Convergence of a discrete Aubry-Mather model to solutions of some Hamilton-Jacobi equations

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Summary of the talk

– I. Motivation: discrete gradient interfaces
– II. Technical results
– III. Some numerical results
– IV. Conclusion
I. Motivation: discrete gradient interfaces
A discrete 1D gradient interface model

- An semi-discrete interface between two media.
- semi-discrete in the sense: $\mathbb{Z} \times \mathbb{R}$ is the lattice
  - the height is continuous: $x_{\tau i} \in \mathbb{R}$
  - the position of each atom is on a lattice of lines: $(\tau i)_{i \in \mathbb{Z}}$
  - the step size is $\tau$
- the interface is supposed to minimize an energy functional

\[
E_\tau(x_i, x_{i+1}) = \frac{1}{2\tau} (x_{\tau(i+1)} - x_{\tau i} - \tau \lambda)^2 + \tau V(x_{\tau i})
\]
A periodic energy functional

\[ E_\tau(x_i, x_{i+1}) = \frac{1}{2\tau} \left( x_{\tau(i+1)} - x_{\tau i} - \tau \lambda \right)^2 + \tau V(x_{\tau i}) \]

- General hypotheses on the energy \( E_\tau(x, y) \)
  - \( E_\tau(x, y) \in C^0(\mathbb{R}^d \times \mathbb{R}^d) \) (continuous)
  - \( E_\tau(x + k, y + k) = E_\tau(x, y), \quad \forall k \in \mathbb{Z} \) (translation periodic)
  - \( \lim_{R \to +\infty} \inf_{\tau \in (0,1)} \frac{\inf_{\|y-x\| \geq \tau R \inf \|y - x\|} E_\tau(x, y)}{E_\tau(x, y)} = +\infty \) (super-linearity)
  - For every \( R > 0 \) there exists a constant \( C(R) \) (uniformly Lipschitz)
    \[ \sup_{\|y-x\| \leq \tau R, \|y'-x\| \leq \tau R} \left| E_\tau(x, y') - E_\tau(x, y) \right| \leq C(R)\|y' - y\| \]

- Hypotheses for the convergence problem \( \tau \to 0 \)

\[ E_\tau(x, y) = \tau L\left( x, \frac{y-x}{\tau} \right) \quad \text{(for instance } L(x, v) = \frac{1}{2}(v - \lambda)^2 + V(x) \text{)} \]

where \( L(x, v) \) is a \( C^2 \) periodic in \( x \) Tonelli Lagrangian
Minimizing and ground interfaces

- First notion of minimizing interface
- Second notion of ground interface

(a minimizing interface at the ground energy; an interface appearing when the system is frozen)

Questions:

What is the limit of the ground interface $\tau i \rightarrow x_{\tau i}$ when the step size of the lattice goes to zero $\tau \rightarrow 0$ and $x_{\tau i} \rightarrow t$?
Minimizing and ground interfaces

- An interface \((x_{\tau i})_{i \in \mathbb{Z}}\) is minimizing if

\[
E_{\tau}(x_{\tau i}, x_{\tau(i+1)}, \cdots, x_{\tau j}) := \sum_{k=i}^{j-1} E_{\tau}(x_{\tau k}, x_{\tau(k+1)})
\]

\[
\leq E_{\tau}(y_{\tau i}, x_{\tau(i+1)}, \cdots, y_{\tau j})
\]

- An interface \((x_{\tau i})_{i \in \mathbb{Z}}\) is at the ground level if

\[
E_{\tau}(x_{\tau i}, x_{\tau(i+1)}, \cdots, x_{\tau j}) - \tau(j - i) \bar{E}_{\tau} = S_{\tau}(x_{\tau i}, x_{\tau j})
\]

\[
:= \inf_{n \geq 1} \inf_{y_0 = x_{\tau i}, \cdots, y_n = x_{\tau j}} \left[ E_{\tau}(y_0, y_1, \cdots, y_n) - \tau n \bar{E}_{\tau} \right]
\]
Mañé potential versus periodic Mañé potential

