Convergence of a discrete Aubry-Mather model to solutions of some Hamilton-Jacobi equations

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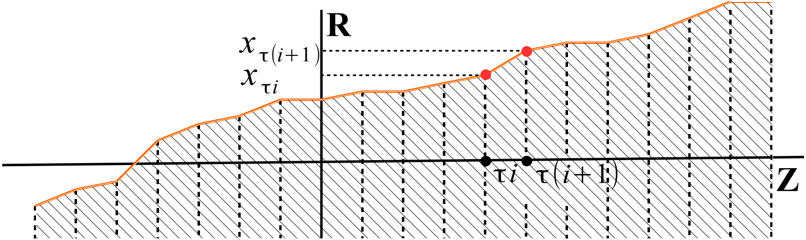
Summary of the talk

- **I.** Motivation: discrete gradient interfaces
- **II.** Technical results
- **III.** Some numerical results
- **IV.** Conclusion

I. Motivation: discrete gradient interfaces

A discrete 1D gradient interface model



- An semi-discrete interface between two media.
- semi-discrete in the sense: $\mathbb{Z} \times \mathbb{R}$ is the lattice
  - the height is continuous: $x_{\tau i} \in \mathbb{R}$
  - the position of each atom is on a lattice of lines: $(\tau i)_{i \in \mathbb{Z}}$
  - the step size is τ
- the interface is supposed to minimize an energy functional

$$E_\tau(x_i, x_{i+1}) = \frac{1}{2\tau}\big(x_{\tau(i+1)} - x_{\tau i} - \tau\lambda\big)^2 + \tau V(x_{\tau i})$$

A periodic energy functional

$$E_\tau(x_i, x_{i+1}) = \frac{1}{2\tau}\big(x_{\tau(i+1)} - x_{\tau i} - \tau\lambda\big)^2 + \tau V(x_{\tau i})$$

- **General hypotheses on the energy $E_\tau(x, y)$**
  - $E_\tau(x, y) \in C^0(\mathbb{R}^d \times \mathbb{R}^d)$ (continuous)
  - $E_\tau(x+k, y+k) = E_\tau(x, y), \quad \forall k \in \mathbb{Z}$ (translation periodic)
  - $\lim_{R\to+\infty} \inf_{\tau\in(0,1)} \inf_{\|y-x\|\geq\tau R} \dfrac{E_\tau(x, y)}{\|y-x\|} = +\infty$ (super-linearity)
  - For every $R >$ there exists a constant $C(R)$ (uniformly Lipschitz)

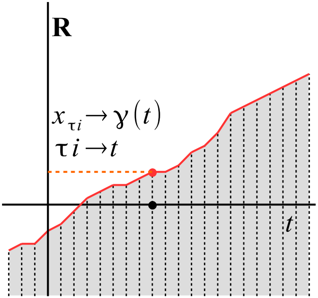
$$\sup_{\|y-x\|\leq\tau R,\ \|y'-x\|\leq\tau R} \big|E_\tau(x, y') - E_\tau(x, y)\big| \leq C(R)\|y'-y\|$$

- **Hypotheses for the convergence problem $\tau \to 0$**

$$E_\tau(x, y) = \tau L\Big(x, \frac{y-x}{\tau}\Big) \qquad \text{(for instance } L(x, v) = \frac{1}{2}(v-\lambda)^2 + V(x)\text{)}$$

where $L(x, v)$ is a C^2 periodic in x Tonelli Lagrangian

Minimizing and ground interfaces



- First notion of

  minimizing interface

- Second notion of

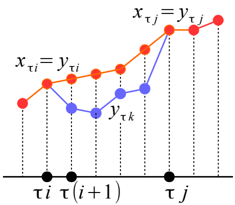
  ground interface

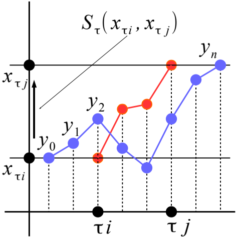
(a minimizing interface at the ground energy; an interface appearing when the system is frozen)

Questions:

What is the limit of the ground interface $\tau i \to x_{\tau i}$ when the step size of the lattice goes to zero $\tau \to 0$ and $x_{\tau i} \to t$?

