \end{aligned}$$

In particular, $\inf_{z\in\mathbb{R}^d}\{T_\tau[v](z) - v(z)\} \leq \bar{E}_\tau$, and by taking the supremum over $v\in\mathcal{L}^\infty(\mathbb{R}^d)$, we obtain $\lambda \leq \bar{E}_\tau$. The argument we have presented above is similar to the maximum principle for viscosity solutions.

Part 2. Let be

$$\lambda := \lim_{k\to+\infty} \inf_{z_0,\dots,z_k\in\mathbb{R}^d} \frac{1}{k}\sum_{i=0}^{k-1} E_\tau(z_i, z_{i+1}).$$

We choose again a discrete weak KAM solution u of (14). On the one hand, since

$$u(y) - u(x) + \bar{E}_\tau \leq E_\tau(x, y), \qquad \forall\, x, y \in \mathbb{R}^d,$$

and u is bounded, we obtain immediately $\bar{E}_\tau \leq \lambda$. On the other hand, for any $x_0 \in \mathbb{R}^d$, one can find an optimal backward point x_{-1} such that

$$u(x_0) + \bar{E}_\tau = u(x_{-1}) + E_\tau(x_{-1}, x_0).$$

Continuing with this procedure, one can find a one-side backward sequence $\{x_{-k}\}_{k=0}^{+\infty}$ such that

$$u(x_{-k}) + \bar{E}_\tau = u(x_{-(k+1)}) + E_\tau(x_{-(k+1)}, x_{-k}).$$

Therefore,

$$\inf_{z_0,\dots,z_k\ \in\ \mathbb{R}^d} \frac{1}{k} \sum_{i=0}^{k-1} E_\tau(z_i, z_{i+1}) \leq \frac{1}{k} \sum_{i=0}^{k-1} E_\tau(x_{-(i+1)}, x_{-i}) = \frac{u(x_0) - u(x_{-k})}{k} + \bar{E}_\tau,$$

and by letting $k \to +\infty$, we obtain $\lambda \leq \bar{E}_\tau$. □

The previous proof has brought to light three important definitions that we recall here (see [CLT01], [GT11]).

Definition 9. Let $\{E_\tau(x, y)\}_{\tau \in (,1]}$ be a family of short-range actions. Let u be a countinuous periodic function. Then

i. u is called *sub-action* for $E_\tau(x, y)$ if

$$u(y) - u(x) \leq E_\tau(x, y) - \bar{E}_\tau \qquad \forall\, x, y \in \mathbb{R}^d,$$

ii. a configuration $(x_k)_{k\in\mathbb{Z}}$ is said to be *calibrated with respect to a sub-action* u if

$$E_\tau(x_k, x_{k+1}) = u(x_{k+1}) - u(x_k) + \bar{E}_\tau \qquad \forall\, k \in \mathbb{Z},$$

iii. u is a discrete weak KAM solution for $E_\tau(x, y)$ if and only if, u is a sub-action and

$$\forall\, y \in \mathbb{R}^d,\ \exists\, x \in \mathbb{R}^d,\ \text{s.t.} \quad u(y) - u(x) = E_\tau(x, y) - \bar{E}_\tau.$$

We also observe the following a priori bound for the effective action which is immediate from (16). For every $\tau > 0$,

$$\begin{gathered} -\infty < \inf_{x,y\in\mathbb{R}^d} E_\tau(x, y) \leq \bar{E}_\tau \leq \inf_{x\in\mathbb{R}^d} E_\tau(x, x) < +\infty, \\ -\infty < \inf_{\tau\in(0,1]} \frac{1}{\tau}\bar{E}_\tau \quad \text{and} \quad \sup_{\tau\in(0,1]} \frac{1}{\tau}\bar{E} < +\infty. \end{gathered} \tag{17}$$

The bounds are finite thanks to the translational periodicity, coercivity and hypothesis (H4).

We now come to the proof of proposition 4. We show that the Lipschitz norm of discrete weak KAM solutions (14) is uniformly bounded with respect to τ. We also show that calibrated configurations $(x_k)_{k\in\mathbb{Z}}$ have uniformly bounded jumps. Notice that item (iii) of theorem 1 is obtained as a particular case by taking as short-range action $E_\tau = \mathcal{L}_\tau$.

Proof of proposition 4. We begin by fixing the constants C and R: let be

$$\begin{aligned} C_1 &:= 2 \sup_{\tau\in(0,1],\ \|y-x\|\le\tau} \frac{E_\tau(x,y)-\bar{E}_\tau}{\tau}, \\ R &:= \inf\Big\{R>1 : \inf_{\tau\in(0,1],\ \|y-x\|>\tau R} \frac{E_\tau(x,y)-\bar{E}_\tau}{\|y-x\|} > C_1\Big\}, \\ C &:= \max\Big(C_1, \sup_{\|y-x\|,\ \|y-x\|\le\tau(R+1)} \frac{E_\tau(x,y)-E_\tau(x,z)}{\|z-y\|}\Big). \end{aligned} \tag{18}$$

Notice that C_1 is finite thanks to (H4) and equation (17), R is finite thanks to (H5) and C is finite thanks to (H6).

*Step 1.*We show a partial proof of item (ia), namely

$$\|y-x\|>\tau \ \Rightarrow\ u_\tau(y)-u_\tau(x)\le C_1\|y-x\|.$$

Indeed, by choosing $n\ge 2$ such that $(n-1)\tau<\|y-x\|\le n\tau$ and by choosing $x_i = x+\frac{i}{n}(y-x)$, we obtain $n\tau\le 2\|y-x\|$,

$$\begin{gathered} u_\tau(x_{i+1})-u_\tau(x_i)\le E_\tau(x_i,x_{i+1})-\bar{E}_\tau, \quad \text{and} \\ u_\tau(y)-u_\tau(x)\le n\tau \sup_{\|y-x\|\le\tau}\frac{E_\tau(x,y)-\bar{E}_\tau}{\tau}\le C_1\|y-x\|. \end{gathered}$$

Step 2. We prove item (ib). Let be $y\in\mathbb{R}^d$. Let x be a calibrated point for u_τ, that is a point satisfying

$$u_\tau(y)-u_\tau(x)=E_\tau(x,y)-\bar{E}_\tau.$$

Choose some $R>1$ as in (18) and assume by contradiction that $\|y-x\|>\tau R$. Then the first part of the proof may be used and we obtain the absurd inequality

$$C_1\|y-x\|\ge u_\tau(y)-u_\tau(x)>C_1\|y-x\|.$$

Step 3. We end the prove of item (ia). Let be $y,z\in\mathbb{R}^d$, either $\|z-y\|>\tau$ and we are done by the step 1, or $\|z-y\|\le\tau$. Let x be a calibrated point for u_τ. Then $\|y-x\|\le\tau R$, $\|z-x\|\le\tau(R+1)$,

$$\begin{gathered} u_\tau(y)-u_\tau(x)=E_\tau(x,y)-\bar{E}_\tau, \quad u_\tau(z)-u_\tau(x)\le E_\tau(x,z)-\bar{E}_\tau, \\ u_\tau(z)-u_\tau(y)\le E_\tau(x,z)-E_\tau(x,y)\le C\|z-y\|. \end{gathered}$$

By permuting z and y, we just have proved that $\mathrm{Lip}(u_\tau) \leq C$.

Step 4. We prove item (ii). Let be $C' > 0$. We define $R' > 0$ as before

$$R' := \inf\Big\{R' > 1 : \inf_{\tau\in(0,1],\ \|y-x\|>\tau R'} \frac{E_\tau(x,y) - E_\tau(y,y)}{\|y-x\|} > C'\Big\}.$$

Let u be a periodic function satisfying $\mathrm{Lip}(u) \leq C'$ and y any point. Let x be a point realizing the minimum of $\min_x \{u(x) + E_\tau(x,y)\}$. Assume by contradiction that $\|y - x\| > \tau R'$, then on the one hand

$$E_\tau(x,y) - E_\tau(y,y) > C'\|y - x\|,$$

and on the other hand $u(x) + E_\tau(x,y) \leq u(y) + E_\tau(y,y)$ and

$$C'\|y - x\| \geq u(y) - u(x) \geq E_\tau(x,y) - E_\tau(y,y),$$

which is impossible. We then estimate $\| T_\tau[u] - u \|$. On the one hand

$$T_\tau[u](y) - u(y) \leq E_\tau(y,y).$$

On the other hand, if x realizes the minimum of $\min_x[u(x) + E_\tau(x,y)]$

$$\begin{aligned} T_\tau[u](y) - u(y) &= u(x) - u(y) + E_\tau(x,y) \\ &\geq -C'\|y - x\| + \inf_{x,y} E\tau(x,y), \\ \tfrac{1}{\tau}[T_\tau[u](y) - u(y)] &\geq -C'R' + \inf_{\tau\in(0,1]} \inf_{x,y} \tfrac{1}{\tau} E_\tau(x,y). \end{aligned}$$

We conclude by taking

$$C'' := C'R' + \sup_{\tau\in(0,1]} \sup_{y} \tfrac{1}{\tau} E_\tau(y,y) - \inf_{\tau\in(0,1]} \inf_{x,y} \tfrac{1}{\tau} E_\tau(x,y). \qquad \square$$

We end this section by showing a third formula for the effective action by using Mather measures. In order to avoid τ in the definition of the set of constraints, we also give the definition of minimizing transshipments. We first recall what is a transshipment measure:

Definition 10. We say that a probability measure μ defined on the Borel sets of $\mathbb{T}^d \times \mathbb{R}^d$ is τ-holonomic if

$$\iint_{\mathbb{T}^d\times\mathbb{R}^d} \varphi(x)\, \mu(dx,dv) = \iint_{\mathbb{T}^d\times\mathbb{R}^d} \varphi(x + \tau v)\, \mu(dx,dv), \quad \forall\varphi \in C^0(\mathbb{T}^d).$$

We say that a sigma-finite measure π defined on the Borel sets of $\mathbb{R}^d \times \mathbb{R}^d$ is a *transshipment measure* if it verifies the following properties:

i. $k_*\pi = \pi$ for any $k \in \mathbb{Z}^d$ (where $k_*\pi$ denotes the push forward of π by the map $(x, y) \mapsto (x + k, y + k)$ on $\mathbb{R}^d \times \mathbb{R}^d$),

ii. π is a probability on each fundamental domain $F \times \mathbb{R}^d$ or $\mathbb{R}^d \times F$ where $F = [0, 1)^d$;

iii. if $pr^1 : \mathbb{R}^d \times F \to \mathbb{R}^d$ and $pr^2 : F \times \mathbb{R}^d \to \mathbb{R}^d$ denotes the two canonical projections, then $pr^1_*(\pi) = pr^2_*(\pi)$.

