# A dynamical approach of some Hamilton-Jacobi equations using weak KAM theory

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# Summary of the talk

- **I.** A summary of weak KAM theory
- **II.** Discrete weak KAM approach
- **III.** Numerical analysis

# I. Summary

# I. Summary - Main goal

**Apply some methods in dynamical systems to solve**

- $\dfrac{\partial u}{\partial t}(t,x) + H(x, \nabla u(t,x)) = 0,$ (non stationary Hamilton-Jacobi),
- $H(x, \nabla \bar{u}(x)) = \bar{H},$ (cell equation),
- $\delta u_\delta(x) + H(x, \nabla u_\delta(x)) = 0,$ (discounted cell equation)

where $H(x,p) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is $C^2$, periodic in $x$, strictly convex and super-linear in $p$, autonomous

- $\left[\frac{\partial^2 H(x,p)}{\partial p_i \partial p_j}\right] \geq \alpha\left[\delta_{ij}\right],$
- $\lim_{R\to+\infty} \ \inf_{x,\|p\|\geq R} \frac{H(x,p)}{\|p\|} = +\infty$

$H(x,p)$ is called a *Tonelli Hamiltonian*

# I. Summary - known results

**Solutions in the viscosity sense**

Consider the cell equation

$$H(x, \nabla \bar{u}(x)) = \bar{H}$$

$u$ is said to be a sub-solution in the viscosity sense if

- $\forall x_0 \in \mathbb{T}^d,\ \forall \phi \in C^1(\mathbb{T}^d, \mathbb{R})$, if $\phi(x_0) = u(x_0)$ and $\forall x,\ \phi(x) \geq u(x)$,
- then $H(x_0, \nabla \phi(x_0)) \leq \bar{H}$.

**Existence of solutions** In all 3 equations a solution exists in the viscosity sense

- $\dfrac{\partial u}{\partial t}(t,x) + H(x, \nabla u(t,x)) = 0,$ $u(t,x)$ is unique,
- $H(x, \nabla \bar{u}(x)) = \bar{H},$ $\bar{u}(x)$ is not unique, $\bar{H}$ is unique,
- $\delta u_\delta(x) + H(x, \nabla u_\delta(x)) = 0,$ $u_\delta(x)$ is unique.

# I. Summary - Solutions

**Explicit solution of the non stationary Hamilton-Jacobi equation**

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + H(x, \nabla u(t,x)) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

the unique solution is given by

$$u(t,x) = \inf_{\substack{\gamma \in C^2([-t,0],\mathbb{R}^d) \\ \gamma(0)=x}} \left[ u_0(\gamma(-t)) + \int_{-t}^{0} L(\gamma, \dot{\gamma})\, ds \right],$$

where $L(x,v)$ is the Legendre transform of $H(x,p)$

$$L(x,v) = \sup_p \{ v \cdot p - H(x,p) \}$$

# I. Summary - Solutions

**Explicit solution of the effective energy**

$$H(x, \nabla \bar{u}(x)) = \bar{H}$$

then the effective energy is given by

$$-\bar{H} := \lim_{t \to +\infty} \inf_{\gamma \in C^2([-t,0],\mathbb{R}^d)} \left[ \frac{1}{t} \int_{-t}^{0} L(\gamma, \dot{\gamma})\, ds \right]$$

**Weak KAM theorem [Fathi, 1997]**

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + H(x, \nabla u(t,x)) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

$$u(t,x) + t\bar{H} \to \bar{u}(x) \quad \text{in the } C^0 \text{ topology}$$

$\bar{u}(x)$ is called weak KAM solution

# I. Summary - Solutions

**Explicit solution for the discounted cell equation**

$$\delta u_\delta(x) + H(x, \nabla u_\delta(x)) = 0$$

the unique solution is given by

$$u_\delta(x) = \inf_{\substack{\gamma \in C^2((-\infty,0],\mathbb{R}^d) \\ \gamma(0)=x}} \int_{-\infty}^{0} e^{s\delta} L(\gamma(s), \dot\gamma(s))\, ds,$$