Minimizing and ground interfaces





- **An interface $(x_{\tau i})_{i\in\mathbb{Z}}$ is minimizing if**

$$E_\tau(x_{\tau i}, x_{\tau(i+1)}, \cdots, x_{\tau j}) := \sum_{k=i}^{j-1} E_\tau(x_{\tau k}, x_{\tau(k+1)})$$
$$\leq E_\tau(y_{\tau i}, x_{\tau(i+1)}, \cdots, y_{\tau j})$$

- **An interface $(x_{\tau i})_{i\in\mathbb{Z}}$ is at the ground level if**

$$E_\tau(x_{\tau i}, x_{\tau(i+1)}, \cdots, x_{\tau j}) - \tau(j-i)\bar{E}_\tau = S_\tau(x_{\tau i}, x_{\tau j})$$
$$:= \inf_{n\geq 1} \inf_{y_0=x_{\tau i},\cdots,y_n=x_{\tau j}} \big[E_\tau(y_0, y_1, \cdots, y_n) - \tau n \bar{E}_\tau\big]$$

Mañé potential versus periodic Mañé potential

- The ground energy $\bar{E}_\tau$ is the largest number such that

$$\inf_{n\geq 1} \inf_{y_0,\cdots,y_n} \left[E_\tau(y_0,y_1,\cdots,y_n) - \tau n\bar{E}_\tau\right] > -\infty$$

- The Mañé potential is the height energy

$$S_\tau(x,y) := \inf_{n\geq 1} \inf_{y_0=x,\cdots,y_n=y} \left[E_\tau(y_0,y_1,\cdots,y_n) - \tau n\bar{E}_\tau\right]$$

- The periodic Mañé potential

$$E_\tau^{\mathrm{per}}(x,y) := \inf_{k\in\mathbb{Z}} E_\tau(x,y+k)$$
$$S_\tau^{\mathrm{per}}(x,y) := \inf_{k\in\mathbb{Z}} S_\tau(x,y+k)$$

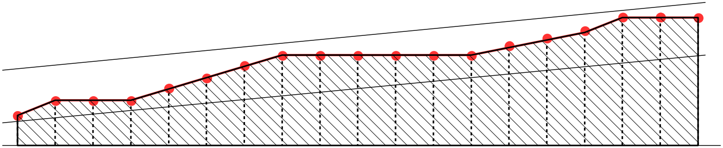
- A (stronger form) of ground interface $(x_{\tau i})_{i\in\mathbb{Z}}$

$$E_\tau^{\mathrm{per}}(x_{\tau i},x_{\tau(i+1)},\cdots,x_{\tau j}) - \tau(j-i)\bar{E}_\tau = S_\tau^{\mathrm{per}}(x_{\tau i},x_{\tau j})$$

What can be proved? $E_\tau(x,y) = \frac{1}{2\tau}(y - x - \tau\lambda)^2 + \tau V(x)$

- Do there exist minimizing interface?
  $\rightarrow$ Yes thanks to Aubry-Mather theory

$$\exists \rho \in \mathbb{R},\ \exists (x_{\tau i})_{i\in\mathbb{Z}},\ \sup_{i\in\mathbb{Z}} |x_{\tau i} - \tau i \rho| < +\infty$$



- Do there exist ground interface?
  $\rightarrow$ Yes thanks to Fathi weak KAM theory. Moreover: $\rho = -\frac{\partial \bar{E}_\tau}{\partial \lambda}$

- Do the discrete interface converges to a continuous interface?
  $\rightarrow$ Yes and it is the content of our result

Convergence of the discrete interface

- **The Frenkel-Kontorova model:**

$$E_\tau(x,y) = \frac{1}{2\tau}(y - x - \tau\lambda)^2 + \tau V(x)$$

$$E_\tau(x,y) = \tau L\Big(x, \frac{y-x}{\tau}\Big), \quad \text{where} \quad L(x,v) := \frac{1}{2}(v-\lambda)^2 + V(x)$$

- **The associated Hamiltonian:** $H(x,p) = \frac{1}{2}(p+\lambda)^2 - \frac{1}{2}\lambda^2 - V(x)$

- **The convergence result:** $x_{\tau i} \to \gamma(t)$ uniformly in $\tau i \to t$

