Non convergence of a 1D Gibbs model at zero temperature

Philippe Thieullen
(Université de Bordeaux)

Joint work with
R. Bissacot (USP, Saõ Paulo) and
E. Garibaldi (UNICAMP, Campinas)

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Summary of the talk

–I. Position of the problem

–II. Previous results of convergence or non-convergence in 1D

–III. The non selection case for a simple Mather set

I. Position of the problem

Chaotic convergence at zero temperature

Main objective Understand the non convergence of the Gibbs measure at zero temperature for the 1D Ising model and for long range Hamiltonian. We want to prescribe the following constraints:

- a 1D model: $\Sigma = \{0,1\}^{\mathbb{Z}}$ with a shift map $\sigma : \Sigma \to \Sigma$
- a family of Hamiltonians per site $H : \Sigma \to \mathbb{R}$ having a summable variation ($\Rightarrow$ uniqueness of the Gibbs measure at $\beta < +\infty$)

$$H_\Lambda(x) = \sum_{k \in \Lambda} H \circ \sigma^k(x)$$

- assume that there exists only two ground states δ_{0^∞} and δ_{1^∞}: more precisely: $H \geq 0$, $H(0^\infty) = H(1^\infty) = 0$, and $\{H = 0\}$ contains no other invariant measure
- detect necessary conditions on the parameters of the class which implies the non convergence of the Gibbs measure

$$\exists \beta_k \nearrow +\infty \quad \mu_{\beta_{2k}} \to \delta_{0^\infty} \quad \text{and} \quad \mu_{\beta_{2k+1}} \to \delta_{1^\infty}$$

The historic of this problem

→ [2003] J. Brémont, Nonlinearity, Vol. 16,
→ [2007] A.C.D. van Enter, W.M. Ruszel, JSP, Vol. 127, No. 3,
→ [2010] J. R. Chazottes, M. Hochman, CMP, Vol. 297,
→ [2012] R. Leplaideur. JSP, Vol. 147,
→ [2012] A. Baraviera, R. Leplaideur, SIAM JADS, VOl. 11,
→ [2013] A. Baraviera, A. O. Lopes, J. K. Mengue, ETDS, Vol. 33,
→ [2015] D. Coronel, J. Rivera-Letelier, JSP, Vol. 160,

Definition of the summability of the variation

$$\sum_{n\geq 1} \mathrm{var}(H,n) < +\infty, \quad \mathrm{var}(H,n) := \sup\Big\{|H(x) - H(y)| \,:\, x \overset{n}{=} y\Big\}.$$

Simplification of the problem

The one sided shift Because of the summability of the variation

- $H = \tilde{H} + V \circ \sigma - V$ is cohomologuous to an Hamiltonian $\tilde{H}(x)$ depending only on the positive coordinates $x_0, x_1, x_2, \ldots$
- $\tilde{\Sigma} = \{0,1\}^{\mathbb{N}}$, $\tilde{\sigma}(x_0, x_1, \cdots) = (x_1, x_2, \cdots)$
- $\tilde{H} : \tilde{\Sigma} \to \mathbb{R}$ is assumed to have also summability variation

The transfer operator For simplification $\tilde{H} = H$, $\tilde{\Sigma} = \Sigma$.

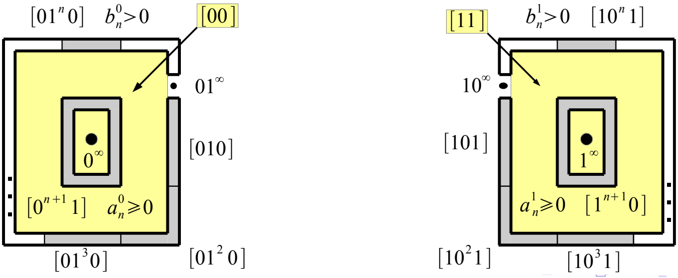
- $\mathcal{L}_\beta[\Psi](x) := e^{-\beta H(0x)}\Psi(0x) + e^{-\beta H(1x)}\Psi(1x), \quad \forall x \in \Sigma$
- $\exists!\ \Phi_\beta > 0, \quad \mathcal{L}_\beta[\Phi_\beta] = \lambda_\beta \Phi_\beta$
- $\exists!\ \nu_\beta$, probability, $\mathcal{L}_\beta^*[\nu_\beta] = \lambda_\beta \nu_\beta$