**Weak KAM theorem [Davini,Fathi,Iturriaga,Zavidovique, 2016]**

$$\lim_{\delta\to 0} \left( u_\delta(x) + \frac{\bar H}{\delta} \right) = \bar u(x) \quad \text{in the } C^0 \text{ topology}$$

$$H(x, \bar u(x)) = \bar H$$

# I. Summary - Characteristics method

**Weierstrass-Tonelli solution** Assume $u_0 \in C^2(\mathbb{T}^d)$

$$\begin{cases} \dfrac{\partial u}{\partial t} + H(x, \nabla u(t,x)) = 0, \\ u(0,x) = u_0(x) \end{cases}$$

Then a $C^2$ solution exists uniformly in $x$ for a short time $0 \le t \le t_0$

$$u(t,x) := u_0(X_{t,x}(0)) + \int_0^t L(X_{t,x}(s), \dot{X}_{t,x}(s))\, ds$$

where $X_{t,x}(s)$, $P_{t,x}(s)$ are solutions of the Hamiltonian flow

$$\begin{aligned} \dot{X}_{x,t} &= \frac{\partial H}{\partial p}(X_{x,t}, P_{x,t}), \qquad X_{x,t}(t) = x \\ \dot{P}_{x,t} &= -\frac{\partial H}{\partial x}(X_{x,t}, P_{x,t}), \qquad P_{x,t}(0) = \nabla u_0(X_{x,t}(0)) \end{aligned}$$

# I. Summary - Characteristics method

**Remark** The Weierstrass-Tonelli solution is a minimizer

$$u(t,x) = \inf_{\substack{\gamma \in C^2([-t,0],\mathbb{R}^d) \\ \gamma(t)=x}} \left\{ u_0(\gamma(-t)) + \int_{-t}^{0} L(\gamma,\dot\gamma)\, ds \right\}$$

**Proof**

$$u(t,\gamma(t)) - u_0(\gamma(0)) = \int_0^t \frac{d}{ds} u(s,\gamma(s))\, ds = \int_0^t \left[ \frac{\partial u}{\partial t} + \nabla u \cdot \dot\gamma \right] ds$$

$$p.v \leq H(x,p) + L(x,v),$$

$$u(t,\gamma(t)) - u_0(\gamma(0)) \leq \int_0^t \left[ \frac{\partial u}{\partial t} + H(\gamma,\nabla u) + L(\gamma,\dot\gamma) \right] ds = \int_0^t L(\gamma,\dot\gamma)\, ds$$

# I. Summary - Dynamic programming

## Dynamic programming for the discounted equation

$$\delta u_\delta(x) + H(x, \nabla u_\delta(x)) = 0$$

$$u_\delta(x) = \inf_{\substack{\gamma \in C^2((-\infty,0],\mathbb{R}^d) \\ \gamma(0)=x}} \int_{-\infty}^{0} e^{s\delta} L(\gamma(s), \dot\gamma(s))\, ds$$

$$u_\delta(x) = \inf_{\substack{\gamma \in C^2([-t,0],\mathbb{R}^d) \\ \gamma(0)=x}} \left\{ e^{-t\delta} u_\delta(\gamma(-t)) + \int_{-t}^{0} e^{s\delta} L(\gamma(s), \dot\gamma(s))\, ds \right\}$$

## Dynamic programming for the cell equation

$$u_\delta(x) + \frac{\bar H}{\delta} \to \bar u(x), \qquad H(\bar u(x), \nabla \bar u(x)) = \bar H$$

$$\bar u(x) - t\bar H = \inf_{\substack{\gamma \in C^2([-t,0],\mathbb{R}^d) \\ \gamma(0)=x}} \left\{ \bar u(\gamma(-t)) + \int_{-t}^{0} L(\gamma, \dot\gamma)\, ds \right\}$$

# I. Summary - Lax-Oleinik operator

**Definition - Lax-Oleinik operator** For every $u_0 \in C^0(\mathbb{T}^d)$

- $T^t[u](x) := \inf_{\substack{\gamma \in C^2([-t,0],\mathbb{R}^d) \\ \gamma(0)=x}} \left\{ u(\gamma(-t)) + \int_{-t}^{0} L(\gamma, \dot\gamma)\, ds \right\}$
- $T^{s+t} = T^s \circ T^t$