$\gamma(t) \in C^2(\mathbb{R})$ and solves the ODE

$$\dot\gamma(t) = \frac{\partial H}{\partial p}\Big(x, Du(\gamma(t))\Big)$$

u is the (unique modulo a constant) viscosity solution of

$$H(x, Du(x))) = \bar{H}(\lambda) = \lim_{\tau\to 0} -\bar{E}_\tau(\lambda)$$

II. Technical results

Main goal

- **Find a discrete scheme solving the 3 equations**

$\rightarrow \dfrac{\partial u}{\partial t}(t,x) + H(x, \nabla u(t,x)) = 0,$ (non stationary Hamilton-Jacobi),

$\rightarrow H(x, \nabla \bar{u}(x)) = \bar{H},$ (cell equation),

$\rightarrow \delta u_\delta(x) + H(x, \nabla u_\delta(x)) = 0,$ (discounted cell equation, $\delta > 0$)

- **Hypotheses on the Hamiltonian:**

$\rightarrow H(x,p) : \mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is C^2, periodic in x, autonomous,

$\rightarrow \Big[\dfrac{\partial^2 H(x,p)}{\partial p_i \partial p_j}\Big] \geq \alpha \big[\delta_{ij}\big]$ (strictly convex)

$H(x,p)$ is called a *Tonelli Hamiltonian*

Summary of known results

• Solutions in the viscosity sense

A continuous function $u(x)$ is solution of the cell equation

$$H(x, \nabla \bar{u}(x)) = \bar{H}$$

if $\forall x_0 \in \mathbb{T}^d,\ \forall \phi \in C^1(\mathbb{T}^d, \mathbb{R})$

$$\phi(x_0) = u(x_0) \text{ and } \forall x,\ \phi(x) \geq u(x) \quad \Longrightarrow \quad H(x_0, \nabla \phi(x_0)) \leq \bar{H}.$$

• Existence of solutions

For the 3 equations, a solution exists in the viscosity sense

$\rightarrow \dfrac{\partial u}{\partial t}(t,x) + H(x, \nabla u(t,x)) = 0,$ $u(t,x)$ is unique,

$\rightarrow H(x, \nabla \bar{u}(x)) = \bar{H},$ $\bar{u}(x)$ is not unique, $\bar{H}$ is unique,

$\rightarrow \delta u_\delta(x) + H(x, \nabla u_\delta(x)) = 0,$ $u_\delta(x)$ is unique.

Summary of known results

- **Explicit solution of the non stationary Hamilton-Jacobi equation**

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + H(x, \nabla u(t,x)) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

the unique solution is given by

$$u(t,x) = \inf_{\substack{\gamma \in C^2([-t,0],\mathbb{R}^d) \\ \gamma(0)=x}} \Big[u_0(\gamma(-t)) + \int_{-t}^{0} L(\gamma, \dot{\gamma})\, ds \Big],$$

where $L(x,v)$ is the Legendre transform of $H(x,p)$

$$L(x,v) = \sup_p \{ v \cdot p - H(x,p) \}$$

Summary of known results

- **Explicit solution of the effective energy**

$$H(x, \nabla \bar{u}(x)) = \bar{H}$$

then the effective energy is given by

$$-\bar{H} := \lim_{t \to +\infty} \inf_{\gamma \in C^2([-t,0],\mathbb{R}^d)} \left[\frac{1}{t} \int_{-t}^{0} L(\gamma, \dot{\gamma})\, ds \right]$$

- **Weak KAM theorem (part 1) [Fathi, 1997]**

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + H(x, \nabla u(t,x)) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

$$u(t,x) + t\bar{H} \to \bar{u}(x) \quad \text{in the } C^0 \text{ topology}$$

$\bar{u}(x)$ is called weak KAM solution (asolution of the cell equation)

