Zero-temperature Gibbs measures for some sub-shifts of finite type

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Summary of the talk

- **I.** Motivations
- **II.** Thermodynamic formalism
- **III.** Selection principle

I. Motivations

Cell equation Consider $H(x,p)$ a Tonelli Hamiltonian: C^2, autonomous, periodic in x, super-linear in p, and definite positive. The cell equation is

$$H(x, d_x u) = \bar{H}, \quad u \text{ as a viscosity solution}$$

Objective The cell equation is very degenerate. There are two approaches:

- a PDE approach (L.P.V.),
- a dynamical approach using Fathi's weak KAM theory.

- A third approach:

$\bar{H}$ and u are thermodynamic objects that
can be obtained as a limit when
the temperature of some the system goes to 0

Discretization in time (Bardi, Capuzzo-Dolcetta, Falcone, 2008)

- solve the discounted cell equation

$$\delta u_\delta + H(x, d_x u_\delta) = 0$$

- use the representation formula

$$u_\delta(x) = \inf_\gamma \int_{-\infty}^{0} e^{-\delta|t|} L(\gamma, \dot\gamma)\, dt$$

where $\gamma : (-\infty, 0] \to \mathbb{R}^d$ is absolutely continuous and $\gamma(0) = x$

- apply the Lions-Papanicolaou-Varadhan theorem (1987)

$$-\delta u_\delta \overset{C^0}{\to} \bar{H}, \quad \left\| u_\delta + \frac{\bar{H}}{\delta} \right\|_\infty \leqslant C, \quad \mathrm{Lip}(u_\delta) \leqslant C.$$

Take a sub-sequence $\delta_i \to 0$ so that $u_\delta + \frac{\bar{H}}{\delta_i} \overset{C^0}{\to} u$

Theorem (Davini, Fathi, Iturriaga, Zavidovique, 2015)

> $u_\delta + \frac{\bar{H}}{\delta} \overset{C^0}{\to} u$, solution of the cell equation $H(x, d_x u) = \bar{H}$

(A true limit as $\delta \to 0$)

Discretization in time

- discretize the representation formula

$$u_\delta(x) = \inf_\gamma \int_{-\infty}^{0} e^{-\delta|t|} L(\gamma, \dot\gamma)\, dt$$

- τ time step

$$u_{\tau,\delta}(x) = \inf_{(v_{-k})_{k\geqslant 0}} \sum_{k=0}^{+\infty} (1-\tau\delta)^k \tau L(x_{-k-1}, v_{-k-1})$$

$\exp(-\delta\tau k) \simeq (1-\tau\delta)^k, \quad x_{-k} = x_{-k-1} + \tau v_{-k-1}$

- Dynamical programming principle

$$u_{\tau,\delta}(x_0) = \inf_{x_{-1}} \left\{(1-\tau\delta) u_{\tau,\delta}(x_{-1}) + \tau L(x_{-1}, \tfrac{x_0 - x_{-1}}{\tau})\right\}$$

- it is easy to see, for fixed $\delta > 0$, $\quad u_{\tau,\delta} \to u_\delta$ as $\tau \to 0$

What happens if τ is fixed and $\delta \to 0$?

Theorem (Xifeng Su, Ph. Thieullen) $H(x,p)$ is a Tonelli Hamiltonian. Let $\tau, \delta > 0$ and $u_{\tau,\delta}$ be a solution of the discrete discounted cell equation

$$u_{\tau,\delta}(y) = \inf_{x \in \mathbb{R}^d} \Big\{ (1-\tau\delta) u_{\tau,\delta}(x) + \tau L\Big(x, \frac{x-y}{\tau}\Big) \Big\}, \quad \forall y \in \mathbb{R}^d$$

Then as $\delta \to 0$

- uniformly in x, $\tau\delta u_{\tau,\delta}(x) \to \bar{E}_\tau$
- uniformly in x, $u_{\tau,\delta}(x) - \frac{\bar{E}_\tau}{\tau\delta} \to u_\tau(x)$, u_τ is Lipschitz
- u_τ is a particular solution of the discrete cell equation

$$u_\tau(y) + \bar{E}_\tau = \inf_{x \in \mathbb{R}^d} \big\{ u_\tau(x) + E_\tau(x,y) \big\}, \quad \forall y \in \mathbb{R}^d$$