**Weak KAM Theorem [Fathi]** for every $u \in C^0(\mathbb{T}^d)$

- $u(t,x) := T^t[u_0](x)$ solves $\left\{ \begin{array}{l} \dfrac{\partial u}{\partial t} + H(x, \nabla u(t,x)) = 0, \\ u(0,x) = u_0(x) \end{array} \right.$
- $T^t[u_0] + t\bar{H} \to \bar{u}(x)$ (in the $C^0$ topology)
- $H(x, \nabla \bar{u}(x)) = \bar{H} \iff T^t[\bar{u}] = \bar{u}(x) - t\bar{H}, \quad \forall t \geq 0$

# II. Discrete weak KAM approach

# II. The discrete Lax-Oleinik operator

**The cell equation**

$$H(x, \nabla \bar{u}(x)) = \bar{H}, \qquad H(x,p) \text{ is a Tonelli Hamiltoinan}$$

**The (continous) Lax-Oleinik operator** For every $u \in C^0(\mathbb{T}^d)$

$$T^t[u](x) := \inf_{\substack{\gamma \in C^2([-t,0],\mathbb{R}^d) \\ \gamma(0)=x}} \Big\{ u(\gamma(-t)) + \int_{-t}^{0} L(\gamma, \dot{\gamma})\, ds \Big\}$$

**The discrete Lax-Oleinik operator**

$$E_\tau(x,y) := \tau L\Big(x, \frac{y-x}{\tau}\Big), \quad \tau > 0 \quad \text{(small)}$$

$$T_\tau[u](y) = \min_{x \in \mathbb{R}^d} \big\{ u(x) + E_\tau(x,y) \big\}, \quad \forall u \in C^0(\mathbb{T}^d)$$

# II. The discrete cell equation

**Notations**

- $H(x,p)$ a Tonelli Hamiltonian
- $L(x,v) = \sup_{p \in \mathbb{R}^d} \{p \cdot v - H(x,p)\}$
- $E_\tau(x,y) = \tau L(x, \frac{y-x}{\tau})$
- $T_\tau[u](y) = \min_x \{u(x) + E_\tau(x,y)\}, \quad \forall u \in C^0(\mathbb{T}^d)$

**Theorem (Su, Th): first part**

- $\exists(\bar{u}_\tau, \bar{E}_\tau), \quad T_\tau[\bar{u}_\tau] = \bar{u}_\tau + \bar{E}_\tau$ (non unique $\bar{u}_\tau \in C^0(\mathbb{T}^d)$)
- $\bar{E}_\tau = \lim_{n \to +\infty} \inf_{x_0, x_1, \ldots, x_n} \frac{1}{n} \sum_{k=1}^{n} E_\tau(x_{k-1}, x_k)$ (unique)
- $\bar{E}_\tau = -\tau \bar{H} + O(\tau^2)$ (as $\tau \to 0$)
- $\bar{u}_\tau \to \bar{u}$, (for some subsequence , $\mathrm{Lip}(\bar{u}_\tau) \leq C$)
- $H(x, \nabla \bar{u}(x)) = \bar{H}$ (in the viscosity sense)

Main drawback: a sub-sequence need be taken for $u$.

# II. The discounted discrete Lax-Oleinik

**The discounted cell equation**

$$\delta u_\delta(x) + H(x, \nabla u_\delta(x))) = 0, \quad \text{(in the viscosity sense)}$$

$$u_\delta(x) = \inf_{\gamma(0)=x} \int_{-\infty}^{0} e^{s\delta} L(\gamma(s), \dot\gamma(s))\, ds$$

**The discounted Lax-Oleinik operator**

$$T_\delta^t[u](x) = \inf_{\gamma(0)=x} \Big\{ e^{-t\delta} u(\gamma(-t)) + \int_{-t}^{0} e^{s\delta} L(\gamma(s), \dot\gamma(s))\, ds \Big\}$$

**The discounted discrete Lax-Oleinik operator**

$$T_{\tau,\delta}[u](y) = \min_x \{(1-\tau\delta)u(x) + E_\tau(x,y)\} \qquad E_\tau(x,y) = \tau L\Big(x, \frac{y-x}{\tau}\Big)$$

