DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume 22, Number 5, July 2017

doi:10.3934/dcdsb.2017115

pp. 1965–1975

TOPOLOGICAL STABILITY IN SET-VALUED DYNAMICS

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Abstract. We propose a definition of topological stability for set-valued maps. We prove that a single-valued map which is topologically stable in the set-valued sense is topologically stable in the classical sense [14]. Next, we prove that every upper semicontinuous closed-valued map which is positively expansive [15] and satisfies the positive pseudo-orbit tracing property [9] is topologically stable. Finally, we prove that every topologically stable set-valued map of a compact metric space has the positive pseudo-orbit tracing property and the periodic points are dense in the nonwandering set. These results extend the classical single-valued ones in [1] and [14].

1. **Introduction.** The topological dynamics of set-valued maps has been studied recently in the literature. For instance, [4], [5] and [8] introduced the metric and topological entropies for set-valued maps. In [11] it is defined the specification and topologically mixing properties for set-valued maps. In [6] it is considered the continuum-wise expansivity for set-valued maps.

In this paper we will propose a definition of topological stability for set-valued maps. We prove that a single-valued map which is topologically stable in the set-valued sense is topologically stable in the classical sense [14]. Next, we prove that every upper semicontinuous closed-valued map which is positively expansive [15] and satisfies the positive pseudo-orbit tracing property [9] is topologically stable. Finally, we prove that every topologically stable set-valued map of a compact metric space has the positive pseudo-orbit tracing property and the periodic points are dense in the nonwandering set. These results extend the classical single-valued ones in [1] and [14].

2010 *Mathematics Subject Classification.* Primary: 54H20; Secondary: 54C60.

Key words and phrases. Topological stability, set-valued map, metric space.

Work partially supported by CNPq from Brazil and MATHAMSUD 15 MATH05-ERGOPTIM, Ergodic Optimization of Lyapunov Exponents.

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2. **Definitions and results.** We start this section by introducing the concept of topologically stable set-valued map. This will require some basic notions of set-valued analysis [2]. Afterwards, we state our results.

Let X denote a metric space. Denote by 2^X the set formed by the subsets of X. By a *set-valued map* of X we mean a map $f : X \to 2^X$. We say that f is *single-valued* if $card(f(x)) = 1$ for every $x \in X$, where $card(\cdot)$ denotes cardinality. There is an obvious correspondence between single-valued maps $f : X \to 2^X$ and maps $f : X \to X$. In what follows all set-valued maps will be assumed to be *strict*, i.e., $f(x) \neq \emptyset$ for every $x \in X$. A set-valued map f is *closed-valued* if $f(x)$ is closed for every $x \in X$. We say that f is *upper semicontinuous* if for every $x \in X$ and every neighborhood $\mathcal{U}$ of $f(x)$ there is $\eta > 0$ such that $f(x') \subset \mathcal{U}$ for every $x' \in X$ satisfying $d(x, x') < \eta$. This definition reduces to the usual continuity in the single-valued case.

The distance between single-valued maps f and g of X is defined by

$$d(f, g) = \sup_{x \in X} d(f(x), g(x)).$$

Next we present the classical definition of topologically stable single-valued map by Walters [14].

Definition 2.1. A continuous single-valued map $f : X \to X$ is topologically stable, in the class of continuous maps (or topologically stable for short), if for every $\epsilon > 0$ there is $\delta > 0$ such that for every continuous map $g : X \to X$ with $d(f, g) < \delta$ there is a continuous map

$$\hat{h} : X \to X$$

such that

$$d(\hat{h}, Id_X) < \epsilon \quad \text{and} \quad f \circ \hat{h} = \hat{h} \circ g,$$

where $Id_X : X \to X$ is the identity.

To extend this definition to the set-valued context we require further notations. Given $A, B \subset X$ we define the distance

$$d(A, B) = \inf\{d(a, b) : (a, b) \in A \times B\},$$

and the *Hausdorff distance*

$$d_H(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in C} d(b, A)\right\}.$$

The distance between the set-valued maps f and g of X is defined by

$$d_H(f, g) = \sup_{x \in X} d_H(f(x), g(x)).$$

Notice that $d_H(f, g)$ reduces to the distance $d(f, g)$ when the involved set-valued maps f and g are single-valued.

In what follows $\mathbb{N}$ *will denote the set of nonnegative integers, i.e.,* $\mathbb{N} = \{0, 1, 2, \dots\}$.