We first notice that the two previous definitions, τ-holonomic and transshipment, are similar. More precisely, by identifying $F = [0, 1)^d$ with $\mathbb{T}^d$, we have

Lemma 11. *Let be $\Psi_\tau : F \times \mathbb{R}^d \to F \times \mathbb{R}^d$, $\Psi_\tau(x, y) := (x, \frac{y-x}{\tau})$. If π is a transshipment measure, then $\mu = (\Psi_\tau)_*\pi|_{F\times\mathbb{R}^d}$ is τ-holonomic. Conversely, if μ is τ-holonomic, then $\pi = \sum_{k\in\mathbb{Z}^d} k_*(\Psi_\tau^{-1})_*\mu$ is a transshipment.*

Proof. Let π be a transshipment measure and $\varphi : \mathbb{T}^d \to \mathbb{R}$ be a bounded Borel function, then

$$\begin{aligned}
\iint_{\mathbb{T}^d\times\mathbb{R}^d} \varphi(x)\, \mu(dx, dv) &= \iint_{F\times\mathbb{R}^d} \varphi(x)\, \pi(dx, dy) \\
&= \int_F \varphi(x)\, pr^1_*\pi(dx) = \int_F \varphi(y)\, pr^2_*\pi(dy) \qquad (19)\\
&= \iint_{\mathbb{R}^d\times F} \varphi(y)\, \pi(dx, dy) = \iint_{F\times\mathbb{R}^d} \varphi(y)\, \pi(dx, dy) \qquad (20)\\
&= \iint_{\mathbb{T}^d\times\mathbb{R}^d} \varphi(x + \tau v)\, \mu(dx, dv)
\end{aligned}$$

which proves that μ is holonomic (equations (19) and (20) correspond to items (iii) and (i) of definition 10). The converse is obtained by reversing the order of the previous equalities. □

Lemma 12. *Let $E_\tau(x, y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a family of short-range actions satisfying the hypotheses (H1)–(H4). Then*

$$\begin{aligned}
\bar{E}_\tau &= \inf\Big\{ \iint_{F\times\mathbb{R}^d} E_\tau(x, y)\, \pi(dx, dy) \ :\ \pi \textit{ is a transshipment} \Big\}, \\
&= \inf\Big\{ \iint_{\mathbb{T}^d\times\mathbb{R}^d} E(x, x + \tau v)\, \mu(dx, dv) \ :\ \mu \textit{ is } \tau\textit{-holonomic} \Big\}.
\end{aligned}$$

See theorem 4.3 in [GT11] for a proof.

The infimum in the previous lemma can be achieved by coerciveness of $E_\tau(x, y)$. Such measures are called τ-minimizing.

Definition 13. Let $E_\tau(x,y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a family of short-range actions satisfying the hypotheses (H1)–(H4).

i. A probability measure μ on $\mathbb{T}^d \times \mathbb{R}^d$ is said to be τ-minimizing, if it is τ-holonomic and satisfies

$$\iint_{\mathbb{T}^d\times\mathbb{R}^d} E_\tau(x, x+\tau v)\, d\mu(x,v) = \bar{E}_\tau.$$

Let $\mathcal{M}_\tau$ be the set of τ-minimizing measures.

ii. A projected τ-minimizing is any $pr^{(1)}_*(\mu)$ where $pr^{(1)} : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d$ is the first projection and μ is any τ-minimizing measure.

iii. We call Mather set, the compact set

$$\mathrm{Mather}(E_\tau) := \cup\{\mathrm{supp}(\mu) \,:\, \mu \text{ is } \tau\text{-minimizing }\}.$$

(There is no need to take the closure since the Mather set is already equal to the support of some measure). The projected Mather set is $pr^{(1)}(\mathrm{Mather}(E_\tau))$.

4 Minimal action

The main purpose of this section is to show that the minimal action satisfies the properties (H1)–(H6) and that the comparison estimate of proposition 5 holds. We first notice that (H1)–(H3) are obvious, and that (H5) is a consequence of the superlinearity of L. Only properties (H4) and (H6) deserve a proof. We will also show that the associated Lax-Oleinik operator satisfies a semi-group property $T^{t+s} = T^t \circ T^s$ and that $\bar{\mathcal{E}}_t = -t\bar{H}(0)$ which is part of the proof of item (ii) of theorem 1. The main novelty of this section is theorem 17 which is valid for every short-range actions satisfying the min-plus convolution property.

Proof of property (H4). Let be $\tau > 0$, $x, y \in \mathbb{R}^d$, $\|y-x\| \leq \tau R$. Since $\gamma(s) := x + s\frac{y-x}{\tau}$ is a particular path joining x to y, we obtain

$$\sup_{\tau>0,\ \|y-x\|\leq\tau R} \tfrac{1}{\tau}\mathcal{E}_\tau(x,y) \leq \sup_{x\in\mathbb{R}^d,\ \|v\|\leq R} L(x,v).$$

Let be $\tau > 0$ and $x, y \in \mathbb{R}^d$. By superlinearity, $L(x,v) \geq \|v\| - C$ for some constant $C > 0$. Then $\int_0^\tau L(\gamma,\dot\gamma)\,ds \geq \|y-x\| - \tau C$ for every absolutely continuous path $\gamma : [0,\tau] \to \mathbb{R}^d$ satisfying $\gamma(0) = x$ and $\gamma(\tau) = y$. One obtains

$$\inf_{\tau>0,\ x,y,\in\mathbb{R}^d} \tfrac{1}{\tau}\mathcal{E}_\tau(x,y) \geq -C. \qquad \square$$

The proof of (H6) (and item (i) of proposition 5) require an *a priori* bound from above of the velocity of a minimizer. We will recall the main arguments of the proof, see [Fat08, Mat91] in the autonomous case, and [BFZ12] in the non autonomous case for more details.

Lemma 14 (A priori compactness for Tonelli minimizers)**.** *We consider a C^2 Tonelli Hamiltonian $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$. For every $R > 0$, there exists a constant $C(R) > 0$ such that, for every $\tau > 0$, $x, y \in \mathbb{R}^d$ satisfying $\|y - x\| \leq \tau R$, and for every minimizer $\gamma : [0, \tau] \to \mathbb{R}^d$ satisfying*

$$\gamma(0) = x,\ \gamma(\tau) = y,\ \int_0^\tau L(\gamma, \dot\gamma)\, ds = \mathcal{E}_\tau(x, y),$$

we have $\|\dot\gamma\| \leq C(R)$ and $\|\ddot\gamma\| \leq C(R)$.

Proof. By Tonelli-Weierstrass theorem, γ is a C^2characteristic. By choosing $\bar\gamma(s) = x + s\frac{y-x}{\tau}$ we obtain the upper bound

$$\mathcal{E}_\tau(x, y) \leq \tau \sup_{x \in \mathbb{T}^d,\ \|v\| \leq R} |L(x, v)| := \tau C_1(R).$$

By superlinearity we obtain a lower bound of L, $L(x, v) \geq \|v\| - C_2$, and

$$\frac{1}{\tau}\int_0^\tau \|\dot\gamma\|\, ds \leq \frac{1}{\tau}\int_0^\tau L(\gamma, \dot\gamma)\, ds + C_2 \leq C_1(R) + C_2.$$

In particular, there exists $s_0 \in [0, \tau]$ such that $\|\dot\gamma(s_0)\| \leq C_1(R) + C_2$. Let be $p = \frac{\partial L}{\partial v}(\gamma, \dot\gamma)$ and

$$C_3(R) := \sup\Big\{H\Big(x, \frac{\partial L}{\partial v}(x, v)\Big) \ :\ x \in \mathbb{T}^d,\ \|v\| \leq C_1(R) + C_2\Big\}.$$

As H is autonomous, $H(\gamma(s), p(s))$ is independent of s and therefor bounded from above by $C_3(R)$. By superlinearity of H, $H(x, p) \geq \|p\| - C_4$, and we obtain $\|p(s)\| \leq C_3(R) + C_4$. Since $\dot\gamma = \frac{\partial H}{\partial p}(\gamma, p)$, we finally obtain

$$\|\dot\gamma\| \leq \sup\Big\{\frac{\partial H}{\partial p}\Big(x, p\Big) \ :\ x \in \mathbb{T}^d,\ \|p\| \leq C_3(R) + C_4\Big\} := C(R).$$

The bound on the second derivative is done similarly using the formula

$$\ddot\gamma = \frac{\partial^2 H}{\partial p \partial x}(\gamma, p)\frac{\partial H}{\partial p}(\gamma, p) - \frac{\partial^2 H}{\partial p \partial p}(\gamma, p)\frac{\partial H}{\partial x}(\gamma, p).$$

∎

Proof of property (H6). Let be $\tau \in (0,1]$, $x, y, z \in \mathbb{R}^d$ such that $\|y - x\| \leq \tau R$ and $\|z - x\| \leq \tau R$. By Tonelli-Weierstrass, there exists a C^2 minimizer $\gamma : [0, \tau] \to \mathbb{R}^d$ starting at x, ending at y, and satisfying $\int_0^\tau L(\gamma, \dot\gamma)\, ds = \mathcal{E}_\tau(x, y)$. Define the path $\xi : [0, \tau] \to \mathbb{R}^d$ by $\xi(s) = \gamma(s) + s\frac{z-y}{\tau}$. By lemma 14, there exists a constant $C(R) > 0$ such that $\|\dot\gamma\| \leq C(R)$. Then

$$\mathcal{E}_\tau(x, z) - \mathcal{E}_\tau(x, y) \leq \int_0^\tau [L(\xi, \dot\xi) - L(\gamma, \dot\gamma)]\, ds \leq \tilde{C}(R)\|z - y\|,$$

where $\tilde{C}(R) = \sup_{x \in \mathbb{R}^d,\ \|v\| \leq C(R)+R} \|DL(x, v)\|$. □

The minimal action actually satisfies a stronger property.

Lemma 15 (Uniformly semi-concave)**.** *Let $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a C^2, time independent, periodic, Tonellli Hamiltonian. Then for every $R > 0$, there exists a constant $C(R) > 0$ such that for every $\tau \in (0, 1]$, $x, y, h \in \mathbb{R}^d$, $\|y - x\|| \leq \tau R$, $\|h\| \leq \tau R$,*

$$\begin{aligned} \mathcal{E}_\tau(x, y + h) + \mathcal{E}_\tau(x, y - h) - 2\mathcal{E}_\tau(x, y) &\leq \frac{C(R)}{\tau} \|h\|^2, \\ \mathcal{E}_\tau(x + h, y) + \mathcal{E}_\tau(x - h, y) - 2\mathcal{E}_\tau(x, y) &\leq \frac{C(R)}{\tau} \|h\|^2. \end{aligned} \tag{21}$$

Proof. For every $R > 0$, $x, y \in \mathbb{R}^d$ with $\|x - y\| \leq \tau R$, by Tonelli-Weierstrass theorem, there exists a C^2 curve $\gamma : [0, \tau] \to \mathbb{R}^d$ such that $\gamma(0) = x$, $\gamma(\tau) = y$, and

$$\mathcal{E}_\tau(x, y) = \int_0^\tau L(\gamma(s), \dot\gamma(s))\, ds.$$

Let be $h \in \mathbb{R}^d$ satisfying $\|h\| \leq \tau R$, and $\xi_+, \xi_- : [0, \tau] \to \mathbb{R}^d$ defined by

$$\xi_+(s) = \gamma(s) + s\frac{h}{\tau} \quad \text{and} \quad \xi_-(s) = \gamma(s) - s\frac{h}{\tau}.$$

In particular, $\dot\xi_+(s) = \dot\gamma(s) + \frac{h}{\tau}$ and $\dot\xi_-(s) = \dot\gamma(s) - \frac{h}{\tau}$. By Lemma 14, there exists a constant $C(R) > 0$ such that $\|\dot\gamma(s)\| \leq C(R)$ for every $s \in [0, \tau]$. Therefore, we have

$$\begin{aligned} &\mathcal{E}_\tau(x, y + h) + \mathcal{E}_\tau(x, y - h) - 2\mathcal{E}_\tau(x, y) \\ &\leq \int_0^\tau [L(\xi_+, \dot\xi_+) + L(\xi_-, \dot\xi_-) - 2L(\gamma, \dot\gamma)]\, ds \\ &= \int_0^\tau \Big[L\Big(\gamma(s) + s\frac{h}{\tau}, \dot\gamma(s) + \frac{h}{\tau}\Big) + L\Big(\gamma(s) - s\frac{h}{\tau}, \dot\gamma(s) - \frac{h}{\tau}\Big) - 2L(\gamma(s), \dot\gamma(s))\Big] ds \\ &\leq 2\tau \sup_{x \in \mathbb{R}^d,\ \|v\| \leq C(R)+R} \|D^2 L(x, v)\| \frac{\|h\|^2}{\tau^2} \leq \frac{\tilde{C}(R)}{\tau}\|h\|^2, \end{aligned}$$

where $\tilde{C}(R) = 2\sup_{x \in \mathbb{R}^d,\ \|v\| \leq C(R)+R} \|D^2 L(x, v)\|$. □