The main tool: Lax-Oleinik operator

- **Definition of the Lax-Oleinik operator** For every $u_0 \in C^0(\mathbb{T}^d)$

$$\to T^t[u](x) := \inf_{\substack{\gamma \in C^2([-t,0],\mathbb{R}^d) \\ \gamma(0)=x}} \left\{ u(\gamma(-t)) + \int_{-t}^{0} L(\gamma,\dot\gamma)\,ds \right\}$$

$$\to T^{s+t} = T^s \circ T^t$$

- **Weak KAM Theorem (part 2) [Fathi, 1997]** for every $u \in C^0(\mathbb{T}^d)$

$$\to u(t,x) := T^t[u_0](x) \qquad \text{solves} \qquad \begin{cases} \dfrac{\partial u}{\partial t} + H(x, \nabla u(t,x)) = 0, \\ u(0,x) = u_0(x) \end{cases}$$

$$\to T^t[u_0] + t\bar H \to \bar u(x) \qquad \text{(in the } C^0 \text{ topology)}$$

$$\to T^t[\bar u] = \bar u(x) - t\bar H, \quad \forall t \geq 0 \quad \Longleftrightarrow \quad H(x, \nabla \bar u(x)) = \bar H$$

Recent results for the discounted cell equation

- **Explicit solution for the discounted cell equation**

$$\delta u_\delta(x) + H(x, \nabla u_\delta(x)) = 0$$

the unique solution is given by

$$u_\delta(x) = \inf_{\substack{\gamma \in C^2((-\infty,0],\mathbb{R}^d) \\ \gamma(0)=x}} \int_{-\infty}^{0} e^{s\delta} L(\gamma(s), \dot{\gamma}(s))\, ds,$$

- **Weak KAM theorem [Davini,Fathi,Iturriaga,Zavidovique, 2016]**

$$\lim_{\delta \to 0} \left(u_\delta(x) + \frac{\bar{H}}{\delta} \right) = \bar{u}^*(x) \quad \text{in the } C^0 \text{ topology}$$

$$H(x, \nabla \bar{u}^*(x)) = \bar{H}$$

The numerical scheme

- **Notations**

→ start with a Tonelli Hamiltonian $H(x,p)$

→ compute the Lagrangian

$$L(x,v) = \sup_{p\in\mathbb{R}^d} \{p\cdot v - H(x,p)\}$$

→ define the interaction energy for each $\tau \in (0,1)$

$$E_\tau(x,y) = \tau L\Big(x, \frac{y-x}{\tau}\Big), \quad \forall x,y \in \mathbb{R}^d$$

→ define the discrete Lax-Oleinik operator

$$T_\tau[u](y) = \min_{x\in\mathbb{R}^d} \{u(x) + E_\tau(x,y)\} \quad \forall u \in C^0(\mathbb{T}^d)$$

$$T^t[u](x) := \inf_{\substack{\gamma\in C^2([-t,0],\mathbb{R}^d)\\ \gamma(0)=x}} \Big\{u(\gamma(-t)) + \int_{-t}^0 L(\gamma,\dot\gamma)\,ds\Big\}$$

Convergence of the discrete weak KAM solution

- **Theorem [Su, Th]: first part**

$\rightarrow \exists(\bar{u}_\tau, \bar{E}_\tau), \quad T_\tau[\bar{u}_\tau] = \bar{u}_\tau + \tau \bar{E}_\tau$ (non unique $\bar{u}_\tau$ Lipschitz)

$\rightarrow \bar{E}_\tau = \lim_{n \to +\infty} \inf_{x_0, x_1, \dots, x_n} \frac{1}{\tau n} \sum_{k=1}^{n} E_\tau(x_{k-1}, x_k)$ (unique)

$\rightarrow \lim_{\tau \to 0} \bar{E}_\tau = -\bar{H}$

$\rightarrow \lim_{\tau \to 0} \bar{u}_\tau = \bar{u}$, (for some subsequence , $\mathrm{Lip}(\bar{u}_\tau) \leq C$)

$\rightarrow H(x, \nabla \bar{u}(x)) = \bar{H}$ (in the viscosity sense)

- Main drawback: a sub-sequence need be taken for u.

The discounted discrete Lax-Oleinik operator

- **The discounted cell equation and its unique solution**

$$\delta u_\delta(x) + H(x, \nabla u_\delta(x))) = 0, \quad \text{(in the viscosity sense)}$$

$$u_\delta(x) = \inf_{\gamma(0)=x} \int_{-\infty}^{0} e^{s\delta} L(\gamma(s), \dot\gamma(s))\, ds$$

- **The discounted Lax-Oleinik operator in the continuous setting**

$$T_\delta^t[u](x) = \inf_{\gamma(0)=x} \Big\{ e^{-t\delta} u(\gamma(-t)) + \int_{-t}^{0} e^{s\delta} L(\gamma(s), \dot\gamma(s))\, ds \Big\}$$