The Gibbs measure

$$\mu_\beta = \Phi_\beta \nu_\beta \quad \text{(is } \sigma\text{-invariant)}$$

The family of long-range Hamiltonians

- $\Sigma = \{0,1\}^{\mathbb{N}} = [0] \cup [1]$
- $H > 0$ on $\Sigma \setminus \big([00] \cup \{01^\infty\}\big)$, $\quad H(0^\infty) = 0$
- $[0] = \Big(\bigcup_{n\geq 1} [0^{n+1}1] \cup \{0^\infty\}\Big) \cup \Big(\bigcup_{n\geq 1} [01^n0] \cup \{01^\infty\}\Big)$
- $H = a_n^0 \geq 0$ on $[0^{n+1}1]$, $\;H = b_n^0 > 0$ on $[01^n0]$, $\quad$ (idem for a_n^1, b_n^1)
- hypotheses on the summability of the variation of H



Example and reduction

Example of Hamiltonians with no chaotic behavior

- $H(x) = d(x, 0^\infty)^{\alpha_0} d(x, 1^\infty)^{\alpha_1}$
- more generally $H(x) = a_n^0 > 0$ for some $x \in [00]$ and some $n \geq 1$

Reduction to a simpler problem

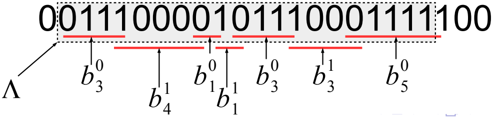
- $H = \tilde{H} + V \circ \sigma - V$ and $\tilde{H}$ has the same structure as H:

$$\tilde{H}(x) = 0, \ \forall x \in [00] \cup [11] \quad \Leftrightarrow \quad \tilde{a}_n^0 = \tilde{a}_n^1 = 0, \ \forall n \geq 1$$

- **from now on**

$$\to H(x) = 0, \ \forall x \in [00] \cup [11]$$

$$\to H(x) = b_n^0, \ \forall x \in [01^n0]$$



II. Previous results of convergence or non-convergence in 1D

General results in 1D thermodynamical formalism

Notations

$$\Sigma = \{0,1\}^{\mathbb{N}}, \quad \sigma : \Sigma \to \Sigma, \quad H : \Sigma \to \mathbb{R}$$

The variational principle (or the minimization of the free energy)

$$F_\beta(H) := \min \Big\{ \int H \, d\mu - \frac{1}{\beta}\mathrm{Ent}(\mu) \ : \ \mu \text{ is } \sigma\text{-invariant} \Big\}$$

$$\mu_\beta \text{ is a Gibbs measure} \quad \Leftrightarrow \int H \, d\mu - \frac{1}{\beta}\mathrm{Ent}(\mu) = F_\beta(H)$$

Minimizing measure ($\supset$ ground state)

$$\mu \text{ is minimizing} \quad \Leftrightarrow \quad \begin{cases} \mu \text{ is } \sigma\text{-invariant} \\ \int H \, d\mu = \min \big\{ \int H \, d\nu : \nu \text{ is } \sigma\text{-invariant} \big\} \end{cases}$$

Lemma

any accumulation point of μ_β as $\beta \to +\infty$ is minimizing

Mather set and Mañé potential

The Mather set

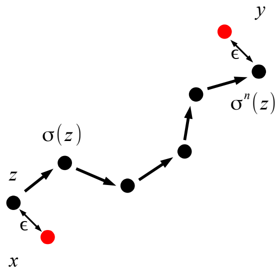
$$\mathrm{Mather}(H) = \cup\{\mathrm{supp}(\mu) \,:\, \mu \text{ is minimizing}\}$$

is a compact set and is σ-invariant (and thus contains limit measures at zero temperature)

The minimizing ergodic value (or ground energy)

$$\bar{H} := \min\Big\{ \int H\, d\mu \,:\, \mu \text{ is } \sigma\text{-invariant} \Big\} = \lim_{n\to+\infty} \inf_{x\in\Sigma} \frac{1}{n}\sum_{k=0}^{n-1} H\circ\sigma^k(x)$$

The Mañé potential



The Mañé potential is the minimal algebraic cost needed to go from one configuration x to another one y.

