A kinetic velocity alignment model in $\mathbb{R}^d$

Philippe Thieullen
University of Bordeaux

Joint work with
V. Ayot, S. Brull

HYKE-Hwarang Day
Department of Mathematical Sciences SNU

30 August 2023

Summary

- I. **BDG model**
- II. Ideas of the proof
- III. Some numerical experiments

I.1. The standard model of velocity alignment

The Cucker-Smale model: A standard model of a kinetic approach of velocity alignment

$$\partial_t f + v \cdot \partial_x f + \mathrm{div}_v((-\nabla_x \Phi + (1 - |v|^2)v)f) + \mathrm{div}_v(\Psi[f]f) = 0$$

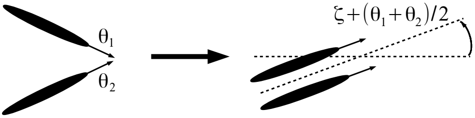
- $-\nabla_x \Phi$: a confinement potential,
- $(1 - |v|^2)v$: a self-propelled term,
- $\Psi[f](x, v) = \int (w - v) K[f](x, dw)$: the velocity alignment
- $K[f](x, dw)$: a probability-valued function

$$\langle K[f](x, \cdot), \varphi \rangle = \frac{\iint \varphi(w)\psi(y - x) f(y, w)\, dydw}{\iint \psi(y - x) f(y, w)\, dydw}$$

  ψ is the communication weight
- no diffusion term

I.2. The BDG model

The BDG model: Bertin, Droz Gregoire (2009)



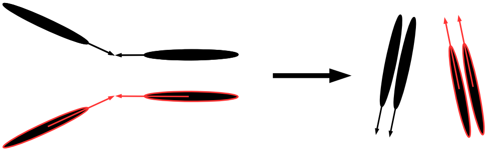
$$\partial_t f + v \cdot \nabla_x f = Q[f,f], \quad f = f(t,x,v)$$

$$Q[f,f] = Q_+[f,f] - Q_-[f,f]$$
$$Q_+[f,f](d\theta) = \iint \beta(\theta_1 - \theta_2)\tau[\Theta_{mid}(\theta_1,\theta_2)]\sharp g(d\theta) f(d\theta_1) f(d\theta_2)$$
$$Q_-[f,f](d\theta) = -f(d\theta)\int \beta(\theta - \theta_1) f(d\theta_1), (\theta_1,\theta_2),$$

- $g(d\zeta)$: the randomness about 0, $\tau[\Theta_{mid}]\sharp g$: about $\tau[\Theta_{mid}$
- $\beta(\theta_1 - \theta_2)$: the scattering cross-section

I.3. The frontal collision

Discontinuity of the mid-point



$$\theta_1 \neq \theta_2 + \pi \quad \Longrightarrow \quad e^{i\Theta_{mid}} = \frac{e^{i\theta_1} + e^{i\theta_2}}{|e^{i\theta_1} + e^{i\theta_2}|},$$

$$\Theta_{mid} = \begin{cases} \frac{\theta_1+\theta_2}{2} & |\theta_1 - \theta_2| < \pi, \\ \frac{\theta_1+\theta_2}{2} + \pi & |\theta_1 - \theta_2| > \pi, \end{cases} \qquad 0 \leqslant \theta_1, \theta_2 \leqslant 2\pi.$$

I.4. References

- E. Bertin; M. Droz; G. Gregoire. Boltzmann and hydrodynamic description for self-propelled particles. Phys. Rev. E, Vol. 74, 022101 (2006).
- E. Bertin; M. Droz; G. Gregoire. Hydrodynamic equations for self-propelled particles: microscopic derivation and stability analysis. J. Phys. A Math. Theor. Vol. 42, 445001 (2009).
- E. Carlen; M. C. Carvalho; P. Degond; B. Wenneberg. A Boltzmann model for rod alignment and schooling fish. Nonlinearity, Vol. 28 (2015), no.6, 1783-1803.
- P. Degond; A. Frouvelle; G. Raoul. Local stability of perfect alignment for a spatially homogeneous kinetic model. J. Stat. Phys. Vol. 157 (2014), no.1, 84-112.