where $E_\tau(x,y) = \tau L(x, \frac{y-x}{\tau})$ is the discrete Lagrangian

- uniformly in τ, $\|u_\tau\|_\infty \leqslant C$, $\mathrm{Lip}(u_\tau) \leqslant C$
- for some sub-sequence $\tau \to 0$,

$$-\frac{\bar{E}_\tau}{\tau} \to \bar{H}$$

$u_\tau \to u$ (some solution of $H(x, d_x u) = \bar{H}$

Conclusion Discretization in time of the cell equation
$\iff$ solving an additive eigenvalue problem of the form:

$$T_-[u] = u + \bar{E}, \quad \text{with} \quad T_-[u](y) = \inf_{x\in\mathbb{R}^d} \{u(x) + E(x,y)\}$$

($T_-[u]$ is called (backward) Lax-Oleinik operator). $E(x,y) = \tau L(x, \frac{y-x}{\tau})$

Proposition If $E(x,y)$ is C^0, coercive $E(x,y) \to +\infty$ as $\|y-x\| \to +\infty$ and periodic $E(x+k, y+k) = E(x,y)$, then

- $\bar{E}$ is the unique additive eigenvalue
- $\exists u$ periodic, but u may not be unique.

Remark The minimization can be taken on $[0,1]^d$

$$\tilde{E}(x,y) = \min_{k\in\mathbb{Z}^d} E(x+k,y) \implies T_-^E[u] = T_-^{\tilde{E}}[u]$$

Program of research Identify the solution $(\bar{E}, u)$ as the ground level of some associated dynamical system. Simplify the problem by assuming, the space is discrete, $\{x_i : i = 1...N\}$ a grid of $[0,1]^d$, and the operator is discrete

$$T_-[u](x_j) = \min_{1\leqslant i\leqslant N} \{u(x_i) + \tilde{E}(x_i, x_j)\}$$

II. Termodynamic formalism

Objective Give a precise meaning to the notion of configurations at equilibrium for some temperature $T > 0$.

$$\text{configuration} \;=\; (\cdots, x_{i_{-2}}, x_{i_{-1}} \mid x_{i_0}, x_{i_1}, \cdots), \qquad x_i \in \text{grid of } [0,1]^d$$

Notations ◆ $S = \{1, \cdots, N\}$, some finite state space

◆ $\Omega = S^{\mathbb{Z}}$, the space of all configurations,

$$\omega = (\cdots, \omega_{-1} \mid \omega_0, \omega_1, \cdots), \quad \omega_k \in S$$

◆ $\sigma : \Omega \to \Omega$, the left shift on the indices

$$\sigma(\omega) = (\cdots, \omega_{-1}, \omega_0 \mid \omega_1, \omega_2, \cdots)$$

◆ $E : \Omega \to \mathbb{R}$, some Hölder function (long range interaction)
A short range interaction: $E(\omega) = E(\omega_0, \omega_1)$
Before we had $E(x_i, x_j) = \tau L\big(x_i, \frac{x_j - x_i}{\tau}\big)$

Important notions

- stationary or invariant measure μ: a probability measure such that if

$$[i_0, \cdots, i_n]_k := \{\omega \in \Omega : \omega_k = i_0, \omega_{k+1} = i_1, \cdots, \omega_{k+n} = i_n\}$$

then $\mu([i_0, \cdots, i_n]_k)$ is independent of k. For example

$$\text{i.i.d. measure} : \mu([i_0, \cdots, i_n]) = \Big(\frac{1}{N}\Big)^{n+1}$$

- the entropy of an invariant measure

$$\mathrm{Ent}_n(\mu) := \sum_{i_0 \cdots i_{n-1}} -\mu([i_0, \cdots, i_{n-1}]) \ln \mu([i_0, \cdots, i_{n-1}])$$

$$\lim_{n \to +\infty} \frac{1}{n} \mathrm{Ent}_n(\mu) = \text{exists} := \mathrm{Ent}(\mu).$$

For example $\mathrm{Ent}(\text{i.i.d.}) = \ln N$

- the free energy, let be $\beta^{-1} > 0$ called the temperature

$$\bar{E}_\beta := \inf_{\mu \text{ invariant}} \Big\{ \int E \, d\mu - \beta^{-1} \mathrm{Ent}(\mu) \Big\}$$