Algebraic in the sense that the cost is measured relatively to the minimizing ergodic value

General results for the Mañé potential

Mañé potential $S(x,y) := \lim_{\epsilon\to 0} \liminf_{n\to+\infty} S_{\epsilon,n}(x,y) \in \mathbb{R} \cup \{+\infty\}$

$$S_{\epsilon,n}(x,y) = \inf \Big\{ \sum_{k=0}^{n-1} (H - \bar H) \circ^k (z) \ : \ d(z,x) < \epsilon, \ d(\sigma^n(z), y) < \epsilon \Big\}$$

****0111001********1100011**** $\quad x = 0111001{*}{*}{*}, \ y = 1100011{*}{*}{*}$

Theorem (Classical)

- $\mathrm{Mather}(H) \subset \{x \ : \ S(x,x) = 0\}$ (= projected Aubry set)
- $\forall x \in \mathrm{Mather}(H)$, $u(y) := S(x,y)$ is a Lipschitz calibrated sub-action (solution of a min-plus transfer operator)

$$T[u](y) = u(y) + \bar H \qquad \text{where} \quad T[u](y) = \min_{\sigma(y')=y} \{u(y') + H(y')\}$$

- Every calibrated sub-action $u(y)$ satisfies

$$u(y) = \min_{y' \in \mathrm{Mather}(H)} \{u(y') + S(y',y)\}$$

Transfer operator at the log scale

The usual transfer operator

$$\mathcal{L}_\beta[\Psi](y) = \sum_{\sigma(y')=y} e^{-\beta H(y')}\Psi(y')$$

The Gibbs measures $\mu_\beta = \Phi_\beta \nu_\beta$

$$\mathcal{L}_\beta[\Phi_\beta] = \lambda_\beta \Phi_\beta, \quad \mathcal{L}^*_\beta[\nu_\beta] = \lambda_\beta \nu_\beta$$

A change to the log-scale $\lambda_\beta = e^{-\beta \bar{H}_\beta}$, $\Phi_\beta(y) = e^{-\beta u_\beta(y)}$

$$T_\beta[u](y) := -\frac{1}{\beta}\log \sum_{\sigma(y')=y} e^{-\beta\{u(y')+H(y')\}}$$

The main observation

$$T_\beta[u](y) \to T[u](y) := \min\{u(0y)+H(0y), u(1y)+H(1y)\}$$

Convergence results at the log scale

$$u_\beta := -\frac{1}{\beta}\log\Phi_\beta, \quad \bar{H}_\beta := -\frac{1}{\beta}\log\lambda_\beta, \quad \mu_\beta := \Phi_\beta\nu_\beta$$

Theorem (Classical)

- $\bar{H}_\beta \to \bar{H} = \min\{\int H\,d\mu \,:\, \mu$ is σ-invariant$\}$
- u_β is uniformly Lipschitz, and if $u_\beta \to u$ for some sub-sequence

$$T[u] = u + \bar{H} = \min_{\sigma(y')=y}\{u(y') + H(y')\}$$

- Every accumulation measure $\mu_\beta \to \mu$ is a minimizing measure of reduced maximal entropy

$$\mathrm{Ent}(\mu) = \mathsf{Ent}(\mathrm{Mather}(H))$$

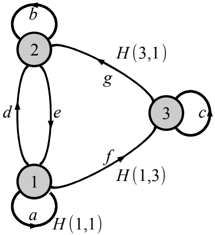
- If the Mather set admits a unique measure $\mu_{\min}$ of maximal entropy, then $\mu_\beta \to \mu_{\min}$. For instance, if the Mather set is reduced to a unique periodic orbit.

The selection case for short range Hamiltonian: notations

Definition A short range potential $H : \Sigma \to \mathbb{R}$ is a potential depending only on a finite number of indices

$$H(x) = H(x_0, \cdots, x_r)$$

We now choose $r = 1$: $\quad H(x) = H(x_0, x_1)$



Reduction of the problem $\quad \Sigma \subset \{1,2,3\}^{\mathbb{N}}$
We may extend the formalism to a sub-shift of finite type. The transfer operator is a matrix

$$\mathcal{L}_\beta = \left[e^{-\beta H(i,j)}\right] = \begin{bmatrix} e^{-\beta a} & e^{-\beta d} & e^{-\beta f} \\ e^{-\beta e} & e^{-\beta b} & 0 \\ 0 & e^{-\beta g} & e^{-\beta c} \end{bmatrix}$$

What is the limit of $\quad \mu_\beta([i]) = \dfrac{\nu_\beta(i)\Phi_\beta(i)}{\sum_j \nu_\beta(j)\Phi_\beta(j)} \quad$?