- The ground energy $\bar{E}_{\tau}$ is the largest number such that

$$\inf_{\tau \geq 1} \inf_{y_0, \cdots, y_\tau} \left[ E_{\tau}(y_0, y_1, \cdots, y_\tau) - \tau \bar{E}_{\tau} \right] > -\infty$$

- The Mañé potential is the height energy

$$S_{\tau}(x, y) := \inf_{\tau \geq 1} \inf_{y_0 = x, \cdots, y_\tau = y} \left[ E_{\tau}(y_0, y_1, \cdots, y_\tau) - \tau \bar{E}_{\tau} \right]$$

- The periodic Mañé potential

$$E_{\tau}^{\text{per}}(x, y) := \inf_{k \in \mathbb{Z}} E_{\tau}(x, y + k)$$

$$S_{\tau}^{\text{per}}(x, y) := \inf_{k \in \mathbb{Z}} S_{\tau}(x, y + k)$$

- A (stronger form) of ground interface $(x_{\tau i})_{i \in \mathbb{Z}}$

$$E_{\tau}^{\text{per}}(x_{\tau i}, x_{\tau(i+1)}, \cdots, x_{\tau j}) - \tau(j - i) \bar{E}_{\tau} = S_{\tau}^{\text{per}}(x_{\tau i}, x_{\tau j})$$
What can be proved? \[ E_\tau(x, y) = \frac{1}{2\tau} (y - x - \tau \lambda)^2 + \tau V(x) \]

- Do there exist minimizing interface?
  → Yes thanks to Aubry-Mather theory

\[ \exists \rho \in \mathbb{R}, \exists (x_{\tau i})_{i \in \mathbb{Z}}, \sup_{i \in \mathbb{Z}} |x_{\tau i} - \tau i \rho| < +\infty \]

- Do there exist ground interface?
  → Yes thanks to Fathi weak KAM theory. Moreover:

\[ \rho = -\frac{\partial \bar{E}_\tau}{\partial \lambda} \]

- Do the discrete interface converges to a continuous interface?
  → Yes and it is the content of our result
Convergence of the discrete interface

- The Frenkel-Kontorova model:
  \[
  E_\tau(x, y) = \frac{1}{2\tau} (y - x - \tau \lambda)^2 + \tau V(x)
  \]

  \[
  E_\tau(x, y) = \tau L\left(x, \frac{y - x}{\tau}\right), \quad \text{where} \quad L(x, v) := \frac{1}{2}(v - \lambda)^2 + V(x)
  \]

- The associated Hamiltonian:
  \[
  H(x, p) = \frac{1}{2}(p + \lambda)^2 - \frac{1}{2}\lambda^2 - V(x)
  \]

- The convergence result:
  \[
  x_{\tau i} \rightarrow \gamma(t) \text{ uniformly in } \tau i \rightarrow t
  \]

  \[
  \gamma(t) \in C^2(\mathbb{R}) \text{ and solves the ODE}
  \]

  \[
  \dot{\gamma}(t) = \frac{\partial H}{\partial p}\left(x, Du(\gamma(t))\right)
  \]

  \[
  u \text{ is the (unique modulo a constant) viscosity solution of}
  \]

  \[
  H(x, Du(x)) = \bar{H}(\lambda) = \lim_{\tau \rightarrow 0} -\bar{E}_\tau(\lambda)
  \]
II. Technical results
Main goal

- Find a discrete scheme solving the 3 equations

\[
\frac{\partial u}{\partial t}(t, x) + H(x, \nabla u(t, x)) = 0, \quad \text{(non stationary Hamilton-Jacobi)},
\]

\[
H(x, \nabla \bar{u}(x)) = \bar{H}, \quad \text{(cell equation)},
\]

\[
\delta u_\delta(x) + H(x, \nabla u_\delta(x)) = 0, \quad \text{(discounted cell equation, } \delta > 0\text{)}
\]