Theorem (Bowen, Ruelle, ∼1970) To simplify: $E(\omega) = E(\omega_0, \omega_1)$

- The infimum in the free energy

$$\bar{E}_\beta := \inf_{\mu \text{ invariant}} \Big\{ \int E \, d\mu - \beta^{-1} \mathrm{Ent}(\mu) \Big\}$$

is reached by a unique invariant measure: called Gibbs measure at temperature β^{-1} and denoted μ_β

- There is an explicit formula

$$\mu_\beta([i_0, \cdots, i_n]) = \frac{\exp\big[-\beta \sum_{k=0}^{n-1} E(i_k, i_{k+1}) \big]}{Z_\beta(i_0, \cdots, i_n)}$$

$$\frac{1}{Z_\beta(i_0, \cdots, i_n)} = \phi_\beta^-(i_0) \exp(\beta n \bar{E}_\beta) \phi_\beta^+(i_n)$$

- $\phi_\beta^\pm$ are backward and forward eigenfunctions of the transfer operator for the largest eigenvalue $\lambda_\beta = \exp[-\beta \bar{E}_\beta]$

$$\mathcal{L}_\beta^-[\phi_\beta^-](j) := \sum_{i=1}^N \phi_\beta^-(i) \exp[-\beta E(i,j)] = \exp[-\beta \bar{E}_\beta] \phi_\beta^-(j)$$

$$\mathcal{L}_\beta^+[\phi_\beta^+](i) := \sum_{j=1}^N \exp[-\beta E(i,j)] \phi_\beta^+(j) = \exp[-\beta \bar{E}_\beta] \phi_\beta^+(i)$$

Transfer operator and Lax-Oleinik

- The backward case

$$\mathcal{L}_\beta^-[\phi](j) := \sum_{i=1}^{N} \phi(i) \exp[-\beta E(i,j)]$$

- The Hopf technique:

$$\phi(i) = \exp[-\beta u(i)], \quad \mathcal{L}_\beta^-[\exp(-\beta u)](j) = \exp[-\beta T_\beta^-[u](j)]$$

- let $\beta \to +\infty$ in

$$\exp\big[-\beta T_\beta^-[u](j)\big] = \sum_{i=1}^{N} \exp\big[-\beta\big(u(i) + E(i,j)\big)\big]$$

- the zero-temperature of the transfer operator = Lax-Oleinik

$$T_\beta^-[u](j) \quad \to \quad T^-[u](j) := \min_{1\leqslant i\leqslant N}\big[u(i) + E(i,j)\big]$$

Important and simple facts

- The eigenvalue problem for transfer operator at temperature β^{-1} and for the Lax-Oleinik operator

$$T_\beta^-[u_\beta] = u_\beta + \bar{E}_\beta \quad \text{versus} \quad T^-[u] = u + \bar{E}$$

- there exists $C > 0$, $\quad \|u_\beta\|_\infty \leqslant C$, $\mathrm{Lip}(u_\beta) \leqslant C$
- $\bar{E}_\beta = \inf_\mu \left\{ \int E\,d\mu - \beta^{-1}\mathrm{Ent}(\mu) \right\} \to \inf_\mu \int E\,d\mu = \bar{E}$
- there exists $\beta_i \to +\infty$, $\quad \mu_{\beta_i} \to \mu_{min}$ and $u_{\beta_i} \to u$

$\mu_{min} \in \arg\min_{\mu \text{ invariant}} \int E\,d\mu, \qquad \text{Mather}(E) = \cup_{\mu_{min}} \mathrm{supp}(\mu_{min})$

Conclusion In the discrete case, both in time and in space

$$H(x, d_x u) = \bar{H} \iff \begin{cases} u & \text{``$=$''} \displaystyle\lim_{\beta\to+\infty} -\frac{1}{\beta}\ln\phi_\beta \\ -\bar{H} & = \displaystyle\lim_{\beta\to+\infty} \bar{E}_\beta = \lim_{\beta\to+\infty} -\frac{1}{\beta}\ln\lambda_\beta \end{cases}$$

- problem: the limit has to be taken along a sub-sequence

III. Selection principle

Question By freezing the system, do the Gibbs measure select a particular configuration?

Counter example (Chazottes-Hochman, 2010) There exists an Hölder energy $E : \Omega \to \mathbb{R}$, (long range), such that μ_β does not converge.