The selection case for short-range Hamiltonian: results

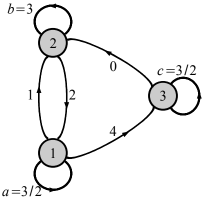
Theorem [Brémont 2003, Leplaideur 2005, Chazottes-Gambaudo-Ugalde 2011, Garibaldi-Thieullen 2012, ...]

If (Σ, σ) is a sub-shift of finite type, if $H : \Sigma \to \mathbb{R}$ has short range, then

- $\mu_\beta \to \mu$ converge to a minimizing measure
- The Mather set is a sub-shift of finite type possibly not irreducible
- All the quantities λ_β, ϕ_β, ν_β admit a Puiseux series expansion

$$\lambda_\beta = c_0 e^{-\beta\gamma_0} + c_1 e^{-\beta\gamma_1} + \cdots, \qquad \gamma_0 < \gamma_1 < \cdots$$

- $\gamma_0 = \bar{H}, \quad \log(c_0) = \mathrm{Ent}(\mathrm{Mather}(H))$



$$\mathcal{L}_\beta = \begin{bmatrix} e^{-\beta\frac{3}{2}} & e^{-\beta} & e^{-\beta 4} \\ e^{-\beta 2} & e^{-\beta 3} & 0 \\ 0 & 1 & e^{-\beta 2} \end{bmatrix} \qquad \mathrm{Mather}(H) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\lambda_\beta \sim \exp\left(\frac{1+\sqrt{5}}{2}\right) e^{-\beta\frac{3}{2}} \qquad \mu_\beta \to \frac{\nu(i)\Phi(i)}{\sum_i \nu(i)\Phi(i)}$$

The non-selection case for long-range Hamiltonian

Theorem [Chazottes-Hochman 2010]

There exists a minimal compact invariant set of zero entropy $K \subset \{0,1\}^{\mathbb{N}}$ such that if

$$H(x) = d(x, K)$$

then the Gibbs measure μ_β defined with respect to H admits at least 2 accumulation measures as $\beta \to +\infty$. Here

- $H = 0$ on K and $H > 0$ outside K
- $\bar{H} = 0, \quad \mathrm{Mather}(H) = K, \quad \mathrm{Ent}(\mathrm{Mather}(H)) = 0$

Results in the non-selection case

Questions

- Do the shape of the Mather set play a role in the problem of non-convergence? In Chazottes-Hochman the Mather set has a large complexity and the potential is simple.
- Do a simple Mather set imply the convergence of the Gibbs measure? In Brémont the Mather set is a finite union of sub-shifts of finite type with equal entropy

Goal

- Find a Mather set which is the simplest one, which contains at least two invariant measure with equal entropy. For example

$$\mathrm{Mather}(H) = \{0^\infty, 1^\infty\}$$

- Find a family of Hamiltonians H which are the simplest one (not short range) and find a necessary and sufficient condition that detects the non-selection case

III. The non-selection case for a simple Mather set

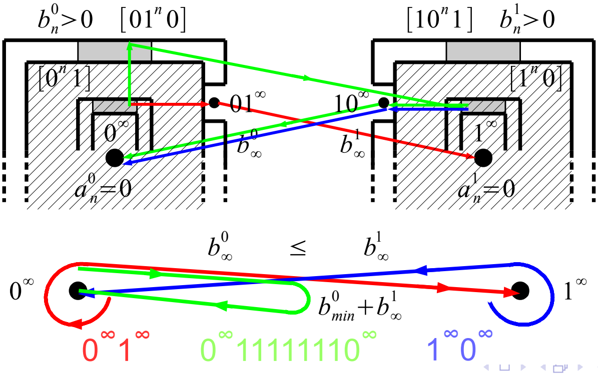
The long-range model

Model

- $H = 0$ on $[00] \cup [11]$
- $H = b_n^0 > 0$ on $[01^n0]$
- $b_{\min}^0 = \inf_n b_n^0$
- $b_\infty^0 = \lim_{n \to +\infty} b_n^0$

4 energy barriers

- barrier: $0^\infty \to 1^\infty = b_\infty^0$
- barrier: $1^\infty \to 0^\infty = b_\infty^1$
- barrier: $0^\infty \to 0^\infty = b_{\min}^0 + b_\infty^1$
- barrier: $1^\infty \to 1^\infty = b_{\min}^1 + b_\infty^0$



The second Puiseux exponent

A Puiseux series of the eignevalue

- $\lambda_\beta \sim c' e^{-\beta\gamma'}$ (always true)
- $\lambda_\beta = c' e^{-\beta\gamma'} + c'' e^{-\beta\gamma''} + \cdots \qquad \gamma' < \gamma''$ ($\Rightarrow$ selection case)
- γ', γ'', first exponent and second exponent are explicit