- Hypotheses on the Hamiltonian:

\[
H(x, p) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R} \text{ is } C^2, \text{ periodic in } x, \text{ autonomous,}
\]

\[
\left[\frac{\partial^2 H(x, p)}{\partial p_i \partial p_j}\right] \geq \alpha [\delta_{ij}] \quad \text{(strictly convex)}
\]

\[H(x, p)\] is called a Tonelli Hamiltonian
Summary of known results

• Solutions in the viscosity sense
A continuous function $u(x)$ is solution of the cell equation

$$H(x, \nabla \bar{u}(x)) = \bar{H}$$

if $\forall x_0 \in \mathbb{T}^d$, $\forall \phi \in C^1(\mathbb{T}^d, \mathbb{R})$

$$\phi(x_0) = u(x_0) \text{ and } \forall x, \phi(x) \geq u(x) \implies H(x_0, \nabla \phi(x_0)) \leq \bar{H}.$$  

• Existence of solutions
For the 3 equations, a solution exists in the viscosity sense

$$\to \frac{\partial u}{\partial t}(t, x) + H(x, \nabla u(t, x)) = 0, \quad u(t, x) \text{ is unique},$$

$$\to H(x, \nabla \bar{u}(x)) = \bar{H}, \quad \bar{u}(x) \text{ is not unique, } \bar{H} \text{ is unique},$$

$$\to \delta u_{\delta}(x) + H(x, \nabla u_{\delta}(x)) = 0, \quad u_{\delta}(x) \text{ is unique}.$$
Summary of known results

- Explicit solution of the non stationary Hamilton-Jacobi equation

\[
\begin{cases}
\frac{\partial u}{\partial t}(t, x) + H(x, \nabla u(t, x)) = 0 \\
u(0, x) = u_0(x)
\end{cases}
\]

the unique solution is given by

\[
u(t, x) = \inf_{\gamma \in C^2([-t, 0], \mathbb{R}^d)} \left[ u_0(\gamma(-t)) + \int_{-t}^0 L(\gamma, \dot{\gamma}) \, ds \right],
\]

where \( L(x, v) \) is the Legendre transform of \( H(x, p) \)

\[
L(x, v) = \sup_p \{ v \cdot p - H(x, p) \}
\]
Summary of known results

- Explicit solution of the effective energy

\[ H(x, \nabla \bar{u}(x)) = \bar{H} \]

then the effective energy is given by

\[ -\bar{H} := \lim_{t \to +\infty} \inf_{\gamma \in C^2([-t,0], \mathbb{R}^d)} \left[ \frac{1}{t} \int_{-t}^{0} L(\gamma, \dot{\gamma}) \, ds \right] \]

- Weak KAM theorem (part 1) [Fathi, 1997]

\[
\begin{cases}
\frac{\partial u}{\partial t}(t, x) + H(x, \nabla u(t, x)) = 0 \\
u(0, x) = u_0(x)
\end{cases}
\]

\[ u(t, x) + t\bar{H} \to \bar{u}(x) \quad \text{in the } C^0 \text{ topology} \]

\( \bar{u}(x) \) is called weak KAM solution (a solution of the cell equation)
The main tool: Lax-Oleinik operator

- **Definition of the Lax-Oleinik operator**
  
  For every $u_0 \in C^0(\mathbb{T}^d)$

  $$
  T^t[u](x) := \inf_{\gamma \in C^2([-t,0],\mathbb{R}^d)} \left\{ u(\gamma(-t)) + \int_{-t}^0 L(\gamma, \dot{\gamma}) \, ds \right\}
  $$

  $$
  T^{s+t} = T^s \circ T^t
  $$

- **Weak KAM Theorem (part 2)** [Fathi, 1997]
  