I.5. Main hypotheses

Homogeneous Maxwell BDG equation in $\mathbb{R}^d$:

$$\partial_t f(dv) = Q[f,f]$$

$$Q[f,f](dv) = \iint \tau\Big[\frac{v_1+v_2}{2}\Big]\sharp g(dv)\, f(dv_1)d(dv_2) - f(dv)\int f(dv_1)$$

$$Q[f,f] = Q_+[f,f] - Q_-[f,f]$$

$$Q_+[f,f] = g \star (\lambda[1/2]\sharp f) \star (\lambda[1/2]\sharp f),$$

- $\tau[w] : v \mapsto v + w$
- $\lambda[1/2] : v \mapsto \frac{1}{2} v$
- $\beta = 1$
- $g(dw)$: a centered probability measure

I.6. Simple observations

Remark

Let $g \in \mathscr{P}(\mathbb{R}^d)$ satisfying $\int v g(dv) = 0$.

- Mass and momentum are preserved

$$\int \begin{bmatrix} 1 \\ v \end{bmatrix} Q[f,f](dv) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

- The energy is not preserved
- Without loss of generality we may assume

$$\int f(dv) = 1, \quad \int v f(dv) = 0$$

- The BDG model has the form

$$\partial_t f = Q_+[f,f] - f$$

I.7. Mild solution

Definition

A mild solution is a continuous function $f \in C^0([0,+\infty), \mathscr{P}_2(\mathbb{R}^d))$ equipped with the Wasserstein 2 metric satisfying

$$f(t,\cdot) = e^{-t} f(0,\cdot) + \int_0^t e^{-(t-s)} Q_+[f(s,\cdot), f(s,\cdot)]\, ds.$$

The Wasserstein 2 metric is

$$W_2^2(f_1,f_2) = \inf\Big\{ \int |v_1 - v_2|^2\, \pi(dv_1,dv_2) : \\ \pi \in \mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d), p_1\sharp\pi = f_1,\ p_2\sharp\pi = f_2 \Big\}$$

I.8. Main result

Theorem (Ayot, Brull, Th)

Let $g \in \mathscr{P}_2(\mathbb{R}^d, 0)$. *Then for every* $f_0 \in \mathscr{P}_2(\mathbb{R}^d, 0)$

1. *There exists a unique mild solution* $f \in C^0([0, +\infty), \mathscr{P}_2(\mathbb{R}^d, 0))$.
2. *There exists a unique equilibrium state*

$$Q[f_\infty, f_\infty] = 0, \quad f_\infty \in \mathscr{P}_2(\mathbb{R}^d, 0)$$

3. *The convergence is exponential*

$$W_2^2(f(t,\cdot), f_\infty) \leqslant e^{-t/2} W_2^2(f_0, f_\infty).$$

4. *Let* $s \geqslant (d+4)/2$. *If in addition* $g, f_0 \in H^s(\mathbb{R}^d)$ *then*

$$\|f(t,\cdot) - f_\infty\|_{L^1(\mathbb{R}^d)} \leqslant C e^{-t/(d+4)}.$$

Summary

II.1. A fixed point approach

Definition

A mild solution $f \in C^0([0,+\infty), \mathscr{P}_2(\mathbb{R}^d,0))$ is a fixed point of

$$f(t,\cdot) = e^{-t} f(0,\cdot) + \int_0^t e^{-(t-s)} T[f(s,\cdot)]\, ds,$$

where by denoting $\lambda[1/2] : v \mapsto v/2$

$$T[f](dv) = Q_+[f,f](dv) = \iint \tau[(v_1+v_2)/2]\sharp g\, f(dv_1) f(dv_2)$$

$$T[f] = g \star (\lambda[1/2]\sharp f) \star (\lambda[1/2]\sharp f).$$

A mild solution is a fixed point

$$\Phi[f] = f$$

II.2. Properties of W_2

Lemma

Let $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathbb{P}_2(\mathbb{R}^d)$*. Then*