The first exponent

- $\gamma' = \bar{H} = \min\{\int H\, d\mu \ : \ \mu$ is invariant$\}$
- $\log(c') = \mathrm{Ent}(\mathrm{Mather}(H))$,

} in general

- $\gamma' = 0$
- $c' = 1$

} here

$H \geq 0$ everywhere and $\{0^\infty, 1^\infty\}$ are the only invariant set of $\{H = 0\}$

$\Rightarrow \ \mathrm{Mather}(H) = \{0^\infty, 1^\infty\}$ and has zero topological entropy

The second exponent We will show

if $\lambda_\beta = 1 + c'' e^{-\beta\gamma''} + o\big(e^{-\beta\gamma''}\big)$, $\gamma'' > 0$, then $\mu_\beta \to \mu_{\min}^{\text{selected}}$

The second Puiseux exponent

In the short-range case, an exponent is a mean of the energy along a cycle. The first exponent is computed along minimizing cycles and gives the Mather set. The second exponent is computed along cycles outside the Mather set.

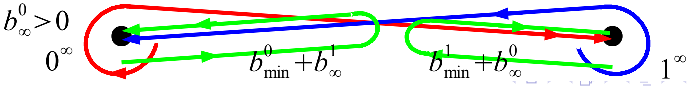
3 “second” intermediate exponents

- $\gamma_{0\leftrightarrow1} = \frac{1}{2}\Big(b^0_\infty + b^1_\infty\Big)$, a cycle of order 2
- $\gamma_{0\leftrightarrow0} = b^0_{\min} + b^1_\infty$, a cycle of order 1 at 0^∞
- $\gamma_{1\leftrightarrow1} = b^1_{\min} + b^0_\infty$, a cycle of order 1 at 1^∞

The a priori second exponent

$$\gamma'' := \min(\gamma_{0\leftrightarrow1}, \gamma_{0\leftrightarrow0}, \gamma_{1\leftrightarrow1})$$

An example with $\gamma'' > 0 \;\Rightarrow\;$ selection case



The second Puiseux exponent described by the Mañé potential

The Mañé potential $S(x,y) := \lim_{\epsilon\to 0} \liminf_{n\to+\infty} S_{\epsilon,n}(x,y) \in \mathbb{R}\cup\{+\infty\}$

$$S_{\epsilon,n}(x,y) = \inf\Big\{ \sum_{k=0}^{n-1}(H-\bar{H})\circ^k(z) \;:\; d(z,x)<\epsilon,\ d(\sigma^n(z),y)<\epsilon \Big\}$$

Energy barriers outside the Mather set

- $S_{00} = \lim_{x\to 0^\infty,\ y\to 0^\infty} S(x,y)$ (similarly S_{01}, S_{11}, S_{10})

Lemma

- $\gamma_{0\leftrightarrow 1} = \frac{1}{2}(S_{01}+S_{10})$, a cycle of order 2
- $\gamma_{0\leftrightarrow 0} = S_{00} \geq S_{10}$, a cycle of order 1 at 0^∞
- $\gamma_{1\leftrightarrow 1} = S_{11} \geq S_{01}$, a cycle of order 1 at 1^∞

The case of non selection $\gamma'' := \min(\gamma_{0\leftrightarrow 1}, \gamma_{0\leftrightarrow 0}, \gamma_{1\leftrightarrow})$

$$\gamma'' = 0 \iff S_{01} = S_{10} = 0$$

The case of selection

Assumption We may assume by symmetry: $S_{10} \geq S_{01}$. Then

$$\gamma'' > 0 \quad \Longrightarrow \quad S_{00} > S_{10} \geq \frac{1}{2}\Big(S_{01} + S_{10}\Big)$$

The coincidence order $\gamma'' := \min\big(S_{11}, \frac{1}{2}(S_{01} + S_{10})\big)$

- $S_{11} \neq \frac{1}{2}(S_{01} + S_{10)} > 0$ (coincidence of zero order)
- $S_{11} = \frac{1}{2}(S_{01} + S_{10)} > 0$ (coincidence of finite order)

  As $S_{11} = b^1_{\min} + S_{01}$, the number of coincidences is

$$\kappa := \mathrm{card}\Big\{n \geq 1 \,:\, b^1_n + S_{01} = \frac{1}{2}\Big(S_{01} + S_{10}\Big)\Big\}$$