  For every $u \in C^0(\mathbb{T}^d)$

  $$
  u(t, x) := T^t[u_0](x)
  $$

  solves

  $$
  \left\{ \begin{array}{l}
  \frac{\partial u}{\partial t} + H(x, \nabla u(t, x)) = 0, \\
  u(0, x) = u_0(x)
  \end{array} \right.
  $$

  $$
  T^t[u_0] + t\bar{H} \to \bar{u}(x)
  $$

  $$(\text{in the } C^0 \text{ topology})
  $$

  $$
  T^t[\bar{u}] = \bar{u}(x) - t\bar{H}, \quad \forall t \geq 0 \iff H(x, \nabla \bar{u}(x)) = \bar{H}
  $$
Recent results for the discounted cell equation

- Explicit solution for the discounted cell equation

\[ \delta u_\delta(x) + H(x, \nabla u_\delta(x)) = 0 \]

the unique solution is given by

\[ u_\delta(x) = \inf_{\gamma \in C^2((-\infty,0], \mathbb{R}^d)} \int_{-\infty}^{0} e^{s\delta} L(\gamma(s), \dot{\gamma}(s)) \, ds, \]

- Weak KAM theorem [Davini,Fathi,Iturriaga,Zavidovique, 2016]

\[ \lim_{\delta \to 0} \left( u_\delta(x) + \frac{\bar{H}}{\delta} \right) = \bar{u}^*(x) \quad \text{in the } C^0 \text{ topology} \]

\[ H(x, \nabla \bar{u}^*(x)) = \bar{H} \]
The numerical scheme

• Notations

→ start with a Tonelli Hamiltonian $H(x, p)$

→ compute the Lagrangian

$$L(x, v) = \sup_{p \in \mathbb{R}^d} \{ p \cdot v - H(x, p) \}$$

→ define the interaction energy for each $\tau \in (0, 1)$

$$E_\tau(x, y) = \tau L\left(x, \frac{y-x}{\tau}\right), \quad \forall x, y \in \mathbb{R}^d$$

→ define the discrete Lax-Oleinik operator

$$T_\tau[u](y) = \min_{x \in \mathbb{R}^d} \{ u(x) + E_\tau(x, y) \} \quad \forall u \in C^0(\mathbb{T}^d)$$

$$T^t[u](x) := \inf_{\gamma \in C^2([-t,0],\mathbb{R}^d) \atop \gamma(0)=x} \left\{ u(\gamma(-t)) + \int_{-t}^0 L(\gamma, \dot{\gamma}) \, ds \right\}$$
Convergence of the discrete weak KAM solution

• Theorem [Su, Th]: first part

\[ \exists (\bar{u}_\tau, \bar{E}_\tau), \quad T_\tau [\bar{u}_\tau] = \bar{u}_\tau + \tau \bar{E}_\tau \]  

(non unique \( \bar{u}_\tau \) Lipschitz)

\[ \bar{E}_\tau = \lim_{n \to +\infty} \inf_{x_0, x_1, \ldots, x_n} \frac{1}{\tau n} \sum_{k=1}^{n} E_\tau(x_{k-1}, x_k) \]  

(unique)

\[ \lim_{\tau \to 0} \bar{E}_\tau = -\bar{H} \]

\[ \lim_{\tau \to 0} \bar{u}_\tau = \bar{u}, \]  

(for some subsequence, \( \text{Lip}(\bar{u}_\tau) \leq C \))

\[ H(x, \nabla \bar{u}(x)) = \bar{H} \]  

(in the viscosity sense)

• Main drawback: a sub-sequence need be taken for \( u \).
The discounted discrete Lax-Oleinik operator

• The discounted cell equation and its unique solution

\[ \delta u_\delta(x) + H(x, \nabla u_\delta(x)) = 0, \quad \text{(in the viscosity sense)} \]

\[ u_\delta(x) = \inf_{\gamma(0)=x} \int_{-\infty}^{0} e^{s\delta} L(\gamma(s), \dot{\gamma}(s)) \, ds \]

• The discounted Lax-Oleinik operator in the continuous setting

\[ T^t_\delta[u](x) = \inf_{\gamma(0)=x} \left\{ e^{-t\delta} u(\gamma(-t)) + \int_{-t}^{0} e^{s\delta} L(\gamma(s), \dot{\gamma}(s)) \, ds \right\} \]