$$W_2^2(\mu_1 \star \mu_2, \nu_1 \star \nu_2) \leqslant W_2^2(\mu_1, \nu_1) + W_2^2(\mu_2, \nu_2),$$

$$W_2^2(\lambda[1/2]\sharp\mu_1, \lambda[1/2]\sharp\nu_1) \leqslant \frac{1}{4} W_2^2(\mu, \nu).$$

Corollary

Let $T[f] = Q_+[f, f] = g \star (\lambda[1/2]\sharp f) \star (\lambda[1/2]\sharp f)$*. Then*

$$W_2^2(T[f_1], T[f_2]) \leqslant \frac{1}{2} W_2^2(f_1, f_2).$$

II.3. Proof of Theorem ABT. Item 1.

Theorem (ABT. Item 1)

Let $g, f_0 \in \mathscr{P}_2(\mathbb{R}^d, 0)$. Then there exists a unique mild solution.

Proof.

Recall $T[f] = g \star (\lambda[1/2]\sharp f) \star (\lambda[1/2]\sharp f)$. Define

$$\Phi[f](t) = e^{-t} f(0) + \int_0^t e^{-(t-s)} T[f(s)]\, ds.$$

Then Φ is contracting in the Wasserstein distance.

$$W_2^2(\Phi[f_1](t), \Phi[f_2](t)) \leqslant e^{-t} W_2^2(f_1(0), f_2(0)) + \frac{1}{2}\int_0^t e^{-(t-s)} W_2^2(f_1(s), f_2(s))\, ds,$$

We obtain the contraction property

$$\sup_{t\geqslant 0} W_2^2(\Phi[f_1](t), \Phi[f_2](t)) \leqslant \frac{1+e^{-t}}{2} \sup_{t\geqslant 0} W_2^2(f_1(t), f_2(t)).$$

II.4. Proof of Theorem ABT. Item 1. last and final

Proof. Last and final.

The previous proof implies existence, uniqueness of the mild solution. The same computations give a stability result

$$W_2^2(f_1(t), f_2(t)) \leqslant e^{-t/2} W_2^2(f_1(0), f_2(0))$$

and exponential convergence to the unique equilibrium state

$$W_2^2(f_1(t), f_\infty) \leqslant e^{-t/2} W_2^2(f_1(0), f_\infty)$$

□

Remark

Compare to CCDW the equation is always stable with a unique equilibrium state.

II.5. Higher regularity in H^s

Lemma

Let $s \geqslant 0$. If $g, f_0 \in H^s(\mathbb{R}^d) \cap \mathscr{P}_2(\mathbb{R}^d, 0)$ then the mild solution belongs to $H^s(\mathbb{R}^d)$ at all time and

$$\|f(t)\|_{H^s(\mathbb{R}^d)} \leqslant e^{-t}\|f_0\|_{H^s(\mathbb{R}^d)} + (1 - e^{-t})\|g\|_{H^s(\mathbb{R}^d)}.$$

Proof.

The L^2 Sobolev norm is defined by

$$Z_R(t) = \int_{\|\xi\| \leqslant R} (1 + \|\xi\|^2)^s |\hat{f}(t,\xi)|^2 \, d\xi$$

Then, thanks to $\beta = 1$,

$$\partial_t \hat{f}(t,\xi) = \hat{g}(\xi)\hat{f}\left(t, \frac{\xi}{2}\right)^2 - \hat{f}(t,\xi),$$

$$\frac{d}{dt} Z_R(t) \leqslant -2Z_R(t) + 2\|g\|_{H^s(\mathbb{R}^d)}\sqrt{Z_R(t)}.$$

The end of the proof uses Gronwall's lemma.

II.6. Fourier-Toscani distance

Remark

The previous estimate does not imply the L^2 convergence to the equilibrium state.