Denote by

$$X^{\mathbb{N}} = \prod_{n \in \mathbb{N}} X$$

the infinite product of copies of X, equipped with the distance

$$d^*((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}) = \sum_{n \in \mathbb{N}} 2^{-n-1} d(x_n, y_n). \tag{1}$$

Another distance to be considered in $X^{\mathbb{N}}$ is

$$D((x)_{n\in\mathbb{N}},(y_n)_{n\in\mathbb{N}})=\sup_{n\in\mathbb{N}} d(x_n,y_n). \tag{2}$$

We say that $(x_n)_{n\in\mathbb{N}}\in X^{\mathbb{N}}$ is an *orbit* of a set-valued map f (or an f*-orbit* for short) if

$$x_{n+1}\in f(x_n), \qquad \forall n\in\mathbb{N}.$$

The set $\varprojlim f$ formed by the f-orbits is often called the *inverse limit space* induced by f (cf. [8]). The name inverse limit system is also used (cf. [1]). Precisely,

$$\varprojlim f=\{(x_n)_{n\in\mathbb{N}}\in X^{\mathbb{N}}:x_{n+1}\in f(x_n),\forall n\in\mathbb{N}\}.$$

It turns out that f induces a map, to be called *left shift*

$$\sigma_f:\varprojlim f\to\varprojlim f,$$

defined by

$$\sigma_f((x_n)_{n\in\mathbb{N}})=(x_{n+1})_{n\in\mathbb{N}}.$$

Let $\pi:X^{\mathbb{N}}\to X$ the projection in the first variable, i.e., $\pi((x_n)_{n\in\mathbb{N}})=x_0$. Define the map $\pi_f:\varprojlim f\to X$ as the restriction of π to $\varprojlim f$.

Now we present our definition of topologically stable set-valued map.

Definition 2.2. An upper semicontinuous closed-valued map f of X is topologically stable, in the class of upper semicontinuous closed-valued maps (or *topologically stable* for short), if for every $\epsilon>0$ there is $\delta>0$ such that for every upper semicontinuous closed-valued map g with $d_H(f,g)<\delta$ there is a continuous map

$$h:(\varprojlim g,d^*)\to(\varprojlim f,d^*)$$

such that

$$D(h,Id_X)<\epsilon \quad\text{ and }\quad \sigma_f\circ h=h\circ\sigma_g,$$

where

$$D(h,Id_X)=\sup\{D(h(\mathbf{x}),\mathbf{x}):\mathbf{x}\in\varprojlim g\}.$$

The following remark holds.

Remark 2.1. An important difference between definitions 2.1 and 2.2 is that the domain of the semiconjugacy h in the latter definition depends on the perturbation g.

Since every continuous single-valued map is upper semicontinuous and closed valued as a set-valued map, it is natural to compare the definitions 2.1 and 2.2 in the single-valued context. This motivates the following result.

Theorem 2.1. *Every continuous single-valued map of a metric space which is topologically stable as a set-valued map (Definition 2.2) is topologically stable in the classical sense (Definition 2.1).*

Unfortunately we do not know if the converse of Theorem 2.1 holds, namely, if a single-valued map which is topologically stable in the classical sense (Definition 2.1) is also topologically stable when regarded as a set-valued map (Definition 2.2). The next theorem (and Example 2.1 below) give some light to this question.

Theorem 2.2. *Every topologically stable single-valued map f of a compact metric space X satisfies the following property:*

- *For every $\epsilon > 0$ there is $\delta > 0$ such that for every continuous single-valued map $g : X \to X$ with $d_H(f, g) < \delta$ there is a continuous map*

$$h : (\lim_{\leftarrow} g, d^*) \to (\lim_{\leftarrow} f, d^*)$$

  such that

$$D(h, Id_X) < \epsilon \quad \text{and} \quad \sigma_f \circ h = h \circ \sigma_g,$$

In [13] Walters proved that every positively expansive map with the positive pseudo-orbit tracing property of a compact metric space is topologically stable. Now we extend this result to the set-valued context. Previously we recall the concepts of positive expansivity and pseudo-orbit tracing property in the set-valued context.

Definition 2.3 ([15])**.** A set-valued map f of a metric space X is *positively expansive* if there is $\epsilon > 0$ (called *positive expansivity constant*) such that $x = y$ whenever $x, y \in X$ satisfy that there there are f-orbits $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ such that $x_0 = x$, $y_0 = y$ and $d(x_n, y_n) \leq \epsilon$ for every $n \in \mathbb{N}$. Sometimes we will say that f is *positively expansive with respect to* d to emphasize the metric d of X.