Question For which energy E do $\bar{E}_\beta$ and u_β converge?

$$\begin{cases} u_\beta = -\frac{1}{\beta} \ln \phi_\beta \\ \bar{E}_\beta = -\frac{1}{\beta} \ln \lambda_\beta \end{cases} \qquad \begin{cases} \mathcal{L}_\beta[\phi_\beta] = \lambda_\beta \phi_\beta \\ \mathcal{L}_\beta(i,j) = \exp\big[-\beta E(i,j)\big], \text{ (short range)} \end{cases}$$

A simpler question: for which E, do the Gibbs measure μ_β converge?

Theorem (Brémont, 2003) For short-range energy $E(i,j)$, $\mu_\beta \to \mu_\infty$ selects a particular minimizing invariant measure ($\mu_\infty \in \arg\min_\mu \int E \, d\mu$).

A simple example Consider $\Omega = \{1,2\}^{\mathbb{Z}}$ and $E(i,j) = \begin{bmatrix} 0 & 100 \\ 1 & 0 \end{bmatrix}$.

The energy is short range, null at the two fixed points, and positive along any other periodic cycle

$$E(1^{\mathbb{Z}}) = 0 = E(2^{\mathbb{Z}}), \quad i_0 = i_n \ \Rightarrow \sum_{k=0}^{n-1} E(i_k, i_{k+1}) > 0$$

As $\mu_\beta([i_0, \cdots, i_n]) = \exp\big[-\beta \sum_{k=0}^{n-1} E(i_k, i_{k+1})\big]/Z_\beta(i_0, \cdots, i_n)$ the Gibbs measure chooses the configurations with the least energy

$$\mu_\beta \to \frac{1}{2}\delta_{1^{\mathbb{Z}}} + \frac{1}{2}\delta_{2^{\mathbb{Z}}}$$

For example $E(i,j) = \begin{bmatrix} 0.1 & 100 \\ 1 & 0 \end{bmatrix}, \quad \mu_\beta \to \delta_{2^{\mathbb{Z}}}$

Question What can we say about the explicit convergence of μ_β when

$$\mathcal{L}_\beta(i,j) = \begin{bmatrix} 1 & e^{-111\beta} & e^{-45\beta} \\ e^{-\beta} & 1 & e^{-63\beta} \\ e^{-\beta} & e^{-\beta} & 1 \end{bmatrix}, \quad \text{or} \quad M_\epsilon(i,j) = \begin{bmatrix} 1 & \epsilon^a & \epsilon^b \\ \epsilon^{a'} & 1 & \epsilon^c \\ \epsilon^{b'} & \epsilon^{c'} & 1 \end{bmatrix}$$

Algorithm (E. Garibaldi, Ph. Thieullen, 2012) By induction on the dimension of the matrix $M_\epsilon(i,j)$. The framework needs to be extended

$$M_\epsilon(i,j) = A(i,j)\epsilon^{a(i,j)} + o(\epsilon^{a(i,j)})$$

$$o(\epsilon^{a(i,j)}) = A_1(i,j)\epsilon^{a_1(i,j)} + A_2(i,j)\epsilon^{a_2(i,j)} + \cdots \quad = \quad \text{a Puiseux series}$$

$$a(i,i) < a_1(i,j) < a_2(i,j) < \cdots \quad (\text{but } a_k(i,j) \text{ are not rational})$$

Let λ_ϵ, $\phi_\epsilon^\pm$ be the eigenvalue and eigenvectors of M_ϵ

$$\sum_i \phi_\epsilon^-(i) M_\epsilon(i,j) = \lambda_\epsilon \phi_\epsilon^-(j), \quad \sum_j M_\epsilon(i,j)\phi_\epsilon^+(j) = \lambda_\epsilon \phi_\epsilon^+(i)$$

Objective: find a Puiseux series expansion of λ_ϵ, ϕ_ϵ, μ_ϵ

Observation 1 (Special case for short-range E) if $\mu_\beta \to \mu_\infty$, for some sub-sequence, then μ_∞ is a barycenter of measures supported on minimizing cycles

Observation 2 (In general) The Mather set $\cup_{\mu_{min}} \text{supp}(\mu_{min})$ may have several components; μ_∞ chooses the one with the largest topological entropy

Proposition $M_\epsilon(i,j) = A(i,j)\epsilon^{a(i,j)} + o(\epsilon^{a(i,j)}) \implies \lambda_\epsilon \sim \bar{\alpha}\epsilon^{\bar{a}}$