- **Definition of a discounted discrete Lax-Oleinik operator**

$$T_{\tau,\delta}[u](y) = \min_{x\in\mathbb{R}^d} \big\{ (1-\tau\delta) u(x) + E_\tau(x,y) \big\}$$

Convergence of the discrete discounted weak KAM solution

- **[DFIZ, 2016]** If u_δ is solution of $\delta u_\delta(x) + H(x, \nabla u_\delta(x)) = 0$ then

$$u_\delta(x) + \frac{\bar{H}}{\delta} \to \bar{u}^*(x), \qquad H(\bar{u}^*(x), \nabla \bar{u}^*(x)) = \bar{H}$$

- **Theorem [Su, Th]: second part**

$\to \exists u_{\tau,\delta} \quad T_{\tau,\delta}[u_{\tau,\delta}] = u_{\tau,\delta}$ (unique Lipschitz solution)

$\to u_{\tau,\delta} = u_\delta + O(\frac{\tau}{\delta})$ as $\tau \to 0$ (for fixed $\delta > 0$)

$\to u_{\tau,\delta} - \dfrac{\bar{E}_\tau}{\delta} \to \bar{u}^*_\tau$ as $\delta \to 0$ (for fixed $\tau > 0$, no speed as in [DFIZ])

$\to \left(u_{\tau,\delta} - \dfrac{\bar{E}_\tau}{\delta}\right) \to \bar{u}^*$ (when $\dfrac{\tau}{\delta} \to 0$)

Main technical points

- **Some previous results of convergence of the discrete model**
  → [Gomes,Oberma, 2004] SIAM J. Control Optim. 43,
  → [Gomes, 2005] Discrete Contin. Dyn. Syst. 13,
  → [Camilli,Cappuzzo Dolcetta,Gomes, 2008] Appl. Math. Optim. 57,
  → [Bouillard,Faou,Zavidovique, 2016] Math. Comput. 85

- **Recent results in discrete selection principles**
  → [Davini,Fathi,Iturriaga,Zavidovique, 2016] Invent. Math. 206
  → [Iturriaga,Lopes,Mengue, 2018], preprint

- **First main issue:** The non-compactness of $\mathbb{R}^d$

$$\to T_\tau[u](y) = \min_{x\in\mathbb{R}^d} \left\{u(x) + E_\tau(x,y)\right\} = \min_{x\in\mathbb{R}^d} \left\{u(x) + E_\tau^{\text{per}}(x,y)\right\}$$

→ There exists a constant $R > 0$ such that for every weak KAM solution, $\forall \tau \in (0,1)$, $\forall y \in \mathbb{R}^d$,

$$x \in \arg\min_{x\in\mathbb{R}^d}\{u_\tau(x) + E_\tau(x,y)\} \implies \|y - x\| \leq \tau R$$

Need of a more general formalism

- **The discrete action and the minimal action**

$$\mathcal{L}_\tau(x,y) = \tau L\Big(x, \frac{y-x}{\tau}\Big), \quad \mathcal{E}_\tau(x,y) := \inf_{\substack{\gamma\in C^2([0,\tau],\mathbb{R}^d)\\ \gamma(0)=x,\gamma(\tau)=y}} \int_0^\tau L(\gamma,\dot\gamma)\,ds$$

- **A general notion of short-range interactions** $E_\tau(x,y)$

  - $E_\tau(x,y) \in C^0(\mathbb{R}^d)$ (continuous)
  - $E_\tau(x+k,y+k) = E_\tau(x,y),\ \forall k\in\mathbb{Z}$ (translation periodic)
  - $\displaystyle\lim_{R\to+\infty}\ \inf_{\tau\in(0,1)}\ \inf_{\|y-x\|\geq\tau R} \frac{E_\tau(x,y)}{\|y-x\|} = +\infty$ (super-linearity)
  - For every $R >$ there exists a constant $C(R)$ (uniformly Lipschitz)

$$\sup_{\|y-x\|\leq\tau R,\ \|y'-x\|\leq\tau R} \big|E_\tau(x,y') - E_\tau(x,y)\big| \leq C(R)\|y'-y\|$$