Proof of item (ii) *of proposition 5.* Let $R > 0$ and $C(R)$ be the constants given by lemma 14. Let be $\tau \in (0, 1]$ and $\|y - x\| \leq \tau R$. We know that $\mathcal{E}_\tau(x, y)$ admits a C^2 minimizer $\gamma : [0, \tau] \to \mathbb{R}^d$ satisfying $\gamma(0) = x$, $\gamma(\tau) = y$, $\mathcal{E}_\tau(x, y) = \int_0^\tau L(\gamma, \dot\gamma)\, ds$, $\|\dot\gamma\| \leq C(R)$ and $\|\ddot\gamma\| \leq C(R)$. Let be $V_0 = \dot\gamma(0))$. Then

$$\|\gamma(s) - x\| = \|\gamma(s) - \gamma(0)\| \leq sC(R) \leq \tau C(R),$$
$$\|\dot\gamma(s) - V_0\| \leq sC(R), \quad \left\|\frac{y - x}{\tau} - V_0\right\| \leq \tau C(R) \text{ and } \left\|\dot\gamma(s) - \frac{y - x}{\tau}\right\| \leq 2\tau C(R).$$

We are now in a position to compare the two actions

$$|\mathcal{E}_\tau(x, y) - \mathcal{L}_\tau(x, y)| \leq \int_0^\tau \Big| L(\gamma(s), \dot\gamma(s)) - L\Big(x, \frac{y - x}{\tau}\Big)\Big|\, ds \leq \tau^2 \tilde C(R),$$

with $\tilde C(R) := 2 \sup_{x \in \mathbb{R}^d,\ \|v\| \leq R + C(R)} \|DL\|\, C(R)$. □

We conclude this section by improving theorem 7 for short-range actions satisfying the min-plus convolution property. Let us first recall the notion of min-plus convolution of two actions $\mathcal{E}_1$ and $\mathcal{E}_2$, denoted by $\mathcal{E}_1 \otimes \mathcal{E}_2$, as follows:

$$\mathcal{E}_1 \otimes \mathcal{E}_2(x, y) := \inf_{z \in \mathbb{R}^d} [\mathcal{E}_1(x, z) + \mathcal{E}_2(z, y)]. \tag{22}$$

The infimum is attained since $\mathcal{E}_1$ and $\mathcal{E}_2$ are coercive. We will show in particular that the effective action $\bar{\mathcal{E}}_\tau$ is linear with respect to τ.

Notice that the discrete Lax-Oleinik operator T_τ defined in (13) for the minimal action $\mathcal{E}_\tau$ corresponds to the Lax-Oleinik semi-group T^t defined in (3) by Fathi,

$$T^t[u](y) = \min_{x \in \mathbb{R}^d} \{u(x) + \mathcal{E}_t(x, y)\}, \quad \forall y \in \mathbb{R},\ \forall t > 0. \tag{23}$$

Lemma 16 (Min-plus convolution property)**.** *Let $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a C^2, autonomous and Tonelli Hamiltonian. Then the minimal action satisfies*

$$\mathcal{E}_{\tau+\sigma} = \mathcal{E}_\tau \otimes \mathcal{E}_\sigma, \quad \forall \tau, \sigma > 0. \tag{24}$$

In particular $T^{\tau+\sigma} = T^\tau \circ T^\sigma$, $\forall \tau, \sigma > 0$.

Proof. On the one hand, for any $x, y, z \in \mathbb{R}^d$, by Tonelli Theorem, one can find absolutely continuous curves $\gamma_1 : [0, \tau] \to \mathbb{R}^d$ with $\gamma_1(0) = x, \gamma_1(\tau) = z$ and $\gamma_2 : [\tau, \tau{+}\sigma] \to \mathbb{R}^d$ with $\gamma_2(\tau) = z, \gamma_2(\tau{+}\sigma) = y$ such that $\mathcal{E}_\tau(x, z) = \int_0^\tau L(\gamma_1(t), \dot\gamma_1(t))\, dt$ and $\mathcal{E}_\sigma(z, y) = \int_\tau^{\tau+\sigma} L(\gamma_2, \dot\gamma_2)\, dt$. Let $\gamma = \gamma_1 \star \gamma_2$ be the concatenation of γ_1 and γ_2. Then

$$\mathcal{E}_\tau(x, z) + \mathcal{E}_\sigma(z, y) = \int_0^{\tau+\sigma} L(\gamma(t), \dot\gamma(t))\, dt \geq \mathcal{E}_{\tau+\sigma}(x, y).$$

Taking infimum with respect to z, we obtain

$$\mathcal{E}_\tau \otimes \mathcal{E}_\sigma(x,y) \geq \mathcal{E}_{\tau+\sigma}(x,y), \quad \forall\, x,y \in \mathbb{R}^d.$$

On the other hand, for any $x,y \in \mathbb{R}^d$, by Tonelli Theorem, there exists an absolutely continuous curve $\gamma : [0,\tau+\sigma] \to \mathbb{R}^d$ with $\gamma(0) = x, \gamma(\tau+\sigma) = y$ and $\mathcal{E}_{\tau+\sigma}(x,y) = \int_0^{\tau+\sigma} L(\gamma,\dot\gamma)\,dt$. Take $z = \gamma(\tau)$, and see $\gamma = \gamma_1 \star \gamma_2$ as the concatenation of two paths as before. Then

$$\mathcal{E}_{\tau+\sigma}(x,y) \geq \mathcal{E}_\tau(x,z) + \mathcal{E}_\sigma(z,y) \geq \mathcal{E}_\tau \otimes \mathcal{E}_\sigma(x,y).$$

The semi-group property $T^{\tau+\sigma} = T^\tau \circ T^\sigma$ follows from (24). □

The following theorem gives a different proof of the existence of a (continuous time) weak KAM solution when the action $E_\tau(x,y)$ is given by the minimal action $\mathcal{E}_\tau(x,y)$. What is new in this theorem is that we prove the existence of a common additive eigenfunction of the Lax-Oleinik equation (14) using a few hypotheses: the main hypotheses (H1) – (H6) and the min-plus convolution property (24). We assume temporarily that E_τ is defined for every $\tau > 0$.

Theorem 17. *Let $\{E_\tau(x,y)\}_{\tau>0}$ be a family of short-range actions satisfying the hypotheses (H1)–(H6). Assume E_τ satisfies the min-plus convolution property, $E_{\tau+\sigma} = E_\tau \otimes E_\sigma$, for every $\tau,\sigma > 0$. Then the following statements hold.*

i. *For every $\tau > 0$, $\bar{E}_\tau = \tau \bar{E}_1$.*

ii. *There exist a subsequence $\tau_i \to 0$ and discrete weak KAM solutions u_{τ_i} for the action $E_{\tau_i}(x,y)$ such that $u_{\tau_i} \to u$ uniformly. Moreover any such limits u satisfy*

$$T_t[u] = u + t\bar{E}_1, \quad \forall t > 0.$$

iii. $\lim_{t\to+\infty} \frac{1}{t} \min_{x,y} E_t(x,y) = \bar{E}_1$.

Proof. Step 1. We prove property (i) for $\tau \in \mathbb{Q}$. We will use the notation

$$\bar{E}_\tau(M) := \min\Big\{ \sum_{j=1}^{M} E_\tau(x_{j-1},x_j) \ :\ x_j \in \mathbb{R}^d \Big\}.$$

It is enough to prove $\bar{E}_{N\tau} = N\bar{E}_\tau$ for every integer N and $\tau > 0$ not necessarily rational. We choose $M > 0$,

$$(z_0,\dots,z_M) \in \arg\min\Big\{ \sum_{i=1}^{M} E_{N\tau}(z_{i-1},z_i) \ :\ z_i \in \mathbb{R}^d \Big\},$$

and by min-plus convolution of $E_{N\tau}$, we choose $(x_{i,0},\dots,x_{i,N})$ so that

$$E_{N\tau}(z_{i-1},z_i)=\sum_{j=1}^N E_\tau(x_{i,j-1},x_{i,j}),\quad x_{i,0}=z_{i-1}\ \text{ and }\ x_{i,N}=z_i.$$

Then $\bar E_{N\tau}(M)=\sum_{i=1}^M\sum_{j=1}^N E_\tau(x_{i,j-1},x_{i,j})\ge \bar E_\tau(MN)$. By dividing by MN and by taking $M\to+\infty$, one obtains $\bar E_{N\tau}\ge N\bar E_\tau$. Conversely, we choose

$$(x_0,\dots,x_{M-1})\in\arg\min\Big\{\sum_{i=1}^{M-1}E_\tau(x_{i-1},x_i)\ :\ x_i\in\mathbb{R}^d\Big\},$$

and N integer translates $k_j\in\mathbb{Z}^d$, $j=1\dots N$, such that $k_0=0$ and

$$\|(x_0+k_j)-(x_{M-1}+k_{j-1})\|\le 1.$$

We define a new chain $(z_0,\dots,z_{MN})$ by concatenating the previous translates

$$z_{i-1+(j-1)M}:=x_{i-1}+k_{j-1}M,\quad i=1,\dots,M,\ \ j=1,\dots,N.$$

Then, using the fact $\|z_{jM}-z_{M-1+(j-1)M}\|\le 1$

$$\begin{aligned}N\bar E_\tau(M-1)&=\sum_{j=1}^N\sum_{i=1}^{M-1}E_\tau(z_{i-1+(j-1)M},z_{i+(j-1)M})\\&\ge\sum_{j=1}^N\sum_{i=1}^{M}E_\tau(z_{i-1+(j-1)M},z_{i+(j-1)M})-N\sup_{\|y-x\|\le1}|E_\tau(x,y)|,\end{aligned}$$

$$\begin{aligned}\sum_{j=1}^N\sum_{i=1}^{M}E_\tau(z_{i-1+(j-1)M},z_{i+(j-1)M})&=\sum_{i=1}^M\sum_{j=1}^{N}E_\tau(z_{j-1+(i-1)N},z_{j+(i-1)N})\\&\ge\sum_{i=1}^M E_{N\tau}(z_{i-1},z_i)\ge\bar E_{N\tau}(M).\end{aligned}$$

By dividing by M and by taking $M\to+\infty$, one obtains $N\bar E_\tau\ge\bar E_{N\tau}$.

Step 2. We prove an intermediate estimate, namely

$$\sup_{\tau>0}\|T_\tau[0]-\bar E_\tau\|\le C,$$

where C is the constant given by the item (ia) of proposition 4. Let $\tau>0$ and N be an integer such that $\tau/N\le1$. Let $u_{\tau/N}$ be a weak KAM solution of $T_{\tau/N}$ that we normalize by $\min u_{\tau/N}=0$. Then

$$\begin{gathered}T_{\tau/N}[u_{\tau/N}]=u_{\tau/N}+\bar E_{\tau/N},\\ T_\tau[u_{\tau/N}]=(T_{\tau/N})^N[u_{\tau/N}]=u_{\tau/N}+N\bar E_{\tau/N}=u_{\tau/N}+\bar E_\tau.\end{gathered}$$

Since $\|u_{\tau/N}\| \leq C$, we obtain

$$T_\tau[0] \leq T_\tau[u_{\tau/N}] \leq C + \bar{E}\tau,$$
$$T_\tau[0] \geq T_\tau[u_{\tau/N} - C] = u_{\tau/N} - C + \bar{E}_\tau \geq -C + \bar{E}_\tau,$$

and finally $\|T_\tau[0] - \bar{E}_\tau\| \leq C$, for every $\tau > 0$.