# II. The discounted discrete cell equation

**Notations**

- $\delta u_\delta(x) + H(x, \nabla u_\delta(x)) = 0$ (the discounted cell equation)
- $H(x, \nabla \bar{u}(x)) = \bar{H}$ (the cell equation)
- $T_{\tau,\delta}[u](y) = \min_x\{(1-\tau\delta)u(x) + E_\tau(x,y)\}$
- $\bar{E}_\tau$ defined uniquely by $T_\tau[\bar{u}_\tau] = \bar{u}_\tau + \bar{E}_\tau$ for some $\bar{u}_\tau$

**Theorem (Su, Th): second part**

- $\exists u_{\tau,\delta} \quad T_{\tau,\delta}[u_{\tau,\delta}] = u_{\tau,\delta}$ (unique)

$$u_{\tau,\delta}(x) = \inf_{\substack{(x_{-k})_{k\geq 0} \\ x_0 = x}} \sum_{k=0}^{\infty} (1-\tau\delta)^k E_\tau(x_{-k-1}, x_{-k})$$

- $u_{\tau,\delta} = u_\delta + O(\frac{\tau}{\delta})$ as $\tau \to 0$ (for fixed $\delta > 0$)
- $u_\delta + \frac{\bar{H}}{\delta} \to \bar{u}$ (DFIZ theorem, no speed)
- $u_{\tau,\delta} - \frac{\bar{E}_\tau}{\tau\delta} \to \bar{u}_\tau$ as $\delta \to 0$ (for fixed $\tau > 0$, no speed)

$$\left[u_{\tau,\delta} - \frac{\bar{E}_\tau}{\tau\delta}\right] - \left[u_\delta + \frac{\bar{H}}{\delta}\right] = O\Big(\frac{\tau}{\delta}\Big)$$

# II. Selection principle

**Recall** $u_{\tau,\delta} - \frac{\bar{E}_\tau}{\tau\delta} \to \bar{u}_\tau$ is a distinguished solution of

$$T_\tau[\bar{u}_\tau](y) = \min_x \{\bar{u}_\tau(x) + E_\tau(x,y)\} = \bar{u}_\tau(y) + \bar{E}_\tau$$

**Stationary plan** A probability measure $\pi(dx,dy)$ on $\mathbb{T}^d \times \mathbb{T}^d$ s.t.

$$\iint \phi(x)\,\pi(dx,dy) = \iint \phi(y)\,\pi(dx,dy) \qquad \forall \phi \in C^0(\mathbb{T}^d)$$

**Minimizing plan** A stationary plan $\pi_{\text{min}}$ is minimizing if

$$\bar{E}_\tau = \min_\pi \iint E_\tau(x,y)\,\pi(dx,dy) = \iint E_\tau(x,y)\,\pi_{\text{min}}(dx,dy)$$

**Mañé Potential** Define on $\mathbb{T}^d \times \mathbb{T}^d$

$$\Phi_\tau(x,y) = \inf_{n \geq 1} \inf_{\substack{x_0, x_1, \cdots, x_n \\ x_0 = x,\ x_n = y}} \sum_{k=0}^{n-1} \left[E_\tau(x_k, x_{k+1}) - \bar{E}_\tau\right]$$

# II. Selection principle bis

**Notation**

- $T_\tau[u_\tau](y) = \min_x \{u_\tau(x) + E_\tau(x,y)\}$
- $T_{\tau,\delta}[u](y) = \min_x \{(1-\tau\delta)u(x) + E_\tau(x,y)\}$

**Theorem: second part** If $T_{\tau,\delta}[u_{\tau,\delta}] = u_{\tau,\delta}$

- $u_{\tau,\delta} - \dfrac{\bar{E}_\tau}{\tau\delta} \to \bar{u}_\tau$ as $\delta \to 0$ where $T_\tau[\bar{u}_\tau] = \bar{u}_\tau + \bar{E}_\tau$
- First characterization

$$\bar{u}_\tau(z) = \sup \Big\{ w(z) \; : \; T_\tau[w] = w + \bar{E}_\tau \text{ and } \iint w(y)\, d\pi(x,y) \leq 0, \quad \forall \pi \Big\}$$

- Second characterization

$$\bar{u}_\tau(z) = \inf \Big\{ \iint \Phi_\tau(x,z)\, d\pi(x,y) \; : \; \pi \text{ minimizing plan} \Big\}$$