  - $\mathcal{E}_{\tau+\sigma}(x,y) = \inf_{z\in\mathbb{R}^d}\{\mathcal{E}_\tau(x,z)+\mathcal{E}_\tau(z,y)\}$ (min-plus convolution)

Comparison lemma

- **The second main issue:** A Weierstrass-Tonelli type estimate

$$\to \mathcal{L}_\tau(x,y) = \tau L\Big(x, \frac{y-x}{\tau}\Big)$$

$$\mathcal{E}_\tau(x,y) := \inf_{\substack{\gamma \in C^2([0,\tau],\mathbb{R}^d) \\ \gamma(0)=x,\gamma(\tau)=y}} \int_0^\tau L(\gamma,\dot\gamma)\,ds$$

$$\to \forall R>0,\ \exists C(R)>0,\ \forall \tau \in (0,1),\ \forall x,y \in \mathbb{R}^d$$

$$\|y-x\| \le \tau R \quad \Longrightarrow \quad |\mathcal{E}_\tau(x,y) - \mathcal{L}_\tau(x,y)| \le \tau^2 C(R)$$

- **The discrete Lax-Oleinik associated to the minimal action**

$$T^\tau[u](y) = \inf_{x\in\mathbb{R}^d}\{u(x) + \mathcal{E}_\tau(x,y)\} \qquad T^{n\tau} = T^\tau \circ \cdots \circ T^\tau$$

$$T^{n\tau}[u](y) := \inf_{\substack{\gamma \in C^2([-n\tau,0],\mathbb{R}^d) \\ \gamma(0)=y}} \Big\{u(\gamma(-n\tau)) + \int_{-t}^0 L(\gamma,\dot\gamma)\,ds\Big\}$$

Selection principle of the discrete discounted weak KAM solution

- **Recall** $u_{\tau,\delta} - \dfrac{\bar{E}_\tau}{\delta} \to \bar{u}^*_\tau$ (a distinguished discrete weak KAM solution)

- **Stationary plan** A probability measure $\pi(dx, dy)$ on $\mathbb{T}^d \times \mathbb{T}^d$ s.t.

$$\iint \phi(x)\, \pi(dx, dy) = \iint \phi(y)\, \pi(dx, dy) \qquad \forall \phi \in C^0(\mathbb{T}^d)$$

- **Minimizing plan** A stationary plan π_{min} is minimizing if

$$\tau \bar{E}_\tau = \min_\pi \iint E_\tau(x, y)\, \pi(dx, dy) = \iint E_\tau(x, y)\, \pi_{\text{min}}(dx, dy)$$

- **Periodic Mañé Potential** Defined on $\mathbb{T}^d \times \mathbb{T}^d$

$$S^{\text{per}}_\tau(x, y) = \inf_{n \geq 1} \inf_{\substack{x_0, x_1, \cdots, x_n \\ x_0 = x,\ x_n = y}} \sum_{k=0}^{n-1} \left[E^{\text{per}}_\tau(x_k, x_{k+1}) - \tau \bar{E}_\tau \right]$$

Characterization of the selected discrete weak KAM solution

- **The two Lax-Oleinik operators**

$$\to T_\tau[u_\tau](y) = \min_{x\in\mathbb{R}^d}\{u_\tau(x) + E_\tau(x,y)\}$$

$$\to T_{\tau,\delta}[u](y) = \min_{x\in\mathbb{R}^d}\{(1-\tau\delta)u(x) + E_\tau(x,y)\}$$

- **Theorem [Su, Th] similar to [DFIZ]** If $T_{\tau,\delta}[u_{\tau,\delta}] = u_{\tau,\delta}$

$$\to \lim_{\delta\to 0} u_{\tau,\delta} - \frac{\bar{E}_\tau}{\delta} = \bar{u}^*_\tau \quad \text{where} \quad T_\tau[\bar{u}^*_\tau] = \bar{u}^*_\tau + \tau\bar{E}_\tau$$

$\to$ First characterization

$$\bar{u}^*_\tau(z) = \sup\Big\{w(z) \,:\, T_\tau[w] = w + \tau\bar{E}_\tau \text{ and } \iint w(y)\,d\pi(x,y) \le 0, \quad \forall\pi\Big\}$$

$\to$ Second characterization

$$\bar{u}^*_\tau(z) = \inf\Big\{\iint S^{\text{per}}_\tau(x,z)\,d\pi(x,y) \,:\, \pi \text{ minimizing plan}\Big\}$$