Step 3. We resume the proof of property (i) for $\tau \notin \mathbb{Q}$. We choose $p_i, q_i \in \mathbb{N}$, $q_i \to +\infty$, such that $p_i < q_i\tau < p_i + 1$. Denote by $\sigma_i = p_i + 1 - q_i\tau$. Then $T_{p_i+1} = T_{\sigma_i} \circ T_{q_i\tau}$. Since $\|T_{q_i\tau}[0] - q_i\bar{E}_\tau\| \leq C$, by applying T_{σ_i}, one obtain on the one hand

$$\|T_{p_i+1}[0] - q_i\bar{E}_\tau\| \leq C + \| T_{\sigma_i}[0] \|.$$

On the other hand $\|T_{p_i+1}[0] - (p_i+1)\bar{E}_1\| \leq C$, which implies

$$\|(p_i+1)\bar{E}_1 - q_i\bar{E}_\tau\| \leq 2C + \sup_{\sigma\in(0,1]} \| T_\sigma[0] \|.$$

Notice that item (ii) of proposition 4 implies that $\| T_\sigma[0] \|$ is uniformly bounded for $\sigma \in (0,1]$. We conclude by dividing by q_i and letting q_i go to infinity.

Step 4. We prove item (ii). From the apriori compactness property of proposition 4, one can find a constant $C > 0$ such that every discrete weak KAM solutions u_τ satisfies $\mathrm{Lip}(u_\tau) \leq C$. Since u_τ is defined up to a constant, we may assume that $\min(u_\tau) = 0$. By choosing a subsequence $\tau_i \to 0$, we may assume that $u_{\tau_i} \to u$ uniformly. Moreover the second part of this proposition implies that $\| T_\sigma[v] - v \| \leq \sigma C$, for every $\sigma \in (0,1]$ and Lipshitz function satisfying $\mathrm{Lip}(v) \leq C$. Let be $t > 0$. There exists integers N_i such that $N_i\tau_i \leq t < (N_i+1)\tau_i$. Let be $\sigma_i = t - N_i\tau_i$. Then

$$T_{\tau_i}[u_{\tau_i}] = u_{\tau_i} + \tau_i\bar{E}_1, \quad T_{N_i\tau_i}[u_{\tau_i}] = u_{\tau_i} + N_i\tau_i\bar{E}_1,$$
$$T_t[u_{\tau_i}] = T_{t-N_i\tau_i}[u_{\tau_i}] + N_i\tau_i\bar{E}_1,$$
$$\| T_t[u_{\tau_i}] - u_{\tau_i} - t\bar{E}_1 \| \leq \| T_{\sigma_i}[u_{\tau_i}] - u_{\tau_i} \| + \sigma_i|\bar{E}_1|.$$

As $\sigma_i \to 0$, $u_{\tau_i} \to u$, $T_{\sigma_i}[u] \to u$, and $\|T_{\sigma_i}[u_{\tau_i}] - T_{\sigma_i}[u]\| \leq \|u_{\tau_i} - u\|$, we obtain $T_t[u] = u + t\bar{E}_1$.

Step 5. We prove item (iii). We first notice

$$\min_{x,y\in\mathbb{R}^d} E_t(x,y) = \min_{y\in\mathbb{R}^d} T_t[0](y).$$

On the one hand,

$$T_t[0] \leq T_t[u - \min(u)] = u + t\bar{E}_1 - \min(u) \leq \max(u) - \min(u) + t\bar{E}_1.$$

On the other hand,

$$T_t[0] \geq T_t[u - \max(u)] = u + t\bar{E}_1 - \max(u) \geq \min(u) - \max(u) + t\bar{E}_1.$$

In particular $\|T_t[0] - t\bar{E}_1\| \leq \mathrm{osc}(u)$ and $\lim_{t\to+\infty} \min_{x,y} \frac{1}{t}E_t(x,y) = \bar{E}_1$. □

5 Proof of theorem 1

Proof of item (i) of theorem 1. The discrete action $\mathcal{L}_\tau(x,y)$ is a particular case of short-range actions. Item (i) is simply theorem 7. $\square$

Proof of item (ii) of theorem 1. The minimal action $\mathcal{E}_\tau(x,y)$ is a particular case of short-range actions satisfying the min-plus convolution property (22). As noticed in (23) the discrete Lax-Oleinik operator T_τ defined in (13) corresponds to the continuous Lax-Oleinik semi-group T^τ defiend in (3). On the one hand theorem 17 implies

$$-\bar{H}(0) = \lim_{t\to+\infty} \frac{1}{t} \min_{x,y\in\mathbb{R}^d} \mathcal{E}_t(x,y) = \bar{\mathcal{E}}_1 = \frac{1}{\tau}\bar{\mathcal{E}}_\tau, \quad \forall\tau > 0.$$

On the other hand, the a priori compactness property in proposition 4 for any short-range actions, and the comparison estimate of proposition 5 implies the existence of constants $R > 0$ and $C > 0$ such that for every $\tau \in (0,1]$,

– any discrete weak KAM solution u_τ of either $\mathcal{E}_\tau$ or $\mathcal{L}_\tau$, that satisfies $\min(u_\tau) = 0$, is uniformly bounded by C (actually uniformly Lipschitz bounded),

– any calibrated configuration $(x_{-k})_{k\geq 0}$ (see definition 9) for any discrete weak KAM solution satisfies $\|x_{-k} - x_{-k-1}\| \leq \tau R$, $\forall k \geq 0$,

– the comparison estimate holds: $|\mathcal{E}_\tau(x,y) - \mathcal{L}_\tau(x,y)| \leq \tau^2 C$, for every x,y satisfying $\|y-x\| \leq \tau R$.

Let us show that the three previous estimates implies

$$|\bar{\mathcal{E}}_\tau - \bar{\mathcal{L}}_\tau| \leq \tau^2 C, \quad \forall\tau \in (0,1].$$

Indeed, we choose a discrete weak KAM solution u_τ of $\mathcal{E}_\tau(x,y)$ and a calibrated configuration $(x_{-k})_{k\geq 0}$ for u_τ. Then

$$\begin{gathered}
\mathcal{E}_\tau(x_{-k-1}, x_{-k}) = u_\tau(x_{-k}) - u_\tau(x_{-k-1}) + \bar{\mathcal{E}}_\tau, \\
\mathcal{L}_\tau(x_{-k-1}, x_{-k}) \leq \mathcal{E}_\tau(x_{-k-1}, x_{-k}) + \tau^2 C, \\
\frac{1}{n}\sum_{k=0}^{n-1} \mathcal{L}_\tau(x_{-k-1}, x_{-k}) \leq \bar{\mathcal{E}}_\tau + \tau^2 C(R) + \frac{2}{n}\|u_\tau\|_\infty.
\end{gathered}$$

By taking the limit $n \to +\infty$, and by using the mean action per site formula (16), we obtain $\bar{\mathcal{L}}_\tau \leq \bar{\mathcal{E}}_\tau + \tau^2 C$. By permuting the roles of $\mathcal{E}_\tau$ and $\mathcal{L}_\tau$ we conclude the proof of item (ii). $\square$

Proof of item (iii) of theorem 1. The proof of this item follow directly from the a priori compactness property of proposition 4. $\square$

Proof of item (iv) of theorem 1. We will use two Lax-Oleinik operators: T_τ, the discrete Lax-Oleinik operator associated to $\mathcal{L}_\tau$, and T^τ, the Lax-Oleinik semi-group associated to $\mathcal{E}_\tau$. We claim there exists a constant $C > 0$ such that, for every small $\tau > 0$, for every discrete weak KAM solution u for $\mathcal{L}_\tau$,

$$\| T^\tau[u] - T_\tau[u] \|_\infty \leq \tau^2 C.$$

Indeed, we know from propositions 4 and 5, there exist positive constants R and C such that, for every $\tau \in (0, 1]$, for every discrete weak KAM solution u for $\mathcal{L}_\tau$,

– $\mathrm{Lip}(u) \leq C$, $\|u\|_\infty \leq C$,
– $\forall y \in \mathbb{R}^d, \quad x \in \arg\min_{x\in\mathbb{R}^d} \{u(x) + \mathcal{L}_\tau(x, y)\} \Rightarrow \|y - x\| \leq \tau R$,
– $\forall y \in \mathbb{R}^d, \quad x \in \arg\min_{x\in\mathbb{R}^d} \{u(x) + \mathcal{E}_\tau(x, y)\} \Rightarrow \|y - x\| \leq \tau R$,
– $\| T^\tau[u] - u \|_\infty \leq \tau C$,
– for every $x, y, \quad \|y - x\| \leq \tau R \;\Rightarrow |\mathcal{E}_\tau(x, y) - \mathcal{L}_\tau(x, y)| \leq \tau^2 C$.

On the one hand, for every y and $x \in \arg\min_{x\in\mathbb{R}^d} \{u(x) + \mathcal{L}_\tau(x, y)\}$,

$$T^\tau[u](y) \leq u(x) + \mathcal{E}_\tau(x, y) \leq u(x) + \mathcal{L}_\tau(x, y) + \tau^2 C,$$
$$T^\tau[u](y) \leq T_\tau[u](y) + \tau^2 C.$$

On the other hand, if $x \in \arg\min_{x\in\mathbb{R}^d} [u(x) + \mathcal{E}_\tau(x, y)]$,

$$T^\tau[u](y) = u(x) + \mathcal{E}_\tau(x, y) \geq u(x) + \mathcal{L}_\tau(x, y) - \tau^2 C,$$
$$T^\tau[u](y) \geq T_\tau[u](y) - \tau^2 C.$$

The claim is proved. Since $\mathrm{Lip}(u_i)$ is uniformly bounded independently of τ, we may choose a sequence of times $\tau_i \to 0$ and discrete weak KAM solutions u_i for $\mathcal{L}_{\tau_i}$ such that $u_i \to u$ uniformly for some periodic Lipschitz function u. Let $t > 0$ be fixed, and N_i integers such that $N_i\tau_i \leq t < (N_i + 1)\tau$. The non-expansiveness property of the Lax-Oleinik operator implies

$$\| T^t[u] - T^{N_i\tau_i}[u_i] \|_\infty \leq \| T^{t-N_i\tau_i}[u] - u \|_\infty + \|u - u_i\|_\infty \to 0.$$

The previous claim $\| T^{\tau_i}[u_i] - T_{\tau_i}[u_i] \|_\infty \leq \tau_i^2 C$ and the estimate proved in item (ii) of theorem 1 $|\bar{\mathcal{E}}_{\tau_i} - \bar{\mathcal{L}}_{\tau_i}| \leq \tau_i^2 C$, imply

$$\| T^{\tau_i}[u_i] - u_i - \tau_i \bar{\mathcal{E}}_1 \|_\infty \leq \tau_i^2 2C.$$

By iterating this inequality, one obtain

$$\| T^{N_i\tau_i}[u_i] - u_i - N_i\tau_i \bar{\mathcal{E}}_1 \|_\infty \leq N_i \tau_i^2 2C \leq t\tau_i 2C.$$

Since $u_i + N_i\tau_i\bar{\mathcal{E}}_1 \to u + t\bar{\mathcal{E}}_1$, one get

$$T^t[u] = u + t\bar{\mathcal{E}}_1, \quad \forall t > 0.$$

The fact that being a solution of the ergodic cell equation (2) is equivalent to being a solution of the PDE cell equation (1) is proved for instance in Fathi [Fat08, theorem 7.6.2]. □

6 Discounted discrete Lax-Oleinik operator

This section is devoted to the proof of theorem 2. Our approach follows the article [DFIZ14] to identify the selected discrete weak KAM solution. We have nevertheless chosen to work with general short-range actions E_τ.

Proof of theorem 2. Items (i), (ii) and (iii), respectively, are consequences of propositions 19, 21 and 26, respectively, applied to the action $E_\tau = \mathcal{L}_\tau$. □

Definition 18. Let $\{E_\tau(x,y)\}_{\tau>0}$ be a family of short-range actions satisfying the hypotheses (H1)–(H6). We call *discounted discrete Lax-Oleinik operator*, the non-linear operator

$$T_{\tau,\delta}[u](y) := \inf_{x\in\mathbb{R}^d} \{(1-\tau\delta)u(x) + E_\tau(x,y)\},$$

defined for every continuous periodic function u, and for every $\tau \in (0,1]$ and $\delta \in (0,1]$. By coerciveness the infimum is actually attained.