Second Puiseux coefficient The largest solution of $X^2 = \kappa X + 1$

$$c := \frac{\kappa + \sqrt{\kappa^2 + 4}}{2}$$

The case of non selection $=$ coincidence of infinite order

$$\gamma'' = 0 \quad \Leftrightarrow \quad \frac{1}{2}(S_{01} + S_{10}) = S_{00} = S_{11} = 0$$

. The main result

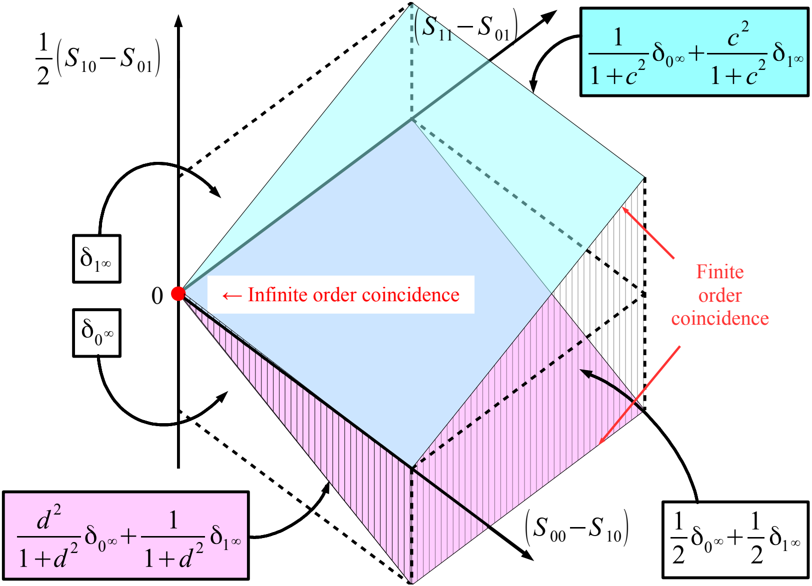
Theorem (Bissacot, Garibaldi, Thieullen, 2016 ETDS)

Assume $S_{10} \geq S_{01}$

- The zero order case, $\gamma'' > 0$ and $S_{11} \neq \frac{1}{2}(S_{01} + S_{10})$:
  - If $S_{11} < \frac{1}{2}(S_{01} + S_{10})$, then
  $$\mu_\beta \to \delta_{1^\infty}$$
  - if $S_{11} > \frac{1}{2}(S_{01} + S_{10})$, then
  $$\mu_\beta \to \frac{1}{2}\Big(\delta_{0^\infty} + \delta_{1^\infty}\Big)$$
- The finite order case, $\gamma'' > 0$ and $S_{11} = \frac{1}{2}(S_{01} + S_{10})$:
$$\mu_\beta \to \frac{1}{1+c^2}\delta_{0^\infty} + \frac{c^2}{1+c^2}\delta_{1^\infty}$$
- The infinite order case, $\gamma'' = 0 \Leftrightarrow b^0_\infty = b^1_\infty = 0$:

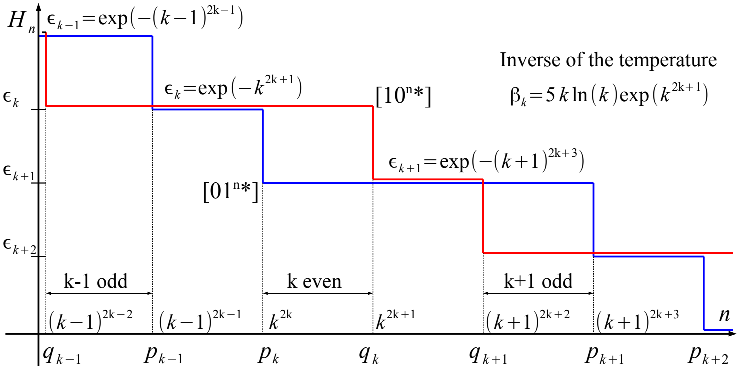
  There exists a Lipschitz potential H, that is a choice of $b^0_n \searrow 0$ and $b^1_n \searrow 0$, and a sub-sequence $\beta_k \to +\infty$ such that
$$\mu_{\beta_{2k}} \to \delta_{0^\infty} \quad \text{and} \quad \mu_{\beta_{2k+1}} \to \delta_{1^\infty}$$

The zero temperature phase diagram



III.c.4. The main result

The non-selection counter example



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