An additional explanation of the second characterization

- **The periodic Aubry set**

$$\mathrm{Aubry}(E_\tau) := \Big\{(x,y) \in \mathbb{T}^d \times \mathbb{T}^d : \big[E_\tau^{\text{per}}(x,y) - \tau \bar{E}_\tau(x,y)\big] + S_\tau^{\text{per}}(y,x) = 0\Big\}$$

- **Aubry classes** $x \sim y \quad \Leftrightarrow \quad S_\tau^{\text{per}}(x,y) + S_\tau^{\text{per}}(y,x) = 0$

- **Propsotion [Su,Th]** If π is an extremal minimizing plan

$\to pr_*^1(\pi)$ belongs to an Aubry class

$\to u_\tau^\pi(y) := \int_{\mathbb{T}^d} S_\tau^{\text{per}}(x,y)\, pr_*^1(\pi)(dx)$ is a weak KAM solution

$\to \int_{\mathbb{T}^d} u_\tau^\pi(y)\, pr_*^1(\pi)(dy) = 0$

$\to u_\tau^* = \sup\{u_\tau^\pi : \pi$ is an extremal minimizing plan $\}$

III. Some numerical results

The approximation scheme

- **Problem:** Solve $\boldsymbol{H}(x, \nabla \bar{u}(x)) = \bar{H}$

$\to$ compute explicitly, $L(x,v) := \sup_{p \in \mathbb{R}^d} \big[p \cdot v - H(x,p) \big]$

$\to$ choose a step time $\tau > 0$ and define $E_\tau(x,y) := \tau L\Big(x, \frac{y-x}{\tau}\Big)$

$\to$ define $T_\tau[u_\tau](y) := \min_{x \in \mathbb{T}^d} \{ u_\tau(x) + E_\tau(x,y) \}$

$\to$ solve $T_\tau[\bar{u}_\tau] = \bar{u}_\tau + \bar{E}_\tau$ (Ishikawa's iterative method)

$$u_\tau^{(0)} = 0, \qquad u_\tau^{(n+1)} := \frac{u_\tau^{(n)} + T_\tau[u_\tau^{(n)}]}{2} - \min\left(\frac{u_\tau^{(n)} + T_\tau[u_\tau^{(n)}]}{2} \right)$$

$\to u_\tau^{(n)} \to \bar{u}_\tau$, $\min\big(T_\tau[u_\tau^{(n)}]\big) \to \tau \bar{E}_\tau$

$\to -\bar{E}_\tau \to \bar{H}$, $\bar{u}_\tau \to \bar{u}$ (for some sub-sequence $\tau \to 0$)

$\to H(x, \nabla \bar{u}(x)) = \bar{H}$, (in the viscosity sense)

Application to the inverse pendulum

→ $H(x,p) = \frac{1}{2}(p+\lambda)^2 - KV(x)$,

→ $V(x) = \frac{1}{(2\pi)^2}\big(1 - \cos 2\pi x\big)$

→ $L(x,v) = \frac{1}{2}v^2 - \lambda v + KV(x)$

→ $E_\tau(x,y) = \tau L\big(x, \frac{y-x}{\tau}\big) = \frac{1}{2\tau}(y-x)^2 - \lambda(y-x) + \tau KV(x)$

→ solve $\bar{u}_\tau(y) + \bar{E}_\tau = \min_x\{\bar{u}_\tau(x) + E_\tau(x,y)\}$

→ $-\bar{E}_\tau \to \bar{H}$

→ $\bar{u}_\tau \to \bar{u}$ as $\tau \to 0$ (uniqueness of Aubry set)

→ $\frac{1}{2}(\nabla\bar{u} + \lambda)^2 - KV(x) = \bar{H}$

Explicit solution for the inverse pendulum

- **Define**

$$\lambda_* := \int_0^1 \sqrt{2KV(s)}\,ds = \frac{2\sqrt{K}}{\pi^2} \simeq 0.203\,\sqrt{K}$$

- **if $\boldsymbol{\lambda > \lambda_*}$ then $\boldsymbol{\bar{H} > 0}$** solution of $\lambda = \int_0^1 \sqrt{2(\bar{H} + KV(s))}\,ds$