We first improve the a priori estimates in time of proposition 4 to obtain an a priori bound of the jumps uniformly with respect to the discounted factor δ.

Proposition 19 (A priori compactness in the discounted case)**.** *Let $\{E_\tau(x,y)\}_{\tau\in(0,1]}$ be a family of short-range actions satisfying the hypotheses (H1)–(H6). Then the following holds.*

i. The operator $T_{\tau,\delta}$ admits a unique fixed point $u_{\tau,\delta}$ in the space of continuous periodic functions

$$u_{\tau,\delta}(x) := \inf_{\{x_{-k}\}_{k=0}^{+\infty}\in(\mathbb{R}^d)^{\mathbb{N}},\, x_0=x} \sum_{k=0}^{\infty} (1-\tau\delta)^k E_\tau(x_{-(k+1)}, x_{-k}).$$

$u_{\tau,\delta}$ is called the discounted discrete weak KAM solution of E_τ.

ii. There exist constants $R>1$ and $C>0$ such that, for every $\tau,\delta\in(0,1]$,

(a) $\inf_{x,y\in\mathbb{R}^d} \dfrac{E_\tau(x,y)}{\tau} \le \delta u_{\tau,\delta} \le \sup_{x\in\mathbb{R}^d} \dfrac{E_\tau(x,x)}{\tau}$,

(b) $u_{\tau,\delta}$ is Lipschitz and $\mathrm{Lip}(u_{\tau,\delta}) \le C$,

(c) $\forall y\in\mathbb{R}^d,\quad [x\in\arg\min_{x\in\mathbb{R}^d}\{(1-\tau\delta)u_{\tau,\delta}(x)+E_\tau(x,y)\}\,] \Rightarrow \|y-x\| \le \tau R$.

Proof. Step 1. The operator $T_{\tau,\delta}$ is contracting in C^0 norm, i.e.

$$\| T_{\tau,\delta}[u] - T_{\tau,\delta}[v] \|_\infty \le (1-\tau\delta)\| u - v \|_\infty, \quad \forall\, u,v\in C^0(\mathbb{T}^d).$$

Moreover, $T_{\tau,\delta}$ preserves the ball $\|u\|_\infty \leq \frac{C_0}{\delta}$ where

$$C_0 := \sup_{\tau\in(0,1]} \Big(\sup_{x\in\mathbb{R}^d} \frac{E_\tau(x,x)}{\tau}, - \inf_{x,y\in\mathbb{R}^d} \frac{E_\tau(x,y)}{\tau} \Big).$$

Indeed, we have

$$\begin{aligned} T_{\tau,\delta}[u](y) &\leq (1-\tau\delta)\max(u) + \max_{x\in\mathbb{R}^d} E_\tau(x,x), \\ T_{\tau,\delta}[u](y) &\geq (1-\tau\delta)\min(u) + \min_{x,y\in\mathbb{R}^d} E_\tau(x,y), \\ \|u\|_\infty \leq \frac{C_0}{\delta} &\Rightarrow \| T_{\tau,\delta}[u] \|_\infty \leq (1-\tau\delta)\|u\|_\infty + \tau C_0 \leq \frac{C_0}{\delta}. \end{aligned}$$

In particular $T_{\tau,\delta}$ admits a unique fixed point $u_{\tau,\delta}$ which is inside $B(0, \frac{C_0}{\delta})$. We have proved item (i). The fixed point satisfies

$$u_{\tau,\delta}(y) = \min_{x\in\mathbb{R}^d} \{(1-\tau\delta)u_{\tau,\delta}(x) + E_\tau(x,y)\}, \quad \forall y \in \mathbb{R}^d.$$

By iterating backward, one obtains the explicit formula for $u_{\tau,\delta}$. Moreover it satisfies the a priori estimate (iia)

$$\inf_{\tau\in(0,1]} \inf_{x,y\in\mathbb{R}^d} \frac{E_\tau(x,y)}{\tau} \leq \delta u_{\tau,\delta} \leq \sup_{\tau\in(0,1]} \sup_{x\in\mathbb{R}^d} \frac{E_\tau(x,x)}{\tau}$$

which are finite thanks to property (H4). Notice that $u_{\tau,\delta}$ explodes as $\frac{1}{\delta}$.

Step 2. To prove part of item (iib), we use the same reasoning as in the proof of proposition 4. For every point x, y satisfying $\|y-x\| \geq \tau$, we have

$$\begin{aligned} &|u_{\tau,\delta}(y) - u_{\tau,\delta}(x)| \leq C_1\|y-x\|, \quad \text{with} \\ &C_1 := \sup_{\tau\in(0,1]} \sup_{\|y-x\|\leq 2\tau} \Big(\frac{E_\tau(x,y)}{\tau} + C_0\Big). \end{aligned}$$

Indeed, we choose $n \geq 1$ so that $n\tau < \|y-x\| \leq (n+1)\tau$ and define $x_i = x + \frac{i}{n}(y-x)$. We then apply n times the inequality

$$u_{\tau,\delta}(x_{i+1}) - u_{\tau,\delta}(x_i) \leq E_\tau(x_i, x_{i+1}) + \tau\delta\|u_{\tau,\delta}\|_\infty \leq \tau C_1$$

to obtain $u_{\tau,\delta}(y) - u_{\tau,\delta}(x) \leq C_1\|y-x\|$.

Step 3. We define R using the uniform super-linearity (H5) by

$$R := \inf \Big\{ R > 1 \ : \ \inf_{\tau\in(0,1]} \inf_{\|y-x\|\geq\tau R} \frac{E_\tau(x,y) - C_0\tau}{\|y-x\|} > C_1 \Big\}.$$

Then any $x \in \arg\min_x\{(1-\tau\delta)u_{\tau,\delta}(x) + E_\tau(x,y)\}$ satisfies $\|y-x\| \leq \tau R$. The proof is done by contradiction. If $\|y-x\| > \tau R > \tau$, by step 2, we know that $u_{\tau,\delta}(y) - u_{\tau,\delta}(x) \leq C_1\|y-x\|$, and by definition of R, we have

$$u_{\tau,\delta}(y) - u_{\tau,\delta}(x) \geq E_\tau(x,y) - \tau\delta\|u_{\tau,\delta}\|_\infty \geq E_\tau(x,y) - \tau C_0 > C_1\|y-x\|.$$

We obtain a contradiction, therefore $\|y-x\| \leq \tau R$, which is item (iic).

Step 4. We conclude the proof of item (iib). If $\|z-y\| \leq \tau$ and x is a point realizing the minimum in the definition of $u_{\tau,\delta}(y)$,

$$u_{\tau,\delta}(z) - u_{\tau,\delta}(y) \leq E_\tau(x,z) - E_\tau(x,y) \leq C\|z-y\|,$$

where

$$C := \max\Big(C_1, \sup_{\tau\in(0,1]} \sup_{\|y-x\|,\|z-x\|\leq\tau(R+1)} \frac{E_\tau(x,z)-E_\tau(x,y)}{\|y-x\|}\Big). \qquad \square$$

We show in the following proposition an extension of item (ii) for general families of short-range actions. We recall the definition of the Mañé potential Φ_τ. We could have used the Peierls Barrier, but the two notions coincide when one of their arguments belongs to the projected Mather set (definition 13).

Definition 20. We call Mañé potential, the doubly periodic function

$$\Phi_\tau(x,y) \equiv \inf_{n\geq1} \inf_{p\in\mathbb{Z}^d} \inf_{\substack{(x_0,\ldots,x_n)\in(\mathbb{R}^d)^{n+1} \\ x_0=x, x_n=y+p}} \sum_{k=0}^{n-1} [E_\tau(x_k,x_{k+1}) - \bar{E}_\tau], \quad \forall\, x,y\in\mathbb{R}^d.$$

From [GT11], we know that $\Phi_\tau(x,y)$ is continuous with respect to (x,y). Moreover

- $\forall x \in pr^{(1)}(\mathrm{Mather}(E_\tau))$, $[y \mapsto \Phi_\tau(x,y)]$ is a discrete weak KAM solution,
- $\forall y \in pr^{(1)}(\mathrm{Mather}(E_\tau))$, $[x \mapsto -\Phi_\tau(x,y)]$ is a discrete weak KAM solution.

Proposition 21. *Let $\{E_\tau(x,y)\}_{\tau\in(0,1]}$ be a family of short-range actions satisfying the hypotheses (H1)–(H6). Let $u_{\tau,\delta}$ be the unique fixed point of the discounted discrete Lax-Oleinik equation. Let $\mathcal{M}_\tau$ be the set of τ-minimizing measures (definition 13). Then, for $\tau \in (0,1]$ fixed,*

$$\lim_{\delta\to0}\Big(u_{\tau,\delta} - \frac{\bar{E}_\tau}{\tau\delta}\Big) = u^*_\tau, \quad \textit{uniformly}.$$

Moreover u^*_τ is a Lipschitz periodic function and is characterized by

$$\begin{aligned} u^*_\tau(y) &= \sup\Big\{w(y) \,:\, T_\tau[w] = w + \bar{E}_\tau \ \textit{and} \ \iint_{\mathbb{T}^d\times\mathbb{R}^d} w(x)\,d\mu(x,v) \leq 0,\ \forall\mu\in\mathcal{M}_\tau\Big\} \\ &= \inf\Big\{\iint_{\mathbb{T}^d\times\mathbb{R}^d} \Phi_\tau(x,y)\,d\mu(x,v) \,:\, \mu\in\mathcal{M}_\tau\Big\}. \end{aligned}$$

We first start by an a priori bound which is uniform in $\tau, \delta \in (0,1]$.

Lemma 22. *Under the hypotheses of proposition 21, there exists a constant $C > 0$ such that, for every $\tau, \delta \in (0,1]$,*

$$\Big\| u_{\tau,\delta} - \frac{\bar{E}_\tau}{\tau\delta} \Big\|_\infty \leq C.$$

Proof. We denote by C' the constant given by proposition 4 which bounds from above the Lipschitz constant of every discrete weak KAM solution independently of $\tau \in (0,1]$.

Part 1. Let u_τ be some discrete weak KAM solution. Let

$$y \in \arg\max_{y\in\mathbb{R}^d} \Big\{ u_{\tau,\delta}(y) - \frac{\bar{E}_\tau}{\tau\delta} - u_\tau(y) \Big\}.$$

As a fixed point of $T_{\tau,\delta}$, the discounted discrete solution satisfies for every x,

$$\begin{aligned} u_{\tau,\delta}(y) - \frac{\bar{E}_\tau}{\tau\delta} - u_\tau(y) \leq (1-\tau\delta)\Big[u_{\tau,\delta}(x) - \frac{\bar{E}_\tau}{\tau\delta} - u_\tau(x) \Big] \\ + \big[E_\tau(x,y) - u_\tau(y) + u_\tau(x) - \bar{E}_\tau\big] - \tau\delta u_\tau(x). \end{aligned}$$

Let x be a backward calibrated point for y, that is a point satisfying

$$E_\tau(x,y) = u_\tau(y) - u_\tau(x) + \bar{E}_\tau.$$

By definition of y, we have

$$\begin{gathered} u_{\tau,\delta}(x) - \frac{\bar{E}_\tau}{\tau\delta} - u_\tau(x) \leq u_{\tau,\delta}(y) - \frac{\bar{E}_\tau}{\tau\delta} - u_\tau(y), \\ u_{\tau,\delta}(y) - \frac{\bar{E}_\tau}{\tau\delta} - u_\tau(y) \leq -u_\tau(x), \end{gathered}$$

or $u_{\tau,\delta}(y) - \frac{\bar{E}_\tau}{\tau\delta} \leq \mathrm{osc}(u_\tau) \leq C'\mathrm{diam}([0,1]^d) := C$.