Definition 2.4 ([9])**.** We say that a set-valued map f of a metric space X has the *positive pseudo-orbit tracing property* (abbrev. $POTP_+$) if for every $\epsilon > 0$ there is $\delta > 0$ such that for each sequence $(p_n)_{n\in\mathbb{N}}$ in X satisfying

$$d(p_{n+1}, f(p_n)) \leq \delta, \qquad \forall n \in \mathbb{N},$$

there is an f-orbit $(q_n)_{n\in\mathbb{N}}$ satisfying

$$d(p_n, q_n) \leq \epsilon, \quad \forall n \in \mathbb{N}.$$

These definitions extend the classical single-valued ones by Utz [12], Eisenberg [7] and Bowen [3]. Using them we obtain the following set-valued version of Walters stability theorem [13].

Theorem 2.3. *Every upper semicontinuous positively expansive closed-valued map with the $POTP_+$ of a compact metric space is topologically stable.*

Let us present two examples where Theorem 2.3 applies.

Example 2.1. Let $f : X \to X$ a continuous positively expansive single-valued map with the POTP_+ of a compact metric space. Then, f is an upper semicontinuous positively expansive closed-valued map with the POTP_+. Hence, by Theorem 2.3, f is topologically stable not only as a single but also as a set-valued map.

A genuine (i.e. not single-valued) example where the theorem applies is as follows.

Example 2.2. Endow the unit interval $[0, 1]$ with the Euclidean metric. Define the set-valued map f of $[0, 1]$ by

$$f(x) = \begin{cases} \{2x\}, & \text{if} \quad 0 \leq x < \frac{1}{2} \\ \{0, 1\}, & \text{if} \quad x = \frac{1}{2} \\ \{2x - 1\}, & \text{if} \quad \frac{1}{2} < x \leq 1. \end{cases}$$

It follows that f is an upper semicontinuous positively expansive closed-valued map with the POTP_+ of $[0, 1]$. Therefore, by Theorem 2.3, f is a topologically stable set-valued map of $[0, 1]$.

Next we present a property of the topologically stable set-valued maps.

Given a set-valued map f of X, we say that $x \in X$ is a *periodic point* if there are an f-orbit $(x_n)_{n\in\mathbb{N}}$ and $m \in \mathbb{N}^+$ such that $x_0 = x$ and $x_{n+m} = x_n$ for every $n \in \mathbb{N}$. The set of periodic points is denoted by $Per(f)$. The *nonwandering set* of f is the set $\Omega(f)$ of those points $x \in X$ such that for every neighborhood U of x there is $m \in \mathbb{N}^+$ satisfying $U \cap f^m(U) \neq \emptyset$. With these definitions we obtain the following result.

Theorem 2.4. *Every topologically stable upper semicontinuous closed-valued map of a compact metric space has the POTP$_+$. Moreover, $Per(f)$ is dense in $\Omega(f)$.*

A short application of this theorem in the single-valued context is as follows. Recall that, on *every compact manifold*, every single-valued map f which is topologically stable *in the classical sense* has the POTP$_+$ and $Per(f)$ is dense in $\Omega(f)$. See for instance Theorem 2.4.8 in [1] or [13].

In the following corollary of Theorem 2.4 and Theorem 2.1 we obtain that, on *every metric space*, every single-valued map f which is topologically stable *as a set-valued map (Definition 1.4)* has the POTP$_+$ and $Per(f)$ is dense in $\Omega(f)$. In other words we have the following result.

Corollary 2.5. *Every continuous single-valued map f of a metric space which is topologically stable as a set-valued map (Definition 2.2) has the POTP$_+$. Moreover, $Per(f)$ is dense in $\Omega(f)$.*

3. **Proof of the theorems.**

In this section we will prove the theorems stated in the previous section. We start with a lemma about the left shift map for single-valued maps.

Lemma 3.1. *If f is a continuous single-valued map of a compact metric space X, then the left shift $\pi_f : (\varprojlim f, d^*) \to (X, d)$ is a homeomorphism.*

Proof. Since f is single-valued, one has $\pi_f((x)_n)_{n\in\mathbb{N}}) = x$ if and only if $x_n = f^n(x)$ for every $n \in \mathbb{N}$. Then, π_f is bijective with inverse $\pi_f^{-1}(x) = (f^n(x))_{n\in\mathbb{N}}$. Also, for fixed $\gamma > 0$, if $d^*((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) < \frac{\gamma}{2}$, then

$$d(\pi_f((x_n)_{n\in\mathbb{N}}, \pi_f((y_n)_{n\in\mathbb{N}}) = d(x_0, y_0) \leq 2d^*((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) < \gamma$$

proving that π_f is continuous.