Definition

If $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^d, m)$ (same mass, same moment of order 1) then

$$d_2(\mu, \nu) = \sup_{\xi \in \mathbb{R}^d} \frac{|\hat{\mu}(\xi) - \hat{\nu}(\xi)|}{|\xi|^2}.$$

Property

- Convexity

$$d_2(\alpha f_1 + (1-\alpha) f_2, \alpha g_1 + (1-\alpha) g_2) \leqslant \alpha d_2(f_1, g_1) + (1-\alpha) d_2(f_2, g_2)$$

- Sub-additivity

$$d_2(f_1 \star f_2, g_1 \star g_2) \leqslant d_2(f_1, g_1) + d_2(f_2, g_2)$$

II.7. Exponential convergence for d_2

Lemma

If $g, f_0 \in \mathscr{P}_2(\mathbb{R}^d, 0)$ and $t \in [0, +\infty) \mapsto f(t) \in \mathscr{P}_2(\mathbb{R}^d, 0)$ is the mild solution, then

$$d_2(f(t), f_\infty) \leqslant \frac{M_2(f_0) + 2M_2(g)}{2} e^{-t/2}.$$

Proof.

let $H(t,\xi) = |\hat{\mu}(\xi) - \hat{\nu}(\xi)|/|\xi|^2$. Then

$$e^{-i\langle v,\xi\rangle} = 1 - i\langle v,\xi\rangle - \langle v,\xi\rangle^2 \int_0^1 (1-s) e^{-is\langle v,\xi\rangle} ds,$$

$$\frac{\partial H}{\partial t} = G(t,\xi) H\left(t, \frac{\xi}{2}\right) - H(t,\xi),$$

$$G(t,\xi) := \frac{\hat{g}(\xi)}{4}\left[\hat{f}\left(t,\frac{\xi}{2}\right) + \hat{f}_m^\infty\left(\frac{\xi}{2}\right)\right]$$

II.8. Exponential convergence in L^2

Lemma

If $s > (d+4)/2$, $f_1, f_2 \in H^s(\mathbb{R}^d)$ *satisfies* $\int f_1(v)dv = \int f_2(v)\,dv$, $\int f_1(v)v\,dv = \int f_2(v)v\,dv$, *then*

$$\|f_1 - f_2\|^2_{L^2(\mathbb{R}^d)} \leqslant C(s) d_2(f_1, f_2) \|f_1 - f_2\|_{H^s(\mathbb{R}^d)}$$

Proof.

A simple application of Cauchy-Schwarz

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\hat{f}_1(\xi) - \hat{f}_2(\xi)|^2 d\xi \\
&\leqslant \int_{\mathbb{R}^d} \frac{|\hat{f}_1(\xi) - \hat{f}_2(\xi)|}{|\xi|^2} \Big(\frac{1 + |\xi|^2}{(1+|\xi|^2)^{s/2}} \Big) (1 + |\xi|^2)^{s/2} |\hat{f}_1(\xi) - \hat{f}_2(\xi)| d\xi \\
&\leqslant d_2(f_1, f_2) \Big(\int_{\mathbb{R}^d} \frac{d\xi}{(1+|\xi|^2)^{s-2}} \Big)^{1/2} \|f_1 - f_2\|_{H^s(\mathbb{R}^d)}.
\end{aligned}$$

$\square$

II.9. Proof of Theorem ABT. Item 4.

Theorem (ABT. Item 4)

Let $s \geqslant (d+4)/2$. If $g, f_0 \in H^s(\mathbb{R}^d) \cap \mathscr{P}_2(\mathbb{R}^d, 0)$ then

$$\|f(t,\cdot) - f_\infty\|_{L^1(\mathbb{R}^d)} \leqslant C e^{-t/(d+4)}.$$

Proof.