Part 2. Let y be a point realizing the minimum of $u_{\tau,\delta}(y) - \frac{\bar{E}_\tau}{\tau\delta} - u_\tau(y)$ and x be a discounted backward calibrated point for y, that is satisfying

$$u_{\tau,\delta}(y) = (1-\tau\delta)u_{\tau,\delta}(x) + E_\tau(x,y).$$

Then similar to what we have done in part 1, we obtain

$$\begin{aligned} u_{\tau,\delta}(y) - \frac{\bar{E}_\tau}{\tau\delta} - u_\tau(y) = (1-\tau\delta)\Big[u_{\tau,\delta}(x) - \frac{\bar{E}_\tau}{\tau\delta} - u_\tau(x) \Big] \\ + \big[E_\tau(x,y) - u_\tau(y) + u_\tau(x) - \bar{E}_\tau\big] - \tau\delta u_\tau(x). \end{aligned}$$

As $E_\tau(x,y) - u_\tau(y) + u_\tau(x) - \bar{E}_\tau \geq 0$, we obtain $u_{\tau,\delta}(y) - \frac{\bar{E}_\tau}{\tau\delta} - u_\tau(y) \geq -u_\tau(x)$ or $u_{\tau,\delta}(y) - \frac{\bar{E}_\tau}{\tau\delta} \geq -\mathrm{osc}(u_\tau) \geq -C'\mathrm{diam}([0,1]^d)$. □

The first main observation is given in the following lemma. See definition 13 for the notion of projected τ-minimizing measures.

Lemma 23. *For every $\tau, \delta \in (0, 1]$*

$$\int_{\mathbb{T}^d} \Big[u_{\tau,\delta}(x) - \frac{\bar{E}_\tau}{\tau\delta} \Big] \, d\mu(x) \leq 0, \quad \forall \, \mu \text{ projected } \tau\text{-minimizing measure}.$$

Proof. By definition of the discounted discrete solution $u_{\tau,\delta}$, we have

$$u_{\tau,\delta}(x + \tau v) \leq (1 - \tau\delta) u_{\tau,\delta}(x) + E_\tau(x, x + \tau v), \quad \forall x \in \mathbb{T}^d.$$

Let μ be a τ-minimizing measure. By integrating the previous inequality, we obtain

$$\begin{aligned} &\iint_{\mathbb{T}^d \times \mathbb{R}^d} u_{\tau,\delta}(x + \tau v) \, d\mu(x, v) \\ &\qquad \leq (1 - \tau\delta) \iint_{\mathbb{T}^d \times \mathbb{R}^d} u_{\tau,\delta}(x) \, d\mu(x, v) + \iint_{\mathbb{T}^d \times \mathbb{R}^d} E_\tau(x, x + \tau v) \, d\mu(x, v). \end{aligned}$$

The τ-holonomic property implies the equality of the two integrals containing $u_{\tau,\delta}$. The τ-minimizing property implies the equality of the last integral to $\bar{E}_\tau$. We just have proved $\tau\delta \iint u_{\tau,\delta} \, d\mu \leq \bar{E}_\tau$ for every τ-minimizing measure. □

The second main observation is given in the following lemma.

Lemma 24. *Let $\tau > 0$ be fixed. Let $\delta_i \to 0$ be a sequence converging to 0. For every δ_i, let u_{τ,δ_i} be a discounted discrete weak KAM solution and $(x^i_{-k})_{k \geq 0}$ be a discounted backward calibrated configuration satisfying*

$$u_{\tau,\delta_i}(x^i_{-k}) = (1 - \tau\delta_i) u_{\tau,\delta_i}(x^i_{-k-1}) + E_\tau(x^i_{-k-1}, x^i_{-k}).$$

Let μ_i be the probability measure on $\mathbb{T}^d \times \mathbb{R}^d$ defined by

$$\mu_i := \sum_{k \geq 0} \tau\delta(1 - \tau\delta)^k \delta_{\left(x^i_{-k-1}, (x^i_{-k} - x^i_{-k-1})/\tau\right)}.$$

Then every weak* accumulation measure of (μ_i) is a τ-minimizing measure.

Proof. We notice from item (iic) of proposition 19 that μ_i has a compact support in $\mathbb{T}^d \times \{\|v\| \leq R\}$. Let μ be a weak* accumultation measure of (μ_i). To simplify the notations we assume that $\mu_i \to \mu$. We first claim that μ is τ-holonomic. Let $\varphi : \mathbb{T}^d \to \mathbb{R}$ be a continuous function. Then

$$\begin{aligned} \iint \varphi(x + \tau v) \, d\mu_i(x, v) &= \sum_{k \geq 0} \tau\delta_i (1 - \tau\delta_i)^k \varphi(x^i_{-k}) \\ &= \tau\delta_i \varphi(x^i_0) + (1 - \tau\delta_i) \sum_{k \geq 0} \tau\delta_i (1 - \tau\delta_i)^k \varphi(x^i_{-k-1}) \\ &= \tau\delta_i \varphi(x^i_0) + (1 - \tau\delta_i) \iint \varphi(x) \, d\mu_i(x, v). \end{aligned}$$

The claim is proved by letting $\delta_i \to 0$. We next claim that μ is τ-minimizing

$$\begin{aligned}
\iint & E_\tau(x, x + \tau v)\, d\mu_i(x, v) \\
&= \sum_{k\geq 0} \tau\delta_i(1-\tau\delta_i)^k E_\tau(x^i_{-k-1}, x^i_{-k}) \\
&= \sum_{k\geq 0} \tau\delta_i(1-\tau\delta_i)^k [u_{\tau,\delta_i}(x^i_{-k}) - (1-\tau\delta_i)u_{\tau,\delta_i}(x^i_{-k-1})] = \tau\delta_i u_{\tau,\delta_i}(x^i_0).
\end{aligned}$$

Lemma 22 implies $\tau\delta_i u_{\tau\delta_i} \to \bar{E}_\tau$ uniformly, and the claim is proved. □

Proof of proposition 21. Let $\delta_i \to 0$ be a sub-sequence converging to 0. Since $\mathrm{Lip}(u_{\tau,\delta_i})$ and $\big\|u_{\tau,\delta_i} - \frac{\bar{E}_\tau}{\tau\delta_i}\big\|_\infty$ are uniformly bounded, there exists a sub-sub-sequence of δ_i, (we still use the same notation) such that, for some u_τ

$$u_{\tau,\delta_i} - \frac{\bar{E}_\tau}{\tau\delta_i} \to u_\tau \text{ holds in } C^0\text{-topology.}$$

Part 1. We claim that u_τ is a discrete weak KAM solution, that is a solution of $T_\tau[u_\tau] = u_\tau + \bar{E}_\tau$. On the one hand, by letting $\delta_i \to 0$ in

$$u_{\tau,\delta_i}(y) - \frac{\bar{E}_\tau}{\tau\delta_i} \leq (1-\tau\delta_i)[u_{\tau,\delta_i}(x) - \frac{\bar{E}_\tau}{\tau\delta_i}] + E_\tau(x,y) - \bar{E}_\tau,$$

one obtains $u_\tau(y) - u_\tau(x) \leq E_\tau(x,y) - \bar{E}_\tau$, for every $x, y \in \mathbb{R}^d$. On the other hand, for every y, there exists $x_i \in \mathbb{R}^d$ such that

$$u_{\tau,\delta_i}(y) - \frac{\bar{E}_\tau}{\tau\delta_i} = (1-\tau\delta_i)\Big[u_{\tau,\delta_i}(x_i) - \frac{\bar{E}_\tau}{\tau\delta_i}\Big] + E_\tau(x_i,y) - \bar{E}_\tau.$$

Proposition 19 implies there exists a constant $R > 0$, independent of δ, such that $\|y - x_i\| \leq \tau R$. By taking possibly a sub-sequence, one may assume $x_i \to x$ for some $x \in \mathbb{R}^d$. One then obtains $u_\tau(y) - u_\tau(x) = E_\tau(x,y) - \bar{E}_\tau$. The claim is proved.

Part 2. We recall that $\mathcal{M}_\tau$ denotes the set of τ-minimizing measures. By letting $\delta \to 0$ in lemma 23 along the sequence δ_i, one obtains

$$\iint u_\tau(x)\, d\mu(x,v) \leq 0, \quad \forall \mu \in \mathcal{M}_\tau.$$

We have proved

$$u_\tau(y) \leq \sup\Big\{w(y) \ :\ T_\tau[w] = w + \bar{E}_\tau,\ \iint w(x)\, d\mu(x,v) \leq 0,\ \forall \mu \in \mathcal{M}_\tau\Big\}.$$

Conversely, let w be a discrete weak KAM solution satisfying $\iint w\,d\mu \leq 0$ for every $\mu \in \mathcal{M}_\tau$. Let $y \in \mathbb{R}^d$. For every δ_i, let $(x^i_{-k})_{k\geq 0}$ be a discounted backward calibrated configuration starting at $y = x^i_0$. Then

$$\begin{aligned} u_{\tau,\delta_i}(x^i_{-k}) - \frac{\bar{E}_\tau}{\tau\delta_i} - w(x^i_{-k}) = (1-\tau\delta_i)\Big[u_{\tau,\delta_i}(x^i_{-k-1}) - w(x^i_{-k-1}) - \frac{\bar{E}_\tau}{\tau\delta_i}\Big] \\ + [E_\tau(x^i_{-k-1}, x^i_{-k}) - w(x^i_{-k}) + w(x^i_{-k-1}) - \bar{E}_\tau] \\ - \tau\delta_i w(x^i_{-k-1}). \end{aligned}$$

As w is a sub-action, we obtain $E_\tau(x^i_{-k-1}, x^i_{-k}) - w(x^i_{-k}) + w(x^i_{-k-1}) - \bar{E}_\tau \geq 0$. By iterating these inequalities, one gets

$$u_{\tau,\delta_i}(y) - \frac{\bar{E}_\tau}{\tau\delta_i} - w(y) \geq \sum_{k\geq 0} -\tau\delta_i(1-\tau\delta_i)^k w(x^i_{-k-1}) = -\iint w(x)\,d\mu_i(x,v),$$

where μ_i is the probability measure defined in lemma 24. As μ_i converges to a τ-minimizing measure μ, one obtains $u_\tau(y) - w(y) \geq -\iint w(x)\,d\mu(x,v) \geq 0$. We have proved

$$u_\tau(y) = \sup\Big\{w(y) \ : \ T_\tau[v] = v + \bar{E}_\tau, \ \iint v\,d\mu \leq 0, \ \forall \mu \in \mathcal{M}_\tau\Big\} := u^*_\tau(y).$$

We also proved that the only accumulation point of $u_{\tau,\delta} - \frac{\bar{E}_\tau}{\tau\delta}$ is the function u^*_τ defined above. Thus

$$\lim_{\delta\to 0} u_{\tau,\delta} - \frac{\bar{E}_\tau}{\tau\delta} = u^*_\tau, \quad \text{in the } C^0 \text{ topology.}$$

Part 3. As u^*_τ is a sub-action satisfying $\iint u^*_\tau d\mu \leq 0$ for every $\mu \in \mathcal{M}_\tau$, one has

$$\begin{gathered} u^*_\tau(y) - u^*_\tau(x) \leq \Phi_\tau(x,y), \quad \forall x,y \in \mathbb{R}^d, \\ u^*_\tau(y) \leq \iint \Phi(x,y)\,d\mu(x,v) + \iint u^*_\tau(x)\,d\mu(x,v) \leq \iint \Phi(x,y)\,d\mu(x,v). \end{gathered}$$

We have proved

$$u^*_\tau(y) \leq \inf\Big\{\iint \Phi(x,y)\,d\mu(x,v) \ : \ \mu \in \mathcal{M}_\tau\Big\}, \quad \forall y \in \mathbb{R}^d.$$