On the other hand, for fixed $\gamma > 0$ we let $diam(X)$ denote the diameter of X and we let $n_0 \in \mathbb{N}$ be such that

$$\sum_{n\geq n_0} 2^{-n-1} diam(X) < \frac{\gamma}{2}.$$

Since f is continuous, there is $\rho > 0$ such that

$$\sum_{n=0}^{n_0-1} 2^{-n-1} d(f^n(x), f^n(y)) < \frac{\gamma}{2} \qquad \text{whenever } d(x, y) < \rho.$$

Then,

$$\begin{aligned} d^*(\pi_f^*(x), \pi_f^*(y)) &= d^*((f^n(x))_{n\in\mathbb{N}}, (f^n(y))_{n\in\mathbb{N}}) \\ &= \sum_{n\in\mathbb{N}} 2^{-n-1} d(f^n(x), f^n(y)) \end{aligned}$$

$$\begin{aligned} &\leq \sum_{n=0}^{n_0-1} 2^{-n-1} d(f^n(x), f^n(y)) + \sum_{n\geq n_0} 2^{-n-1} diam(X) \\ &< \frac{\gamma}{2} + \frac{\gamma}{2} = \gamma \end{aligned}$$

proving that π_f^{-1} is continuous. Then, $\pi_f : (\varprojlim f, d^*) \to (X, d)$ is a homeomorphism and the proof follows. □

With this lemma we can prove Theorem 2.1.

Proof of Theorem 2.1. Let f be a continuous map of a metric space X which is topologically stable as a set-valued map (Definition 2.2).

Fix $\epsilon > 0$ and let δ be given by that property. Take $g : X \to X$ continuous such that $d(f, g) < \delta$. Since f and g are single-valued, $d_H(f, g) = d(f, g)$ and so $d_H(f, g) < \delta$. Then, there is $h : (\varprojlim g, d^*) \to (\varprojlim f, d^*)$ continuous such that $D(h, Id_X) \leq \epsilon$ and $\sigma_f \circ h = h \circ \sigma_g$.

By Lemma 3.1, since both f and g are single-valued, we have that the maps $\pi_f : (\varprojlim f, d^*) \to (X, d)$ and $\pi_g : (\varprojlim g, d^*) \to (X, d)$ are homeomorphisms. Then, the composition $\hat{h} = \pi_f \circ h \circ \pi_g^{-1}$ defines a continuous map $\hat{h} : X \to X$. Since

$$d(\hat{h}(x), x) = d(\pi_f(h(\pi_g^{-1}(x))), x) = d(\pi_f(h((g^n(x)_{n\in\mathbb{N}})), x) \leq D(h, Id_X) \leq \epsilon$$

for every $x \in X$, one has $d(\hat{h}, Id_X) \leq \epsilon$.

In addition, since $f \circ \pi_f = \pi_f \circ \sigma_f$, one has

$$\begin{aligned} (f \circ \hat{h})(x) = f(\hat{h}(x)) &= f(\pi_f(h(\pi_g^{-1}(x)))) \\ &= f(\pi_f(h((g^n(x))_{n\in\mathbb{N}}))) \\ &= \pi_f(\sigma_f(h((g^n(x))_{n\in\mathbb{N}}))) \\ &= \pi_f(h(\sigma_g((g^n(x))_{n\in\mathbb{N}}))) \\ &= \pi_f(h((g^{n+1}(x))_{n\in\mathbb{N}})) \\ &= (\pi_f \circ h \circ \pi_g^{-1})(g(x)) = (\hat{h} \circ g)(x) \end{aligned}$$

i.e., $f \circ \hat{h} = \hat{h} \circ g$. Then, f is topologically stable according to Definition 2.1. □

Next we prove Theorem 2.2.

Proof of Theorem 2.2. Fix $\epsilon > 0$ and let δ be given by the topological stability of f. Take $g : X \to X$ continuous such that $d_H(f, g) < \delta$. Then, $d(f, g) < \delta$ and so there is $\hat{h} : X \to X$ continuous such that $d(\hat{h}, Id_X) \leq \epsilon$ and $f \circ \hat{h} = \hat{h} \circ g$.

On the other hand, by Lemma 3.1 we have that $\pi_f : (\varprojlim, d^*) \to (X, d)$ and $\pi_g : (\varprojlim g, d^*) \to (X, d)$ are homeomorphisms. Then, since $\hat{h}$ is continuous, the composition $h = \pi_f^{-1} \circ \hat{h} \circ \pi_g$ defines a continuous map $h : (\varprojlim g, d^*) \to (\varprojlim f, d^*)$.