• Definition of a discounted discrete Lax-Oleinik operator

\[ T_{\tau,\delta}[u](y) = \min_{x \in \mathbb{R}^d} \{(1 - \tau\delta) u(x) + E_\tau(x, y)\} \]
Convergence of the discrete discounted weak KAM solution

• [DFIZ, 2016] If $u_\delta$ is solution of $\delta u_\delta(x) + H(x, \nabla u_\delta(x)) = 0$ then

$$u_\delta(x) + \frac{H}{\delta} \to \bar{u}^*(x), \quad H(\bar{u}^*(x), \nabla \bar{u}^*(x)) = \bar{H}$$

• Theorem [Su, Th]: second part

$$\to \exists u_{\tau,\delta}, \quad T_{\tau,\delta}[u_{\tau,\delta}] = u_{\tau,\delta} \quad \text{(unique Lipschitz solution)}$$

$$\to u_{\tau,\delta} = u_\delta + O\left(\frac{\tau}{\delta}\right) \quad \text{as } \tau \to 0 \quad \text{(for fixed } \delta > 0)$$

$$\to u_{\tau,\delta} - \frac{\bar{E}_\tau}{\delta} \to \bar{u}_\tau^* \quad \text{as } \delta \to 0 \quad \text{(for fixed } \tau > 0, \text{ no speed as in [DFIZ])}$$

$$\to \left(u_{\tau,\delta} - \frac{\bar{E}_\tau}{\delta}\right) \to \bar{u}^* \quad \text{(when } \frac{\tau}{\delta} \to 0)$$
Main technical points

• Some previous results of convergence of the discrete model
  → [Gomes, Oberma, 2004] SIAM J. Control Optim. 43,
  → [Gomes, 2005] Discrete Contin. Dyn. Syst. 13,

• Recent results in discrete selection principles
  → [Davini, Fathi, Iturriaga, Zavidovique, 2016] Invent. Math. 206
  → [Iturriaga, Lopes, Mengue, 2018], preprint

• First main issue: The non-compactness of $\mathbb{R}^d$

  → $T_\tau[u](y) = \min_{x \in \mathbb{R}^d} \{u(x) + E_\tau(x, y)\} = \min_{x \in \mathbb{R}^d} \{u(x) + E^\text{per}_\tau(x, y)\}$

  → There exists a constant $R > 0$ such that for every weak KAM solution, $\forall \tau \in (0, 1)$, $\forall y \in \mathbb{R}^d$,

  \[ x \in \arg\min_{x \in \mathbb{R}^d} \{u_\tau(x) + E_\tau(x, y)\} \implies \|y - x\| \leq \tau R \]
Need of a more general formalism

- The discrete action and the minimal action

\[
\mathcal{L}_\tau(x, y) = \tau L \left( x, \frac{y-x}{\tau} \right), \quad \mathcal{E}_\tau(x, y) := \inf_{\gamma \in C^2([0, \tau], \mathbb{R}^d)} \int_0^\tau L(\gamma, \dot{\gamma}) \, ds \\
\gamma(0) = x, \gamma(\tau) = y
\]

- A general notion of short-range interactions \( E_\tau(x, y) \)

- \( E_\tau(x, y) \in C^0(\mathbb{R}^d) \) (continuous)

- \( E_\tau(x + k, y + k) = E_\tau(x, y), \forall k \in \mathbb{Z} \) (translation periodic)

- \( \lim_{R \to +\infty} \inf_{\tau \in (0,1)} \inf_{\|y-x\| \geq \tau R} \frac{E_\tau(x, y)}{\|y-x\|} = +\infty \) (super-linearity)

- For every \( R > 0 \) there exists a constant \( C(R) \) (uniformly Lipschitz)

\[
\sup_{\|y-x\| \leq \tau R, \|y'-x\| \leq \tau R} \left| E_\tau(x, y') - E_\tau(x, y) \right| \leq C(R) \|y' - y\|
\]