Conversely, let y be fixed, by using a discounted backward calibrated configuration $(x^i_{-k})_{k\geq 0}$ starting at $y = x^i_0$, one obtains

$$u_{\tau,\delta_i}(y) - \frac{\bar{E}_\tau}{\tau\delta_i} = \sum_{k\geq 0}(1-\tau\delta_i)^k[E_\tau(x^i_{-k-1}, x^i_{-k}) - \bar{E}_\tau].$$

As $x \mapsto -\Phi_\tau(x,y)$ is a sub-action,

$$E_\tau(x^i_{-k-1}, x^i_{-k}) - \bar{E}_\tau \geq -\Phi_\tau(x^i_{-k}, y) + \Phi_\tau(x^i_{-k-1}, y), \quad \forall k \geq 1,$$
$$E_\tau(x^i_{-1}, x^i_0) - \bar{E}_\tau \geq \Phi_\tau(x^i_{-1}, x^i_0), \quad (k = 0).$$

(The second inequality comes from the definition of the Mañé potential written as an infimum). By substituting these inequalities in the previous equality, we obtain

$$\begin{aligned}
u_{\tau,\delta_i}(y) - \frac{\bar{E}_\tau}{\tau\delta_i} &\geq \sum_{k\geq 1}(1-\tau\delta_i)^k[\Phi_\tau(x^i_{-k-1}, y) - \Phi_\tau(x^i_{-k}, y)] + \Phi_\tau(x^i_{-1}, y), \\
&\geq \sum_{k\geq 0}(1-\tau\delta_i)^k\Phi_\tau(x^i_{-k-1}, y) - \sum_{k\geq 1}(1-\tau\delta_i)^k\Phi_\tau(x^i_{-k}, y), \\
&= \sum_{k\geq 0}\,[(1-\tau\delta_i)^k - (1-\tau\delta_i)^{k+1}]\Phi_\tau(x^i_{-k-1}, y), \\
&= \sum_{k\geq 0}\,\tau\delta_i(1-\tau\delta_i)^k\Phi_\tau(x^i_{-k-1}, y) = \iint \Phi_\tau(x,y)\,d\mu_i(x,v),
\end{aligned}$$

where μ_i is defined in lemma 24. By taking a sub-sequence, μ_i converges to a τ-minimizing measure μ. We have proved

$$u^*_\tau(y) \geq \inf\Big\{\iint \Phi_\tau(x,y)\,d\mu(x,v) \ : \ \mu \in \mathcal{M}_\tau\Big\}, \quad \forall y \in \mathbb{R}^d.$$

This completes the proof of proposition 21. □

We now prove the last part of theorem 2. The main tool is the existence of a backward calibrated path, associated to the discounted weak KAM solution u_δ, having a regularity at least $C^{1,1}$ independent of the discounted factor δ. In the case of the ergodic cell equation, there exists a C^2 minimizer which realizes the infimum in (3) by Tonelli-Weierstrass (see [Fat08] for instance). In the case of the discounted cell equation (8), we are not aware of a reference giving this regularity. Neither [DFIZ14] nor [ISM11] gives the regularity $C^{1,1}$.

Proposition 25. *Let $H(x,p) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a C^2 Tonelli Hamiltonian satisfying the hypotheses (L1) and (L2). There exists a constant $C > 0$ such that for every $x \in \mathbb{R}^d$, for every $\delta > 0$, if u_δ denotes the unique solution of the discounted cell equation* (8)*, then there exists a $C^{1,1}$ path $\gamma^x_\delta : (-\infty, 0] \to \mathbb{R}^d$ with $\gamma^x_\delta(0) = x$, $\|\dot\gamma^x_\delta\|_\infty \leq C$ and $\mathrm{Lip}(\dot\gamma^x_\delta) \leq C$, such that*

$$u_\delta(x) - e^{t\delta}u_\delta(\gamma^x_\delta(t)) = \int_t^0 e^{s\delta}L(\gamma^x_\delta(s), \dot\gamma^x_\delta(s))\,ds, \quad \forall t \leq 0. \tag{25}$$

Moreover $u_{\tau,\delta} \to u_\delta$ uniformly on $\mathbb{R}^d$.

Proof. Part 1. Let be $\tau > 0$, and $(x_n^{\tau,\delta})_{n\le 0}$ a discounted backward calibrated configuration for the discrete action $\mathcal{L}_\tau$ ending at x. We note

$$v_n^{\tau,\delta} := \frac{1}{\tau}\Big(x_{n+1}^{\tau,\delta} - x_n^{\tau,\delta}\Big), \quad \forall n \le -1.$$

We show in this part there exists a constant $C > 0$, independant of n, δ and x, such that $\|v_n^{\tau,\delta} - v_{n-1}^{\tau,\delta}\| \le C\tau$ for all $n \le -1$. Let be $x_n := x_n^{\tau,\delta}$ and $v_n := v_n^{\tau,\delta}$. By definition of calibration we have

$$\begin{aligned}
u_{\tau,\delta}(x_{n+1}) &= (1-\tau\delta)u_{\tau,\delta}(x_n) + \mathcal{L}_\tau(x_n, x_{n+1}) \\
&= (1-\tau\delta)^2 u_{\tau,\delta}(x_{n-1}) + (1-\tau\delta)\mathcal{L}_\tau(x_{n-1}, x_n) + \mathcal{L}_\tau(x_n, x_{n+1}) \\
&\le (1-\tau\delta)u_{\tau,\delta}(x) + \mathcal{L}_\tau(x, x_{n+1}), \quad \forall x \in \mathbb{R}^d \\
&\le (1-\tau\delta)^2 u_{\tau,\delta}(x_{n-1}) + (1-\tau\delta)\mathcal{L}_\tau(x_{n-1}, x) + \mathcal{L}_\tau(x, x_{n+1}), \quad \forall x \in \mathbb{R}^d.
\end{aligned}$$

In other words $(x_n^{\tau,\delta})_{n\le 0}$ is minimizing in the following sense

$$(1-\tau\delta)\mathcal{L}_\tau(x_{n-1}, x_n) + \mathcal{L}_\tau(x_n, x_{n+1}) \le (1-\tau\delta)\mathcal{L}_\tau(x_{n-1}, x) + \mathcal{L}_\tau(x, x_{n+1}), \quad \forall x \in \mathbb{R}^d,$$

and satisfies the *discounted discrete Euler-Lagrange equation*

$$\begin{aligned}
& (1-\tau\delta)\frac{\partial \mathcal{L}_\tau}{\partial y}(x_{n-1}, x_n) + \frac{\partial \mathcal{L}_\tau}{\partial x}(x_n, x_{n+1}) = 0, \\
\iff\quad & (1-\tau\delta)\frac{\partial L}{\partial v}(x_{n-1}, v_{n-1}) - \frac{\partial L}{\partial v}(x_n, v_n) + \tau\frac{\partial L}{\partial x}(x_n, v_n) = 0, \\
\iff\quad & \frac{1}{\tau}\Big[\frac{\partial L}{\partial v}(x_n, v_n) - \frac{\partial L}{\partial v}(x_{n-1}, v_{n-1})\Big] = \frac{\partial L}{\partial x}(x_n, v_n) - \delta\frac{\partial L}{\partial v}(x_{n-1}, v_{n-1}.
\end{aligned} \tag{26}$$

Proposition 19 shows there exists $R > 0$ such that $\|v_n^{\tau,\delta}\| \le R$, $\forall n \le -1$. The property of positive definiteness (L1) implies the existence of a constant $\alpha(R) > 0$ such that

$$\frac{\partial L}{\partial v \partial v}(x, v).(h, h) \ge \alpha(R)\|h\|^2, \quad \forall x \in \mathbb{R}^d,\ \forall \|v\| \le R,\ \forall h \in \mathbb{R}^d.$$

By integrating over $t \in [0, 1]$

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial v}(x_{n-1} + t(x_n - x_{n-1}), v_{n-1}, t(v_n - v_{n-1}))\Big)$$

and by taking the scalar product with $(v_n - v_{n-1})$, one obtains

$$\alpha(R)\|v_n - v_{n-1}\| \le \Big\|\frac{\partial L}{\partial x \partial v}\Big\|\ \|x_n - x_{n-1}\| + \tau\Big(\ \Big\|\frac{\partial L}{\partial x}\Big\| + \delta\Big\|\frac{\partial L}{\partial v}\Big\|\Big)$$

where all norms $\|\cdot\|$ are taken over $\mathbb{T}^d \times \{v \in \mathbb{R}^d \ : \text{ and } \|v\| \le R\|\}$. As $\|x_n - x_{n-1}\| \le \tau R$ thanks to item (iic) of proposition 19, one obtains $\|v_n - v_{n-1}\| \le \tau C$, for some constant $C > 0$, uniformly in n, δ and x.

Part 2. Let $\gamma^x_{\tau,\delta} : (-\infty, 0] \to \mathbb{R}^d$ be the piecewise affine path interpolating the points x_n at time $n\tau$. We show that $\gamma^x_{\tau,\delta}$ is Lipschitz uniformly in n, δ and $(x^{\tau,\delta}_n)_{n\le 0}$. To simplify we write $\gamma = \gamma^x_{\tau,\delta}$. Let be $s < t < 0$. Either s, t belongs to the same interval $((n-1)\tau, n\tau]$. As γ is affine with speed bounded by R, we obtain $\|\gamma(t) - \gamma(s)\| \le |t-s|R$. Or s, t belong to different intervals. By introducing the points x_n corresponding to the intermediate times $s \le n\tau \le t$, one obtains again the same estimate.

Part 3. We choose a subsequence $\tau_i \to 0$ and a discounted backward calibrated configuration $(x^i_n)_{n\le 0}$ such that $\gamma_i := \gamma^x_{\tau_i,\delta} \to \gamma^x_\delta$ uniformly on any compact interval of $(-\infty, 0]$ for some Lipschitz function γ^x_δ. We claim there exists a uniformly Lipschitz function $V : (-\infty, 0] \to \mathbb{R}^d$ such that

$$\int_t^0 V(s)\,ds = x - \gamma^x_\delta(t), \quad \forall t \le 0.$$

Let $T \subset (-\infty, 0)$ be a countable dense subset. Let be $V_i : (-\infty, 0) \to \mathbb{R}^d$ such that

$$V_i(t) := \frac{1}{\tau_i}\Big(x^i_n - x^i_{n-1}\Big), \quad \forall t \in [(n-1)\tau_i, n\tau_i),\ \forall n \le 0.$$

By compactness of the ball $\{v \ : \ \|v\| \le R\}$, by taking a subsequence if needed, we may assume $V_i(t) \to V(t)$ exists for every $t \in T$. Let be $s < t < 0$ and $m \le n$ non positive integers such that $(m-1)\tau_i \le s < m\tau_i$ and $(n-1)\tau_i \le t < n\tau_i$. Part 1 implies,

$$\|V_i(t) - V_i(s)\| = \|v^i_{n-1} - v^i_{m-1}\| \le (n-m)\tau_i C \le |t-s|C + \tau_i C.$$

By letting $\tau_i \to 0$, one obtains $\|V(t) - V(s)\| \le |t-s|C$ for every $s, t \in T$. Let $V : (-\infty, 0) \to \mathbb{R}^d$ be the unique Lipschitz extension of V. Then $V_i(t) \to V(t)$ for every $t \in (-\infty, 0)$. Since

$$\int_t^0 V_i(s)\,ds = x - \gamma_i(t), \quad \forall t < 0,$$

the claim is proved and γ^x_δ is a $C^{1,1}$ path.