Since g is single-valued, $x_n = g^n(x_0)$ for all $(x_n)_{n\in\mathbb{N}} \in \varprojlim g$ and $n \in \mathbb{N}$. Then,

$$D(h((x_n)_{n\in\mathbb{N}}), (x_n)_{n\in\mathbb{N}}) = \sup_{n\in\mathbb{N}} d(\hat{h}(g^n(x_0)), g^n(x_0)) \leq \epsilon$$

for all $(x_n)_{n\in\mathbb{N}} \in \varprojlim g$ proving $D(h, Id_X) \leq \epsilon$.

Moreover,

$$\begin{aligned}(\sigma_f\circ h)((x_n)_{n\in\mathbb{N}}) &= \sigma_f(h((x_n)_{n\in\mathbb{N}})))\\ &= \sigma_f(\pi_f^{-1}(\hat{h}(\pi_g((x_n)_{n\in\mathbb{N}})))\\ &= \pi_f^{-1}(f(\hat{h}x_0)))\\ &= \pi_f^{-1}(\hat{h}(g(x_0)))\\ &= (\pi_f^{-1}\circ\hat{h}\circ\pi_g)(\sigma_g((x_n)_{n\in\mathbb{N}}))\\ &= (h\circ\sigma_g)((x_n)_{n\in\mathbb{N}})\end{aligned}$$

proving $\sigma_f\circ h = h\circ\sigma_g$. Since ϵ is arbitrary, f satisfies the required property and the proof follows. □

To prove the remainder theorems we need some short preliminars. The first one is a basic property of the upper semicontinuous closed valued maps (see Proposition 1.4.8 in [2]).

Lemma 3.2. *Let f be an upper semicontinuous closed-valued map of a compact metric space X. If $(a^k)_{k\in\mathbb{N}}$ and $(b^k)_{k\in\mathbb{N}}$ are sequences such that $a^k\to a$, $b^k\to b$ and $a^k\in f(b^k)$ for all $k\in\mathbb{N}$, then $a\in f(b)$.*

Since $\lim\limits_{\leftarrow} f = \pi_f^{-1}(X)$ we obtain the following lemma.

Lemma 3.3. *The limit inverse space $(\lim\limits_{\leftarrow} f, d^*)$ of an upper semicontinuous closed-valued map f of a compact metric space X is a compact subset of $(X^{\mathbb{N}}, d^*)$.*

For the next lemma we will use an auxiliary definition.

Definition 3.1. We say that a set-valued map f of a metric space X has the *finite shadowing property* if for every $\epsilon>0$ there is $\delta>0$ such that for every finite set $\{p_0,\cdots,p_m\}$ satisfying $d(p_{n+1}, f(p_n))<\delta$ for every $0\leq n\leq m-1$ there is a finite set $\{q_0,\cdots,q_m\}$ such that $q_{n+1}\in f(q_n)$ and $d(p_n,q_n)<\epsilon$ for every $0\leq n\leq m-1$.

With this definition we obtain the following result.

Lemma 3.4. *An upper-semicontinuous closed-valued map of a compact metric space has the $POTP_+$ if and only if if has the finite shadowing property.*

Proof. We only need to prove the sufficiency. Let f be an upper semicontinuous closed-valued map with the finite shadowing property of a compact metric space X. Let $\epsilon>0$ be given. Find a corresponding $\delta>0$ given by the finite shadowing property. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence satisfying $d(p_{n+1}, f(p_n))\leq\delta$ for every $n\in\mathbb{N}$. Then, by finite shadowing, for every $m\in\mathbb{N}$ there is a sequence $\{q_0^m,\cdots,q_m^m\}$ such that $q_{n+1}^m\in f(q_n^m)$ and $d(p_n,q_n^m)\leq\epsilon$ for every $0\leq n\leq m$. Since X is compact, we can assume by passing to subsequences if necessary that there is a sequence $(q_n)_{n\in\mathbb{N}}$ such that $q_n^m\to q_n$ as $m\to\infty$ for every $n\in\mathbb{N}$. Since f is upper semicountinuous, closed-valed and X is compact, Lemma 3.2 implies $(q_n)_{n\in\mathbb{N}}\in\lim\limits_{\leftarrow} f$. By fixing n in $d(p_n,q_n^m)\leq\epsilon$ and letting $m\to\infty$ we obtain $d(p_n,q_m)\leq\epsilon$ for every $n\in\mathbb{N}$. Then, f has the POTP_+ proving the result. □

The next lemma is about the expansivity of the shift map for positively expansive set-valued maps.