Let $f : [0, +\infty) \mapsto \mathscr{P}(\mathbb{R}^d, 0)$

$$\|f\|_{L^1} \leqslant C \|f\|_{L^2}^{4/(d+4)} M_2(f)^{d/(4+d)} \tag{1}$$

$$\sup_{t \geqslant 0} \|f(t)\|_{H^s} < +\infty \tag{2}$$

$$\|f(t) - f_\infty\|_{L^2}^2 \leqslant C d_2(f(t), f_\infty) \|f(t) - f_\infty\|_{H^s} \tag{3}$$

$$\sup_{t \geqslant 0} M_2(f(t)) < +\infty \tag{4}$$

$$\|f(t) - f_\infty\|_{L^1} \leqslant C d_2(f(t), f_\infty)^{2/(4+d)} \tag{5}$$

$$d_2(f(t), f_\infty) \leqslant C e^{-t/2} \tag{6}$$

$$\|f(t) - f_\infty\|_{L^1} \leqslant C e^{-t/(d+4)} \tag{7}$$

Summary

- I. BDG model
- II. Ideas of proof
- **III. Some numerical experiments**

III.1. A general test case

Recall: BDG with a non zero scattering cross section

$$\partial_t f(dv) = Q[f,f]$$
$$Q[f,f](dv) = \iint \tau\Big[\frac{v_1+v_2}{2}\Big]\sharp g(dv)\beta(v_1-v_2)\, f(dv_1)d(dv_2) - f(dv)\int \beta(v-v_1)f(dv_1)$$

Lemma

Assume

$$g(v) = \frac{1}{\sqrt{2\pi\sigma_g^2}}\exp(-\frac{v^2}{2\sigma_g^2}), \quad \beta(v) = \exp(-\frac{v^2}{2b^2}).$$

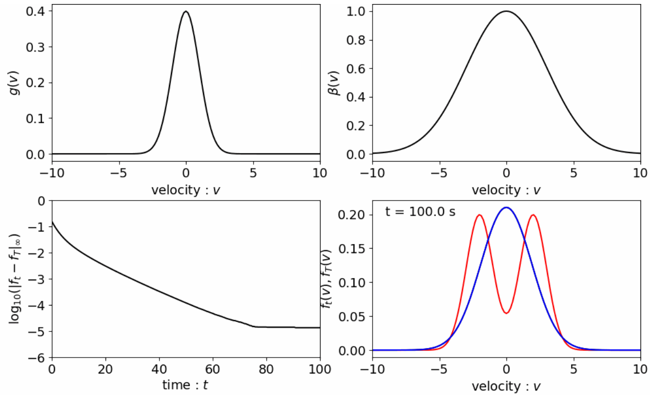
Assume $b^2 > 4\sigma_g^2$. *Then there exists a Gaussian equilibrium state* f_∞ *given by*

$$f_\infty(v) = \frac{1}{\sqrt{2\pi\sigma_f^2}}\exp\Big(-\frac{v^2}{2\sigma_f^2}\Big) \quad \textit{with} \quad \frac{1}{\sigma_f^2}+\frac{2}{b^2} = \frac{1}{2\sigma_g^2}.$$

III.2. Experiment 1.

$$g(v) = \frac{1}{\sqrt{2\pi(1.0)^2}} \exp\left(-\frac{v^2}{2(1.0)^2}\right), \quad \beta(v) = \exp\left(-\frac{v^2}{2(3.0)^2}\right)$$

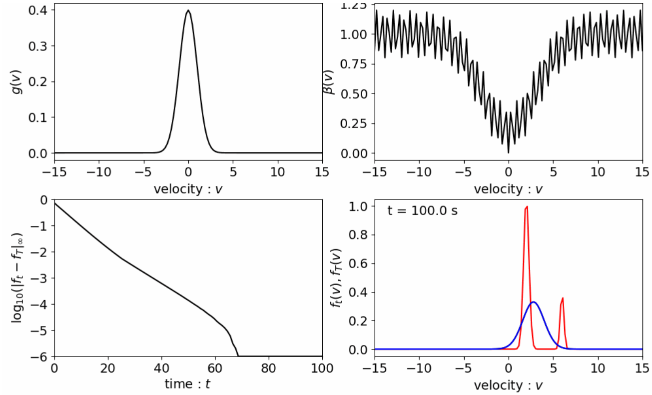
$$f_0(v) \propto \left[\exp\left(-\frac{(v-2)^2}{2(1.0)^2}\right) + \exp\left(-\frac{(v+2)^2}{2(1.0)^2}\right)\right]$$