- \( \mathcal{E}_{\tau+\sigma}(x, y) = \inf_{z \in \mathbb{R}^d} \left\{ \mathcal{E}_\tau(x, z) + \mathcal{E}_\sigma(z, y) \right\} \) (min-plus convolution)
Comparison lemma

- The second main issue: A Weierstrass-Tonelli type estimate

\[ \mathcal{L}_\tau(x, y) = \tau L\left(x, \frac{y-x}{\tau}\right) \]

\[ \mathcal{E}_\tau(x, y) := \inf_{\gamma \in C^2([0,\tau], \mathbb{R}^d)} \int_0^\tau L(\gamma, \dot{\gamma}) \, ds \]

\[ \forall R > 0, \ \exists C'(R) > 0, \ \forall \tau \in (0, 1), \ \forall x, y \in \mathbb{R}^d \]

\[ \|y - x\| \leq \tau R \implies |\mathcal{E}_\tau(x, y) - \mathcal{L}_\tau(x, y)| \leq \tau^2 C'(R) \]

- The discrete Lax-Oleinik associated to the minimal action

\[ T^n\tau [u](y) = \inf_{x \in \mathbb{R}^d} \{u(x) + \mathcal{E}_\tau(x, y)\} \quad T^n\tau = T^n \circ \cdots \circ T^n \]

\[ T^n\tau [u](y) := \inf_{\gamma \in C^2([-n\tau, 0], \mathbb{R}^d)} \left\{ u(\gamma(-n\tau)) + \int_{-\tau}^0 L(\gamma, \dot{\gamma}) \, ds \right\} \]
Selection principle of the discrete discounted weak KAM solution

• Recall \( u_{\tau, \delta} - \frac{\bar{E}_\tau}{\delta} \to \bar{u}_\tau^* \) (a distinguished discrete weak KAM solution)

• Stationary plan A probability measure \( \pi(dx, dy) \) on \( \mathbb{T}^d \times \mathbb{T}^d \) s.t.

\[
\iint \phi(x) \pi(dx, dy) = \iint \phi(y) \pi(dx, dy) \quad \forall \phi \in C^0(\mathbb{T}^d)
\]

• Minimizing plan A stationary plan \( \pi_{\text{min}} \) is minimizing if

\[
\tau \bar{E}_\tau = \min \pi \iint E_\tau(x, y) \pi(dx, dy) = \iint E_\tau(x, y) \pi_{\text{min}}(dx, dy)
\]

• Periodic Mañé Potential Defined on \( \mathbb{T}^d \times \mathbb{T}^d \)

\[
S^\text{per}_\tau(x, y) = \inf_{n \geq 1} \inf_{x_0, x_1, \ldots, x_n} \sum_{k=0}^{n-1} [E^\text{per}_\tau(x_k, x_{k+1}) - \tau \bar{E}_\tau]
\]
Characterization of the selected discrete weak KAM solution

- The two Lax-Oleinik operators

\[ T_{\tau}[u_{\tau}](y) = \min_{x \in \mathbb{R}^d} \{u_{\tau}(x) + E_{\tau}(x, y)\} \]

\[ T_{\tau,\delta}[u](y) = \min_{x \in \mathbb{R}^d} \{(1 - \tau \delta)u(x) + E_{\tau}(x, y)\} \]

- Theorem [Su, Th] similar to [DFIZ] If \( T_{\tau,\delta}[u_{\tau,\delta}] = u_{\tau,\delta} \)

\[ \lim_{\delta \to 0} u_{\tau,\delta} - \frac{E_{\tau}}{\delta} = \bar{u}_{\tau}^* \quad \text{where} \quad T_{\tau}[\bar{u}_{\tau}^*] = \bar{u}_{\tau}^* + \tau \bar{E}_{\tau} \]

- First characterization

\[ \bar{u}_{\tau}^*(z) = \sup \left\{ w(z) : T_{\tau}[w] = w + \tau \bar{E}_{\tau} \text{ and } \iint w(y) \, d\pi(x, y) \leq 0, \ \forall \pi \right\} \]