Lemma 3.5. *If f is a positively expansive set-valued map of a metric space X, then the left shift $\sigma_f : \varprojlim f \to \varprojlim f$ is positively expansive with respect to the metric d^* in (1).*

Proof. Let ϵ be a positive expansivity constant of f. Take $(x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}} \in \varprojlim f$ such that

$$d^*(\sigma_f^k((x_n)_{n\in\mathbb{N}}), \sigma_f^k((x'_n)_{n\in\mathbb{N}})) \leq 2^{-1}\epsilon, \qquad \forall k \in \mathbb{N}.$$

It follows that

$$\sum_{n\in\mathbb{N}} 2^{-n-1} d(x_{n+k}, x'_{n+k}) \leq 2^{-1}\epsilon, \qquad \forall k \in \mathbb{N}.$$

Since

$$2^{-1} d(x_k, x'_k) \leq \sum_{n\in\mathbb{N}} 2^{-n-1} d(x_{n+k}, x'_{n+k})$$

we obtain

$$d(x_k, x'_k) \leq \epsilon, \qquad \forall k \in \mathbb{N}.$$

Since ϵ is a positive expansivity constant of f, $(x_k)_{k\in\mathbb{N}} = (x'_k)_{k\in\mathbb{N}}$ so σ_f is positively expansive. □

The following result is the positively expansive version of Lemma 2 in [13] (with similar proof).

Lemma 3.6. *Let $r : Y \to Y$ be a positively expansive continuous map of a compact metric space Y. Then, for every positive expansivity constant $\hat{e}$ and every $\Delta > 0$ there is $N \geq 1$ such that $d(x, y) \leq \Delta$ whenever $x, y \in Y$ satisfy $d(r^k(x), r^k(y)) \leq \hat{e}$ for every $0 \leq k \leq N$.*

Next we prove the continuity of the left shift.

Lemma 3.7. *For every set-valued map g of a metric space X, the left shift $\sigma_g : (\varprojlim g, d^*) \to (\varprojlim g, d^*)$ is continuous.*

Proof. If $(x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}} \in \varprojlim g$, then

$$\begin{aligned} d^*(\sigma_g((x_n)_{n\in\mathbb{N}})\sigma_g((x'_n)_{n\in\mathbb{N}}) &= d^*((x_{n+1})_{n\in\mathbb{N}}, (x'_{n+1})_{n\in\mathbb{N}}) \\ &= \sum_{n\in\mathbb{N}} 2^{-n-1} d(x_{n+1}, x'_{n+1}) \\ &= \sum_{n\geq 1} 2^{-n} d(x_n, x'_n) \\ &\leq 2\sum_{n\in\mathbb{N}} 2^{-n-1} d(x_n, x'_n) \\ &= 2d^*((x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}}) \end{aligned}$$

proving

$$d^*(\sigma_g((x_n)_{n\in\mathbb{N}}), \sigma_g((x'_n)_{n\in\mathbb{N}}) \leq 2d^*((x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}}), \quad \forall (x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}} \in \varprojlim g.$$

This completes the proof. □

Proof of Theorem 2.3. Let f be an upper semicontinuous positively expansive closed-valued map with the POTP$_+$ of a compact metric space X. It follows from Lemma 3.5 that the left shift $\sigma_f : \varprojlim f \to \varprojlim f$ is positively expansive with respect to the metric d^* in (1). Let $\hat{e}$ be the corresponding positive expansivity constant.

Fix $\epsilon > 0$ and let δ be given from POTP$_+$ for the constant $\epsilon_0 = \frac{\min\{\epsilon, e, \hat{e}\}}{8}$, where e is the positive expansivity constant of the set-valued map f. Fix a set-valued map g such that $d_H(f, g) \leq \frac{\delta}{8}$.

Let $(x_n)_{n\in\mathbb{N}}$ be a g-orbit. Since $d_H(g(x_0), f(x_0)) \leq \frac{\delta}{8}$ (by hypothesis) and $x_1 \in g(x_0)$, we have

$$d(x_1, f(x_0)) < \delta.$$

Similarly, since $d_H(g(x_1), f(x_1)) \leq \frac{\delta}{8}$ and $x_2 \in g(x_1)$, we have

$$d(x_2, f(x_1)) < \delta.$$

Repeating this argument we conclude that

$$d(x_{n+1}, f(x_n)) < \delta, \qquad \forall n \in \mathbb{N}.$$

Then, by the POTP$_+$ and the choice of δ, there is an f-orbit $(y_n)_{n\in\mathbb{N}}$ such that

$$d(x_n, y_n) \leq \epsilon_0, \qquad \forall n \in \mathbb{N}. \tag{3}$$

It turns out that this f-orbit is unique. Indeed, any other f-orbit $(y'_n)_{n\in\mathbb{N}}$ satisfying

$$d(x_n, y'_n) \leq \epsilon_0, \qquad \forall n \in \mathbb{N},$$

must satisfy

$$d(y_n, y'_n) \leq 2\epsilon_0 = \frac{\min\{\epsilon, e, \hat{e}\}}{4} < e, \qquad \forall n \in \mathbb{N},$$

and so $(y_n)_{n\in\mathbb{N}} = (y'_n)_{n\in\mathbb{N}}$ because e is a positive expansivity constant of f.