II.3. Experiment 2.

$$g(v) = \frac{1}{\sqrt{2\pi(1.0)^2}} \exp(-\frac{v^2}{2(1.0)^2}), \quad \beta(v) = 1 - 0.8\exp\left(-\frac{v^2}{2(5.0)^2}\right) - 0.2\cos(10v)$$

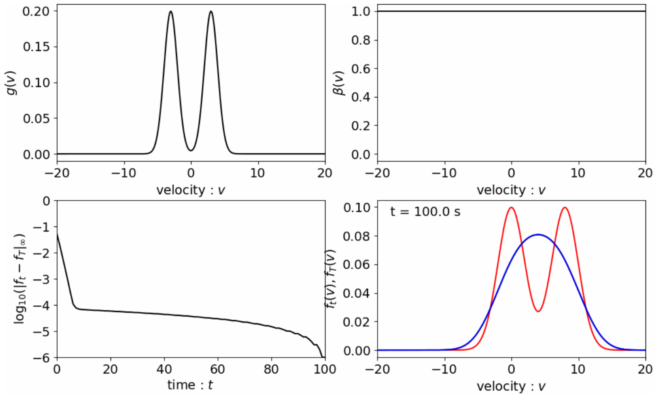
$$f(0,v) \propto \left(\frac{1}{\sqrt{2\pi(0.2)^2}} \exp\left(-\frac{(v-6)^2}{2(0.2)^2}\right) + \frac{4}{\sqrt{2\pi(0.3)^2}} \exp\left(-\frac{(v-2)^2}{2(0.3)^2}\right)\right)$$



II.4. Experiment 3.

$$g(v) = \frac{1/2}{\sqrt{2\pi(1.0)^2}}\left[\exp(-\frac{(v-3)^2}{2(1.0)^2}) + \exp(-\frac{(v+3)^2}{2(1.0)^2})\right], \quad \beta(v) = 1$$

$$f(0,v) \propto \left(\exp\left(-\frac{v^2}{2(2.0)^2}\right) + \exp\left(-\frac{(v-8)^2}{2(2.0)^2}\right)\right)$$



II.5. Conclusion and perspective

Recall: BDG non homogeneous with a non zero scattering cross section

$$\partial_t f(dv) + v \cdot \nabla_x f = Q[f,f]$$
$$Q[f,f](dv) = \iint \tau\Big[\frac{v_1+v_2}{2}\Big]\sharp g(dv)\beta(v_1-v_2)\, f(dv_1)d(dv_2) - f(dv)\int \beta(v-v_1)f(dv_1)$$

- We only studied the homogeneous case and in the Maxwell setting $\beta = 1$
- Using Schauder fixed point theorem, there exists an equilibrium

$$Q[f_\infty, f_\infty] = 0, \quad f_\infty \in \mathscr{P}_2(\mathbb{R}^d, m), \quad M_2(f_\infty) \leqslant \frac{2M_2(g)}{1-\|1-\beta\|_\infty} + m^2$$

II.6. Conclusion and perspective

- We don't know the regularity of f_∞ for general β
- We can't prove the convergence (even in the homogeneous case)
- We have not studied the Cauchy problem for the non homogeneous case (even for $\beta = 1$)
- For the true model where the velocities have norm 1, the Cauchy problem and the convergence to some equilibrium is known only for $g = \delta_0$ and $\beta = 1$
- The hydrodynamics limit would be interesting
- The comparison with traffic flow, opinion formation model should be done

Thank you