- Second characterization

\[ \bar{u}_{\tau}^*(z) = \inf \left\{ \iint S_{\tau}^{\text{per}}(x, z) \, d\pi(x, y) : \pi \text{ minimizing plan} \right\} \]
An additional explanation of the second characterization

- The periodic Aubry set

\[ \text{Aubry}(E_\tau) := \{ (x, y) \in \mathbb{T}^d \times \mathbb{T}^d : \left[ E^\text{per}_\tau(x, y) - \tau \bar{E}_\tau(x, y) \right] + S^\text{per}_\tau(y, x) = 0 \} \]

- Aubry classes

\[ x \sim y \iff S^\text{per}_\tau(x, y) + S^\text{per}_\tau(y, x) = 0 \]

- Proposition [Su, Th]

If \( \pi \) is an extremal minimizing plan

\[ \rightarrow \text{pr}^1_*(\pi) \text{ belongs to an Aubry class} \]

\[ \rightarrow u^\pi_\tau(y) := \int_{\mathbb{T}^d} S^\text{per}_\tau(x, y) \text{pr}^1_*(\pi)(dx) \text{ is a weak KAM solution} \]

\[ \rightarrow \int_{\mathbb{T}^d} u^\pi_\tau(y) \text{pr}^1_*(\pi)(dy) = 0 \]

\[ \rightarrow u^*_\tau = \sup \{ u^\pi_\tau : \pi \text{ is an extremal minimizing plan} \} \]
III. Some numerical results
The approximation scheme

- **Problem:** Solve \( H(x, \nabla \tilde{u}(x)) = \tilde{H} \)

\[ \rightarrow \text{compute explicitly,} \quad L(x, v) := \sup_{p \in \mathbb{R}^d} [p \cdot v - H(x, p)] \]

\[ \rightarrow \text{choose a step time} \; \tau > 0 \; \text{and define} \; E_{\tau}(x, y) := \tau L\left(x, \frac{y-x}{\tau}\right) \]

\[ \rightarrow \text{define} \quad T_{\tau}[u_{\tau}](y) := \min_{x \in \mathbb{T}^d} \{u_{\tau}(x) + E_{\tau}(x, y)\} \]

\[ \rightarrow \text{solve} \quad T_{\tau}[\tilde{u}_{\tau}] = \tilde{u}_{\tau} + E_{\tau} \text{ (Ishikawa’s iterative method)} \]

\[ u_{\tau}^{(0)} = 0, \quad u_{\tau}^{(n+1)} := \frac{u_{\tau}^{(n)} + T_{\tau}[u_{\tau}^{(n)}]}{2} - \min \left( \frac{u_{\tau}^{(n)} + T_{\tau}[u_{\tau}^{(n)}]}{2} \right) \]

\[ \rightarrow u_{\tau}^{(n)} \rightarrow \tilde{u}_{\tau}, \quad \min \left( T_{\tau}[u_{\tau}^{(n)}] \right) \rightarrow \tau \tilde{E}_{\tau} \]

\[ \rightarrow -\tilde{E}_{\tau} \rightarrow \tilde{H}, \quad \tilde{u}_{\tau} \rightarrow \tilde{u} \quad \text{(for some sub-sequence} \; \tau \rightarrow 0) \]

\[ \rightarrow H(x, \nabla \tilde{u}(x)) = \tilde{H}, \quad \text{(in the viscosity sense)} \]
Application to the inverse pendulum