From this uniqueness, we obtain a map $h : \underleftarrow{\lim}\, g \to \underleftarrow{\lim}\, f$ given by $h((x_n)_{n\in\mathbb{N}}) = (y_n)_{n\in\mathbb{N}}$. It follows from (3) that

$$D(h, Id_X) \leq \epsilon.$$

On the other hand, replacing n by $n+1$ in (3) we get $d(x_{n+1}, y_{n+1}) \leq \epsilon_0$ for every $n \in \mathbb{N}$. Then, $(y_{n+1})_{n\in\mathbb{N}} = h((x_{n+1})_{n\in\mathbb{N}})$ and so

$$\sigma_f(h((x_n)_{n\in\mathbb{N}})) = (y_{n+1})_{n\in\mathbb{N}} = h((x_{n+1})_{n\in\mathbb{N}}) = h(\sigma_g((x_n)_{n\in\mathbb{N}})), \quad \forall (x_n)_{n\in\mathbb{N}} \in \underleftarrow{\lim}\, g.$$

This proves

$$\sigma_f \circ h = h \circ \sigma_g.$$

It remains to prove that h is continuous.

Fix $\Delta > 0$.

By lemmas 3.3 and 3.5 the map $\sigma_f : \underleftarrow{\lim}\, f \to \underleftarrow{\lim}\, f$ is a positively expansive map of the compact metric space $Y = (\underleftarrow{\lim}\, f, d^*)$. But $\sigma_f : (\underleftarrow{\lim}\, f, d^*) \to (\underleftarrow{\lim}\, f, d^*)$ is also continuous by Lemma 3.7. Then, we can apply Lemma 3.6 to obtain an integer $N \geq 1$ for the given Δ. Since $\sigma_g : (\underleftarrow{\lim}\, g, d^*) \to (\underleftarrow{\lim}\, g, d^*)$ is continuous and $(\underleftarrow{\lim}\, g, d^*)$ compact by lemmas 3.7 and 3.3 respectively, there is $\gamma > 0$ such that

$$d^*(\sigma_g^k((x_n)_{n\in\mathbb{N}}), \sigma_g^k((x'_n)_{n\in\mathbb{N}})) < \frac{\hat{e}}{4}, \qquad \forall 0 \leq k \leq N,$$

whenever $(x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}} \in \underleftarrow{\lim}\, g$ satisfy $d^*((x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}}) < \gamma$.

Then, whenever $(x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}} \in \lim\limits_{\leftarrow} g$ satisfy $d^*((x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}}) < \gamma$, one has for $(y_n)_{n\in\mathbb{N}} = h((x_n)_{n\in\mathbb{N}})$ and $(y'_n)_{n\in\mathbb{N}} = h((x'_n)_{n\in\mathbb{N}})$ that

$$\begin{aligned} d^*(\sigma_f^k((y_n)_{n\in\mathbb{N}}), \sigma_f^k((y'_n)_{n\in\mathbb{N}})) &= d^*(h(\sigma_g^k((x_n)_{n\in\mathbb{N}}), h(\sigma_g^k((x'_n)_{n\in\mathbb{N}})) \\ &\le d^*(h(\sigma_g^k((x_n)_{n\in\mathbb{N}}), \sigma_g^k((x_n)_{n\in\mathbb{N}})) + \\ &\quad\ d^*(\sigma_g^k((x_n)_{n\in\mathbb{N}}), \sigma_g^k((x'_n)_{n\in\mathbb{N}})) + \\ &\quad\ d^*(h(\sigma_g^k((x'_n)_{n\in\mathbb{N}}), \sigma_g^k((x'_n)_{n\in\mathbb{N}})) \\ &\le \frac{\hat{e}}{4} + \frac{\hat{e}}{4} + \frac{\hat{e}}{4} \\ &= \frac{3\hat{e}}{4} \\ &< \hat{e}, \qquad \forall 0 \le k \le N. \end{aligned}$$

Therefore, by Lemma 3.6,

$$d^*(h((x_n)_{n\in\mathbb{N}}, h((x'_n)_{n\in\mathbb{N}}) < \Delta,$$

whenever $(x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}} \in \lim\limits_{\leftarrow} g$ satisfy $d^*((x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}}) < \gamma$. This proves the continuity of h and completes the proof of the theorem. □

Proof of Theorem 2.4. Let $f : X \to X$ be a topologically stable upper semicontinuous closed-valued map of a compact metric space X.