- Remember $\bar{E} = \inf_\mu \int E\, d\mu = \lim_{\beta\to+\infty} \bar{E}_\beta = \lim_{\epsilon\to 0} \dfrac{\ln \lambda_\epsilon}{\ln \epsilon} = \bar{a}$

Definitions

- $\bar{a} = \inf_{\text{cycles}} \frac{1}{n}\sum_{k=0}^{n-1} a(i_k, i_{k+1}),$ cycle $= (i_0, \cdots, i_n)$ with $i_n = i_0$

For example

$$M_\epsilon \sim \begin{bmatrix} \epsilon^a & \epsilon^b \\ \epsilon^c & \epsilon^d \end{bmatrix}, \quad \implies \quad \bar{a} = \min\left(a, d, \frac{b+c}{2}\right)$$

- $G_{min} \subset S \times S$ the sub-graph of minimizing cycles.
- $A_{min} = [A(i,j)]_{(i,j)\in G_{min}}$ the restriction of A to G_{min}

For example

$$\frac{b+c}{2} < \min(a,d) \quad \implies \quad A_{min} = \begin{bmatrix} 0 & A(1,2) \\ A(2,1) & 0 \end{bmatrix}$$

- $\bar{\alpha}$ = the spectral radius of A_{min}

Reduction I Let $M_\epsilon(i,j) = A(i,j)\epsilon^{a(i,j)} + o(\epsilon^{a(i,j)})$
Then solve the discrete Lax-Oleinik equation for $a(i,j)$, that is find $u(i)$

$$\begin{cases} a(i,j) & \geqslant u(j) - u(i) + \bar{a} \quad \forall i,j \\ a(i,j) & = u(j) - u(i) + \bar{a} \quad \iff \quad (i,j) \in G_{min} \end{cases}$$

- Let $\Delta_\epsilon = \mathrm{diag}(\epsilon^{u(1)}, \cdots, \epsilon^{u(N)})$, then

$$M_\epsilon^{\mathrm{I}} := \Delta_\epsilon M_\epsilon \Delta_\epsilon^{-1}/\epsilon^{\bar{a}} = \begin{bmatrix} A'_{min} & 0 & 0 \\ 0 & A''_{min} & 0 \\ 0 & 0 & 0 \end{bmatrix} + N_\epsilon, \quad N_\epsilon = o(\mathrm{Id})$$

with $\rho(A''_{min}) < \rho(A'_{min}) = \bar{\alpha}$

$$M_\epsilon^{\mathrm{I}} = \begin{bmatrix} A'_{min} & 0 \\ 0 & D \end{bmatrix} + N_\epsilon, \quad A'_{min} = \mathrm{diag}(A^1_{min}, \cdots, A^r_{min})$$

with $\rho(A^1_{min}) = \cdots = \rho(A^r_{min}) = \bar{\alpha}$

From now on M_ϵ is reduced to its normal form

Reduction II $M_\epsilon = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + N_\epsilon,\ \ A = \mathrm{diag}(A^1, \cdots, A^r)$ with $\rho(D) < \rho(A^1) = \cdots = \rho(A^r) = \bar{\alpha}$. We already know $\lambda_\epsilon \sim \bar{\alpha}$

- R_ϵ, L_ϵ the right and left eigenvectors of M_ϵ for λ_ϵ

$$\begin{cases} M_\epsilon^{AA} R_\epsilon^A + M_\epsilon^{AD} R_\epsilon^D & = & \lambda_\epsilon R_\epsilon^A \\ M_\epsilon^{DA} R_\epsilon^A + M_\epsilon^{DD} R_\epsilon^D & = & \lambda_\epsilon R_\epsilon^D \end{cases}$$

Extract $R_\epsilon^D = (\lambda_\epsilon - M_\epsilon^{DD})^{-1} M_\epsilon^{DA} R_\epsilon^A$ and substitute

- We are left to study M_ϵ of the form

$$M_\epsilon = \begin{bmatrix} A^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A^r \end{bmatrix} + N_\epsilon, \quad \rho(A^1) = \cdots \rho(A^r) = \bar{\alpha}$$

- $R_\epsilon^i, L_\epsilon^i$ the restriction of R_ϵ, L_ϵ to the indices of A^i
- R^i, L^i the right and left eigenvectors of A^i for $\bar{\alpha}$