\[ H(x, p) = \frac{1}{2} (p + \lambda)^2 - KV(x), \]
\[ V(x) = \frac{1}{(2\pi)^2} \left(1 - \cos 2\pi x\right), \]
\[ L(x, v) = \frac{1}{2} v^2 - \lambda v + KV(x), \]
\[ E_\tau(x, y) = \tau L\left(x, \frac{y-x}{\tau}\right) = \frac{1}{2\tau} (y-x)^2 - \lambda(y-x) + \tau KV(x), \]
\[ \text{solve} \quad \bar{u}_\tau(y) + \bar{E}_\tau = \min_x \{\bar{u}_\tau(x) + E_\tau(x, y)\} \]
\[ -\bar{E}_\tau \to \bar{H} \]
\[ \bar{u}_\tau \to \bar{u} \quad \text{as} \quad \tau \to 0 \quad \text{(uniqueness of Aubry set)} \]
\[ \frac{1}{2} (\nabla \bar{u} + \lambda)^2 - KV(x) = \bar{H} \]
Explicit solution for the inverse pendulum

- Define

\[ \lambda_* := \int_0^1 \sqrt{2KV(s)} \, ds = \frac{2\sqrt{K}}{\pi^2} \simeq 0.203 \sqrt{K} \]

- if \( \lambda > \lambda_* \) then \( \bar{H} > 0 \) solution of

\[ \lambda = \int_0^1 \sqrt{2(\bar{H} + KV(s))} \, ds \]

\[ \bar{u}(x) = -\lambda x + \int_0^x \sqrt{2(\bar{H} + KV(s))} \, ds \]

- If \( \lambda \leq \lambda_* \) then \( \bar{H} = 0 \),

\[ \exists! \; x_* \in \left[ \frac{1}{2}, 1 \right], \quad \lambda = \int_0^{x_*} \sqrt{2KV(s)} \, ds - \int_{x_*}^1 \sqrt{2KV(s)} \, ds \]

\[ \begin{cases} \bar{u}(x) = -\lambda x + \int_0^x \sqrt{2KV(s)} \, ds, & \text{if } 0 \leq x \leq x_* \\ \bar{u}(x) = -\lambda x + \int_0^{x_*} \sqrt{2KV(s)} \, ds - \int_{x_*}^x \sqrt{2KV(s)} \, ds, & \text{if } x_* \leq x \leq 1 \end{cases} \]
The discrete weak KAM solution for $\bar{H} = 0$

Case $\lambda \in [0, \lambda_* = 0.20264]$: In blue, the continuous weak KAM solution $u(x)$ for $\lambda = 0.14329$ and $K = 1$. The theoretical effective energy is $\bar{H} = 0$. The discontinuity of $\nabla u(x)$ is located at $x_*(\lambda) = 0.75$. In red, the backward discrete weak KAM solution for different values of $\tau$. The mesh grid is $10^{-3}$, the precision is $10^{-6}$. On the left hand side, $\tau = 1$, $N_{iter} = 32$, elapsed time = 2s. On the right hand side, $\tau = 0.1$, $N_{iter} = 196$, elapsed time = 1s.
**The discrete weak KAM solution for $\bar{H} > 0$**

Case $\lambda > \lambda_* = 0.20264$: In blue, the continuous weak KAM solution $u(x)$ for $\lambda = 0.25$ and $K = 1$. The theoretical effective energy is $\bar{H} = 0.008643$. In red, the backward discrete weak KAM solution for different values of $\tau$. The mesh grid is $10^{-3}$, the precision is $10^{-6}$. On the left hand side: $\tau = 1$, $-\bar{E}_\tau = 0.009$, $N_{iter} = 92$, elapsed time = 4s. On the right hand side: $\tau = 0.1$, $-\bar{E}_\tau = 0.008642$, $N_{iter} = 2207$, elapsed item = 9s.
IV. Conclusion
• For the $1+1$ interface problem

The use of weak KAM theory is well adapted to prove the existence of a ground interface. The Lax-Oleinik operator is the natural operator obtained as a limit of the Ruelle operator as the temperature goes to zero.

• For the $d+1$ interface problem

For general codimensional-one problems, the theory of minimizing interface is well understood both in the discrete and continuous setting.

→ U. Bessi, D. Massart, CPAM, Vol. 64 (2011)

The notion of ground interface remains to be defined. The lack of a Lax-Oleinik operator is the main difficulty in codimension-one.