Part 4. Item (iib) of proposition 19 shows there exists a constant $C > 0$ such that $\mathrm{Lip}(u_{\tau_i,\delta}) \le C$. By taking a subsequence if necessary, we may assume that $u_i := u_{\tau_i,\delta} \to u$ uniformly for some Lipschitz function u. We claim that

$$u(x) - e^{t\delta}u(\gamma^x_\delta(t)) = \int_t^0 e^{s\delta} L(\gamma^x_\delta(s), \dot\gamma^x_\delta(s))\,ds, \quad \forall x \in \mathbb{R}^d,\ \forall t \le 0.$$

Indeed using the notations in part 3, we have for every $n \le -1$,

$$u_i(x) = (1-\tau_i\delta)^{-n} u_i \circ \gamma_i(n\tau_i) + \sum_{k=n}^{-1} (1-\tau_i\delta)^{-k-1} \tau_i L(\gamma_i(k\tau_i), V_i(k\tau_i))).$$

Let be $t < 0$ fixed, $n \le 0$ such that $(n-1)\tau_i \le t < n\tau_i$. Then

$$I := \Big| \sum_{k=n}^{-1} (1-\tau_i\delta)^{-k-1} \tau_i L(\gamma_i(k\tau_i), V_i(k\tau_i)) - \int_{n\tau_i}^{0} e^{s\delta} L(\gamma_i(s), V_i(s))\, ds \Big|$$

can be bound from above by the following three terms I_1, I_2, I_3

$$\begin{aligned}
I_1 &= \sum_{k=n}^{-1} (1-\tau_i\delta)^{-k-1} \int_{k\tau_i}^{(k+1)\tau_i} \Big| L(\gamma_i(k\tau_i), V_i(k\tau_i)) - L(\gamma_i(s), V_i(s)) \Big|\, ds \\
&\le R \Big\| \frac{\partial L}{\partial x} \Big\| \frac{\tau_i}{\delta}, \\
I_2 &= \sum_{k=n}^{-1} \Big[(1-\tau_i\delta)^{-k-1} - (1-\tau_i\delta)^{-k} \Big] \int_{k\tau_i}^{(k+1)\tau_i} \Big| L(\gamma_i(s), V_i(s)) \Big|\, ds \\
&\le \tau_i \|L\| \Big(1 - (1-\tau_i\delta)^{-n} \Big) \le \tau_i \|L\|, \\
I_3 &= \sum_{k=n}^{-1} \int_{k\tau_i}^{(k+1)\tau_i} \Big[e^{s\delta} - (1-\tau_i\delta)^{-k} \Big] \Big| L(\gamma_i(s), V_i(s)) \Big|\, ds \\
&\le \|L\| \Big[\int_{n\tau_i}^{0} e^{s\delta}\, ds - \tau_i \sum_{k=n}^{-1} (1-\tau_i\delta)^{-k} \Big], \\
&\le \|L\| \Big[\int_{n\tau_i}^{0} e^{s\delta}\, ds - \tau_i \sum_{k=n}^{-1} (1-\tau_i\delta)^{-k-1} \Big] + \tau_i \|L\|, \\
&\le \|L\| \frac{(1-\tau_i\delta)^{-n} - e^{n\tau_i\delta}}{\delta} + \tau_i \|L\| \le +\tau_i \|L\|.
\end{aligned}$$

We finally obtain

$$I \le R \Big\| \frac{\partial L}{\partial x} \Big\| \frac{\tau_i}{\delta} + 2\tau_i \|L\|,$$

and the claim is proved by letting $\tau_i \to 0$, since $n\tau_i \to t$, $u_i \to u$ uniformly on $\mathbb{R}^d$, and both $\gamma_i \to \gamma_\delta^x$ and $V_i \to \dot\gamma_\delta^x$ uniformly on any compact set of $(-\infty, 0]$.

Part 5. We claim that

$$u(x) - e^{td} u(x+tv) \le \int_t^0 e^{s\delta} L(x+sv, v)\, ds, \quad \forall x \in \mathbb{R}^d,\ \forall t \le 0,\ \forall v \in \mathbb{R}^d.$$

We choose as before $n \le 0$ such that $(n-1)\tau_i \le t < n\tau_i$. Let be $x^i_k := x + k\tau_i v$, $\forall k \in \{n, \dots, -1, 0\}$. By definition of the discounted discrete weak KAM solution $u_i = u_{\tau_i,\delta}$,

$$u_i(x) \le (1-\tau_i\delta)^{-n} u_i(x^i_n) + \sum_{k=n}^{-1} (1-\tau_i\delta)^{-k-1} \tau_i L(x^i_k, v).$$

Then the expression $| \sum_{k=n}^{-1} (1-\tau_i\delta)^{-k-1} \tau_i L(x^i_k, v) - \int_{n\tau_i}^0 e^{s\delta} L(x+sv, v)\, ds|$ is estimated in the same way as before, and the claim is proved.

Part 6. We show that u is a viscosity solution of $\delta u + H(x, du(x)) = 0$. We first show u is a subsolution. Let be $x \in \mathbb{R}^d$ fixed and $\phi : \mathbb{R}^d \to \mathbb{R}$ a C^1 function, such that $u(x) = \phi(x)$ and $u \le \phi$. Using part 5, for every $v \in \mathbb{R}^d$ and $t \le 0$,

$$\phi(x) - e^{t\delta}\phi(x+tv) \le u(x) - e^{t\delta} u(x+tv) \le \int_t^0 L(x+sv, v)\, ds.$$

By dividing by $-t$ and taking the limit $t \to 0$, one obtains $\delta\phi(x) + d\phi(x).v \le L(x, v)$. By taking the supremum in v and using the definition of the Legendre-Fenchel transform $H(x,p) := \sup_v\{p.v - L(x,v)\}$, on obtains $\delta\phi(x) + H(x, d\phi(x)) \le 0$.

We next show u is a suppersolution. Let be $x \in \mathbb{R}^d$ fixed and $\phi : \mathbb{R}^d \to \mathbb{R}$ a C^1 function, such that $u(x) = \phi(x)$ and $u \ge \phi$. Part 4 implies $\delta\phi(x) + d\phi(x).v \ge L(x,v)$ for $v = \dot\gamma^x_\delta(0)$ and in particular $\delta\phi(x) + H(x, d\phi(x)) \ge 0$.

Part 7. We have proved that any accumulation function u of $u_{\tau,\delta}$, as $\tau \to 0$, is necessarily equal to u_δ, the unique viscosity solution of $\delta u + H(x, du(x)) = 0$. Thus $u_{\tau,\delta} \to u_\delta$ uniformly in $\mathbb{R}^d$ as $\tau \to 0$, and the equation (25) is proved. □

Proposition 26. *Let $H(x,p) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a C^2 Tonelli Hamiltonian which is time-independent and periodic in x. Then there exists a constant $C > 0$ such that, for every $\tau, \delta \in (0,1]$, if u_δ is the unique viscosity solution of* (8) *and $u_{\tau,\delta}$ is the unique discrete solution of* (11)*, then*

$$\|u_{\tau,\delta} - u_\delta\|_\infty \le C\frac{\tau}{\delta}.$$

Proof. Part 1. We first show $u_{\tau,\delta} - u_\delta \le C\frac{\tau}{\delta}$. Thanks to proposition 25, there exists a constant $C_1 > 0$ such that, for every $x \in \mathbb{R}^d$, there exists a $C^{1,1}$ curve $\gamma^x_\delta : (-\infty, 0] \to \mathbb{R}^d$, satisfying $\gamma^x_\delta(0) = x$, $\|\dot\gamma^x_\delta\| \le C_1$ and $\mathrm{Lip}(\dot\gamma^x_\delta) \le C_1$ uniformly on $(-\infty, 0]$, and

$$u_\delta(x) = \int_{-\infty}^0 e^{s\delta} L(\gamma^x_\delta(s), \dot\gamma^x_\delta(s))\, ds.$$

Let be $x_{-k} := \gamma_\delta^x(-k\tau)$, $v_{-k} := (x_{-k+1} - x_{-k})/\tau$, for every $k \geq 0$. Then

$$u_{\tau,\delta}(x) \leq \sum_{k\geq 0}(1-\tau\delta)^k \mathcal{L}_\tau(x_{-k-1}, x_{-k}),$$

$$(1-\tau\delta)u_{\tau,\delta}(x) - u_\delta(x) \leq \sum_{k\geq 0}\int_{-(k+1)\tau}^{-k\tau}\Big[(1-\tau\delta)^{k+1} - e^{s\delta}\Big]L(x_{-k-1}, v_{-k-1})$$
$$+ \sum_{k\geq 0}\int_{-(k+1)\tau}^{-k\tau} e^{s\delta}\big[L(x_{-k-1}, v_{-k-1}) - L(\gamma_\delta(s), \dot\gamma_\delta(s))\big]\,ds.$$

For every $s \in [-(k+1)\tau, -k\tau]$,

$$\|\gamma_\delta(s) - x_{-k-1}\| \leq C_1\tau, \quad \|\dot\gamma_\delta(s) - v_{-k-1}\| \leq C_1\tau,$$
$$|L(x_{-k-1}, v_{-k-1}) - L(\gamma_\delta(s), \dot\gamma_\delta(s))| \leq \|DL\|C_1\tau,$$

(where $\|DL\|$ is computed by taking the supremum of $\|DL(x,v)\|$ over $x \in \mathbb{R}^d$ and $\|v\| \leq C_1$). Moreover

$$\sum_{k\geq 0}\int_{-(k+1)\tau}^{-k\tau}\Big[e^{s\delta} - (1-\tau\delta)^{k+1}\Big] \leq \frac{1}{\delta} - \frac{\tau(1-\tau\delta)}{\tau\delta} = \tau.$$

Let be $C_2 := \sup_{x\in\mathbb{R}^d} L(x,0)$. Then item (iia) of proposition 19 implies

$$u_{\tau,\delta}(x) - u_\delta(x) \leq \tau C_2 + \tau\|L\| + \|DL\|C_1\frac{\tau}{\delta} \leq \big(C_2 + \|L\| + \|DL\|C_1\big)\frac{\tau}{\delta} := C\frac{\tau}{\delta}.$$

Part 2. We next show $u_{\tau,\delta} - u_\delta \geq -C\frac{\tau}{\delta}$. Let be $x \in \mathbb{R}^d$ and $(x_{-k})_{k\geq 0}$ a discounted backward calibrated configuration for $\mathcal{L}_\tau$ starting at x. then

$$u_{\tau,\delta}(x) = \sum_{k\geq 0}(1-\tau\delta)^k \mathcal{L}_\tau(x_{-k-1}, x_{-k}).$$

Let $\gamma : (-\infty, 0] \to \mathbb{R}^d$ be the piecewise linear path interpolating the points x_{-k} at the times $-k\tau$. Then, thanks to the property (9), or to part 5 in the proof of proposition 25,

$$u_\delta(x) \leq \int_{-\infty}^0 e^{s\delta}L(\gamma(s), \dot\gamma(s))\,ds.$$

We follow the same estimates as in part 1. Using item (iic) of proposition 19, we notice that for every $s \in [-(k+1)\tau, -k\tau]$,

$$\|\gamma(s) - x_{-k-1}\| \leq \|x_{-k} - x_{-k-1}\| \leq R\tau, \quad \dot\gamma(s) = (x_{-k} - x_{-k-1})/\tau := v_{-k-1},$$
$$|L(x_{-k-1}, v_{-k-1}) - L(\gamma(s), \dot\gamma(s))| \leq \Big\|\frac{\partial L}{\partial x}\Big\|R\tau,$$

(where $\big\|\frac{\partial L}{\partial x}\big\|$ is computed by taking the supremum of $\big\|\frac{\partial L}{\partial x}(x,v)\big\|$ over $x \in \mathbb{R}^d$ and $\|v\| \leq R$). Let be $C_3 := \inf_{x,v\in\mathbb{R}^d} L(x,v)$. Then item (iia) of proposition 19 implies

$$u_{\tau,\delta}(x) - u_\delta(x) \geq \Big(C_3 - \|L\| - \Big\|\frac{\partial L}{\partial x}\Big\|R\Big)\frac{\tau}{\delta} := -C\frac{\tau}{\delta}. \qquad \square$$

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