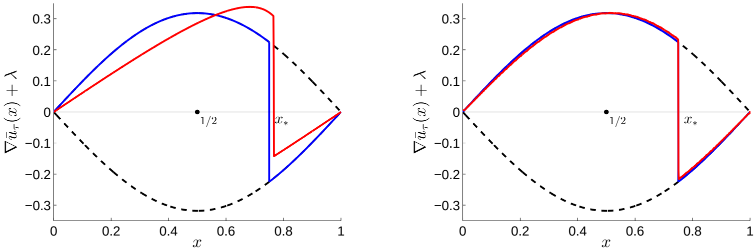
$$\bar{u}(x) = -\lambda x + \int_0^x \sqrt{2(\bar{H} + KV(s))}\,ds$$

- **If $\boldsymbol{\lambda \leq \lambda_*}$ then $\boldsymbol{\bar{H} = 0}$**,

$$\exists!\; x_* \in [\tfrac{1}{2}, 1], \qquad \lambda = \int_0^{x_*} \sqrt{2KV(s)}\,ds - \int_{x_*}^1 \sqrt{2KV(s)}\,ds$$

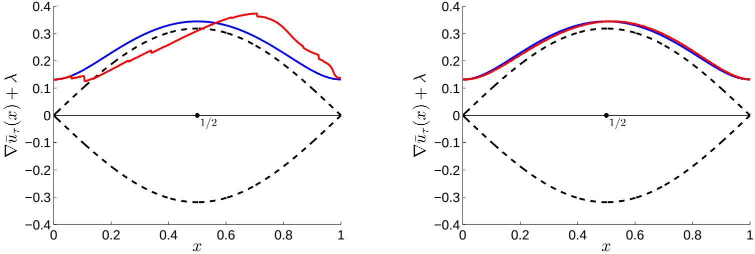
$$\begin{cases} \bar{u}(x) = -\lambda x + \int_0^x \sqrt{2KV(s)}\,ds, & \text{if } 0 \leq x \leq x_* \\ \bar{u}(x) = -\lambda x + \int_0^{x_*} \sqrt{2KV(s)}\,ds - \int_{x_*}^x \sqrt{2KV(s)}\,ds, & \text{if } x_* \leq x \leq 1 \end{cases}$$

The discrete weak KAM solution for $\bar{H} = 0$



Case $\boldsymbol{\lambda \in [0, \lambda_* = 0.20264]}$: In blue, the continuous weak KAM solution $u(x)$ for $\lambda = 0.14329$ and $K = 1$. The theoretical effective energy is $\bar{H} = 0$. The discontinuity of $\nabla u(x)$ is located at $x_*(\lambda) = 0.75$. In red, the backward discrete weak KAM solution for different values of τ. The mesh grid is 10^{-3}, the precision is 10^{-6}. On the left hand side, $\tau = 1$, $N_{iter} = 32$, elapsed time $= 2$s. On the right hand side, $\tau = 0.1$, $N_{iter} = 196$, elapsed time $= 1$s.

The discrete weak KAM solution for $\bar{H} > 0$



Case $\boldsymbol{\lambda > \lambda_* = 0.20264}$: In blue, the continuous weak KAM solution $u(x)$ for $\lambda = 0.25$ and $K = 1$. The theoretical effective energy is $\bar{H} = 0.008643$. In red, the backward discrete weak KAM solution for different values of τ. The mesh grid is 10^{-3}, the precision is 10^{-6}. On the left hand side: $\tau = 1$, $-\bar{E}_\tau = 0.009$, $N_{iter} = 92$, elapsed time $= 4$s. On the right hand side: $\tau = 0.1$, $-\bar{E}_\tau = 0.008642$, $N_{iter} = 2207$, elapsed item $= 9$s.

IV. Conclusion

- **For the $1+1$ interface problem**

The use of weak KAM theory is well adapted to prove the existence of a ground interface. The Lax-Oleinik operator is the natural operator obtained as a limit of the Ruelle operator as the temperature goes to zero

- **For the $d+1$ interface problem**

For general codimensional-one problems, the theory of minimizing interface is well understood both in the discrete and continuous setting

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The notion of ground interface remains to be defined. The lack of a Lax-Oleinik operator is the main difficulty in codimension-one.

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