# II. The discrete Aubry set

**Effective energy and Mañé potential**

$$\bar{E}_\tau = \lim_{n \to +\infty} \inf_{x_0, x_1, \ldots, x_n} \frac{1}{n} \sum_{k=1}^{n} E_\tau(x_{k-1}, x_k)$$

$$\Phi_\tau(x, y) = \inf_{n \geq 1} \inf_{\substack{x_0, x_1, \cdots, x_n \\ x_0 = x, \ x_n = y}} \sum_{k=0}^{n-1} \big[ E_\tau(x_k, x_{k+1}) - \bar{E}_\tau \big]$$

**The projected Aubry set** $\Phi_\tau(x, x) \geq 0, \quad \forall x \in \mathbb{T}^d$

$$\mathrm{ProjectedAubry}(E_\tau) := \{ x \in \mathbb{T}^d : \Phi_\tau(x, x) = 0 \}$$

**The Aubry set** $\Phi_\tau(x, y) + \Phi_\tau(y, x) \geq 0, \quad \forall (x, y) \in \mathbb{T}^d \times \mathbb{T}^d$

$$\mathrm{Aubry}(E_\tau) := \{ (x, y) \in \mathbb{T}^d \times \mathbb{T}^d : \Phi_\tau(x, y) + \Phi_\tau(y, x) = 0 \}$$

# II. Minimizing configuration

**Definition** A configuration $(x_k)_{k=-\infty}^{+\infty}$ is minimizing if

$$\sum_{k=m+1}^{m+n} \big[E_\tau(x_{k-1},x_k) - \bar{E}_\tau\big] \leq \sum_{k=m+1}^{m+n} \big[E_\tau(y_{k-1},y_k) - \bar{E}_\tau\big],$$

$$\forall m \in \mathbb{Z},\ n \geq 0 \quad \forall (y_k)_{k=-\infty}^{+\infty} \quad y_{m+1} = x_{m+1},\ y_{m+n} = x_{m+n}$$

**Existence** Let $T_\tau[\bar{u}_\tau] = \bar{u}_\tau + \bar{E}_\tau$, $y \in \mathbb{R}^d$, and

$$x_{-n-1} \in \arg\min_{x\in\mathbb{R}^d}\{u_\tau(x) + E_\tau(x, x_{-n})\}, \quad x_0 = y$$

- $(x_{-n})_{n\geq 0}$ is a backward minimizing configuration $\to \mathrm{Aubry}(E_\tau)$
- the Aubry set is a set of uniqueness for $\bar{u}_\tau$

$$\bar{u}_\tau(y) = \min_{x\in\mathrm{ProjectedAubry}(E_\tau)} \{\bar{u}_\tau(x) + \Phi_\tau(x,y)\}$$

# III. Numerical analysis

# III. The approximation scheme

**Solve** $\boldsymbol{H}(x, \nabla \bar{u}(x)) = \bar{H}$

- compute explicitly, $L(x,v) := \sup_{p \in \mathbb{R}^d} \big[ p \cdot v - H(x,p) \big]$,
- choose a step time $\tau > 0$ and define $E_\tau(x,y) := \tau L\Big(x, \frac{y-x}{\tau}\Big)$
- define $T_\tau[u_\tau](y) := \min_{x \in \mathbb{T}^d} \{ u_\tau(x) + E_\tau(x,y) \}$
- solve $T_\tau[\bar{u}_\tau] = \bar{u}_\tau + \bar{E}_\tau$ using Ishikawa's iterative method

$$u_\tau^{(0)} = 0, \qquad u_\tau^{(n+1)} := \frac{u_\tau^{(n)} + T_\tau[u_\tau^{(n)}]}{2} - \min\Big( \frac{u_\tau^{(n)} + T_\tau[u_\tau^{(n)}]}{2} \Big)$$

- $u_\tau^{(n)} \to \bar{u}_\tau$, $\quad \min\big(T_\tau[u_\tau^{(n)}]\big) \to \bar{E}_\tau$
- $-\frac{\bar{E}_\tau}{\tau} \to \bar{H}$, $\quad \bar{u}_\tau \to \bar{u}$ for some sub-sequence $\tau \to 0$
- $H(x, \nabla \bar{u}(x)) = \bar{H}$, in the viscosity sense