First we prove that f has the finite shadowing property. Fix $\epsilon > 0$ and let $\delta > 0$ be given by the topological stability of f. Let $\{p_0, \cdots, p_m\}$ be a finite set satisfying

$$d(p_{n+1}, f(p_n)) \le \frac{\delta}{8}, \qquad \forall 0 \le n \le m-1.$$

Define the set-valued map

$$g(x) = \begin{cases} f(x), & \text{if} \quad x \notin \{p_0, p_1, \cdots, p_m\} \\ B[f(p_n), \frac{\delta}{4}], & \text{if} \quad x = p_n \text{ for some } n \in \{0, \cdots, m\}. \end{cases}$$

Clearly $d_H(f, g) \le \delta$. Moreover, since f is closed-valued, g also is. Furthermore, since $\{p_0, \cdots, p_m\}$ is a finite set and f is upper semicontinuous, we have that g is upper semicontinuous. Then, by the choice of δ, there exists $h : (\lim\limits_{\leftarrow} g, d^*) \to (\lim\limits_{\leftarrow} f, d^*)$ continuous such that $D(h, Id_X) \le \epsilon$ and $\sigma_f \circ h = h \circ \sigma_g$. On the other hand, it follows from the definition that $p_{n+1} \in g(p_n)$ for every $0 \le n \le m-1$. Then, since f (and so g) are strict, we can complete $\{p_0, \cdots, p_m\}$ to a g-orbit $(p_n)_{n\in\mathbb{N}}$ and so $(q_n)_{n\in\mathbb{N}} = h((p_n)_{n\in\mathbb{N}})$ is a well-defined f-orbit. Since $D(h, Id_X) \le \epsilon$ we have $d(p_n, q_n) \le \epsilon$ for every $n \in \mathbb{N}$. In particular, $q_{n+1} \in f(q_n)$ and $d(p_n, q_n) < \epsilon$ for every $0 \le n \le m-1$ proving the finite shadowing property. Then, f has the POTP$_+$ by Lemma 3.4.

Next we prove that $Per(f)$ is dense in $\Omega(f)$. Fix $\epsilon > 0$ and $x \in \Omega(f)$. For this ϵ we let $\delta > 0$ be given by topological stability. Since $x \in \Omega(f)$, there are $m \in \mathbb{N}^+$ and a finite sequence $\{z_0, z_1, \cdots, z_m\}$ such that $z_0, z_m \in B(x, \frac{\delta}{4})$ and $z_{n+1} \in f(z_n)$ for every $0 \le n \le m-1$. Define the set-valued map

$$g(x) = \begin{cases} f(x), & \text{if} \quad x \notin \{z_0, z_1, \cdots, z_m\} \\ B[f(z_n), \frac{\delta}{2}], & \text{if} \quad x = z_n \text{ for some } n \in \{0, \cdots, m\}. \end{cases}$$

As before we have that g is upper semicontinuous, closed-valued and $d_H(f, g) \le \delta$. Then, by the choice of δ, there is $h : (\lim\limits_{\leftarrow} g, d^*) \to (\lim\limits_{\leftarrow} f, d^*)$ continuous such that $D(h, Id_X) \leq \epsilon$ and $\sigma_f \circ h = h \circ \sigma_g$. Now define the sequence $(x_n)_{n\in\mathbb{N}}$ by $x_{lm+r} = z_r$ whenever $l \in \mathbb{N}$ and $0 \leq r \leq m-1$. It follows that $(x_n)_{n\in\mathbb{N}} \in \underleftarrow{\lim}\, g$. Moreover, since for all $n \in \mathbb{N}$ there are $l \in \mathbb{N}$ and $0 \leq r \leq m-1$ such that $n = lm + r$, one has $x_{n+m} = x_{(l+1)m+r} = z_r = x_{lm+r} = x_{n+m}$. It follows that $\sigma_g^m((x_n)_{n\in\mathbb{N}}) = (x_n)_{n\in\mathbb{N}}$. Therefore, the f-orbit $(y_n)_{n\in\mathbb{N}} = h((x_n)_{n\in\mathbb{N}})$ is well defined. Since $\sigma_g^m((x_n)_{n\in\mathbb{N}}) = (x_n)_{n\in\mathbb{N}}$, one has $\sigma_f^m((y_n)_{n\in\mathbb{N}}) = (y_n)_{n\in\mathbb{N}}$ and so $y_0 \in Per(f)$. Moreover, since $D(h, Id_X) \leq \epsilon$, we have $d(y_0, x) \leq \epsilon$ proving the result. □

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Received November 2016; revised January 2017.

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