# III. The inverse pendulum

**Frenkel-Kontorova model**

- $H(x,p) = \frac{1}{2}(p+\lambda)^2 - KV(x), \quad V(x) = \frac{1}{(2\pi)^2}\big(1-\cos 2\pi x\big)$
- $L(x,v) = \frac{1}{2}v^2 - \lambda v + KV(x)$
- $E_\tau(x,y) = \tau L\big(x, \frac{y-x}{\tau}\big) = \frac{1}{2\tau}(y-x)^2 - \lambda(y-x) + \tau KV(x)$
- solve $\quad \bar{u}_\tau(y) + \bar{E}_\tau = \min_x\{\bar{u}_\tau(x) + E_\tau(x,y)\}$
- $-\bar{E}_\tau/\tau \to \bar{H}$
- $\bar{u}_\tau \to \bar{u}$ for some subsequence
- $\frac{1}{2}(\nabla\bar{u}+\lambda)^2 - KV(x) = \bar{H}$

# III. The continuous weak KAM

## Explicit solutions

- $\frac{1}{2}(\nabla \bar{u} + \lambda)^2 - KV(x) = \bar{H} \quad \Leftrightarrow \quad \nabla \bar{u} = -\lambda \pm \sqrt{2(\bar{H} + KV(x))}$
- $\lambda_* := \int_0^1 \sqrt{2KV(s)}\, ds = \frac{2\sqrt{K}}{\pi^2} \simeq 0.203\ \sqrt{K}$
- if $\lambda > \lambda_*$ then $\exists!\ \bar{H}$ s.t. $\lambda = \int_0^1 \sqrt{2(\bar{H} + KV(s))}\, ds$

$$\forall 0 \le x \le 1, \quad \bar{u}(x) = -\lambda x + \int_0^x \sqrt{2(\bar{H} + KV(s))}\, ds$$

- If $\lambda \le \lambda_*$ then $\bar{H} = 0$,

$$\exists!\ x_* \in [\tfrac{1}{2}, 1], \qquad \lambda = \int_0^{x_*} \sqrt{2KV(s)}\, ds - \int_{x_*}^1 \sqrt{2KV(s)}\, ds$$

$$\begin{cases} \bar{u}(x) = -\lambda x + \int_0^x \sqrt{2KV(s)}\, ds, & \text{if } 0 \le x \le x_* \\ \bar{u}(x) = -\lambda x + \int_0^{x_*} \sqrt{2KV(s)}\, ds - \int_{x_*}^x \sqrt{2KV(s)}\, ds, & \text{if } x_* \le x \le 1 \end{cases}$$

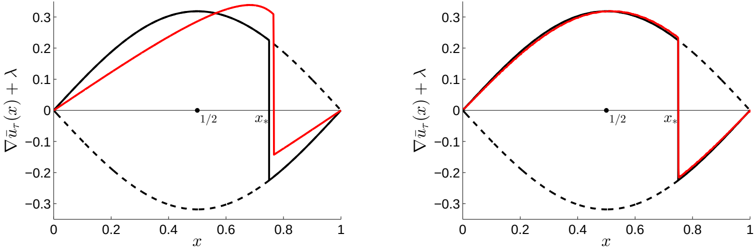
# III. The discrete weak KAM: $\bar{H} = 0$

## The cell equation

- $\frac{1}{2}(\nabla \bar{u}(x) + \lambda)^2 - KV(x) = \bar{H}, \quad V(x) = \frac{1}{(2\pi)^2}\big(1 - \cos 2\pi x\big)$
- $\bar{u}_\tau(y) + \bar{E}_\tau = \min_x \left\{ \bar{u}_\tau(x) + \frac{1}{2\tau}(y-x)^2 - \lambda(y-x) + \tau KV(x) \right\}$
- $K = 1, \;\; x_* = \frac{3}{4}, \;\; \lambda = 0.143 < \lambda_* = 0.202, \;\; N_{\text{grid}} = 1000$

## Numerical approximation

In black: $\bar{u}$, in red: $\bar{u}_\tau$




$\tau = 1, \;\; \bar{E}_\tau \sim 10^{-12}, \;\; N_{\text{iter}} = 32 \qquad \tau = 0.1, \;\; \bar{E}_\tau \sim 10^{-20}, \;\; N_{\text{iter}} = 196$

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