Proposition $\dfrac{R_\epsilon^i(x)}{R_\epsilon^i(y)} \sim \dfrac{R^i(x)}{R^i(y)}, \quad \forall x \overset{i}{\sim} y$ but $\dfrac{R_\epsilon^i(x)}{R_\epsilon^j(y)} \to ??$ for $i \neq j$

Reduction II $M_\epsilon = \mathrm{diag}(A^1, \cdots, A^r) + N_\epsilon, \ \ \rho(A^1) = \cdots = \rho(A^r) = \bar\alpha$
Write a system of r equations by blocs

$$(A^i + N_\epsilon^{ii})R_\epsilon^i + \sum_{j \neq i} N_\epsilon^{ij} R_\epsilon^j = \lambda_\epsilon R_\epsilon^i$$

Take the scalar product with the left fixed eigenvector L^i of A^i

$$\Big(\frac{L^i N_\epsilon^{ii} R_\epsilon^i}{L^i R_\epsilon^i}\Big) L^i R_\epsilon^i + \sum_{j \neq i} \Big(\frac{L^i N_\epsilon^{ij} R_\epsilon^j}{L^i R_\epsilon^j}\Big) L^i R_\epsilon^j = (\lambda_\epsilon - \bar\alpha) L^i R_\epsilon^i$$

$$M_\epsilon^{\mathrm{II}}(i,j) := \Big[\frac{L^i N_\epsilon^{ij} R_\epsilon^j}{L^i R_\epsilon^j}\Big], \quad M_\epsilon^{\mathrm{II}} R_\epsilon^{\mathrm{II}} = (\lambda_\epsilon - \bar\alpha) R_\epsilon^{\mathrm{II}}$$

$$M_\epsilon^{\mathrm{II}}(i,j) = A^{\mathrm{II}}(i,j)\epsilon^{a^{\mathrm{II}}(i,j)} + o(\epsilon^{a^{\mathrm{II}}(i,j)}),$$

$$a^{\mathrm{II}}(i,j) > 0, \quad \lambda_\epsilon - \bar\alpha \sim \bar\alpha^{(2)} \epsilon^{\bar a^{(2)}} \quad \text{(by induction)}$$

The missing case There is a problem if $r = N$

$$M_\epsilon = \bar\alpha \mathrm{Id} + N_\epsilon, \quad (M_\epsilon - \bar\alpha \mathrm{Id}) = N_\epsilon = B(i,j)\epsilon^{b(i,j)} + o(\epsilon^{b(i,j)})$$

$$\mathrm{diag}(N_\epsilon) = \mathrm{diag}(B^1)\epsilon^{b^1} + \cdots + \mathrm{diag}(B^s)\epsilon^{b^s} + N_\epsilon^{\mathrm{II}}$$

the minimizing sub-graph of N_ϵ^{II} either contains a cycle of order at least 2, or the number of cycles of order 1 est less than the dimension of N_ϵ^{II}

An example $N = 3$, $M_\epsilon(i,j) = \exp[-\beta E(i,j)]$. Reduction I gives

- A unique dominant irreducible component: $\bar\alpha = 1$,

$$M_\epsilon^{\mathsf{I}} = \begin{bmatrix} \epsilon^a & 1 & \epsilon^{b'} \\ \epsilon^{c'} & \epsilon^b & 1 \\ 1 & \epsilon^{a'} & \epsilon^c \end{bmatrix} \text{ or } \begin{bmatrix} \epsilon^a & 1 & \epsilon^d \\ 1 & \epsilon^b & \epsilon^e \\ \epsilon^{d'} & \epsilon^{e'} & \epsilon^c \end{bmatrix} \text{ or } \begin{bmatrix} 1 & \epsilon^a & \epsilon^c \\ \epsilon^{a'} & \epsilon^b & \epsilon^d \\ \epsilon^{c'} & \epsilon^{d'} & \epsilon^e \end{bmatrix}.$$

- Two irreducible components with equal dominant spectral radius

$$\bar\alpha = 1, \quad M_\epsilon^{\mathsf{I}} = \begin{bmatrix} 1 & \epsilon^a & \epsilon^b \\ \epsilon^{a'} & \epsilon^c & 1 \\ \epsilon^{b'} & 1 & \epsilon^d \end{bmatrix} \text{ or } \begin{bmatrix} 1 & \epsilon^a & \epsilon^b \\ \epsilon^{a'} & 1 & \epsilon^c \\ \epsilon^{b'} & \epsilon^{c'} & \epsilon^d \end{bmatrix}.$$

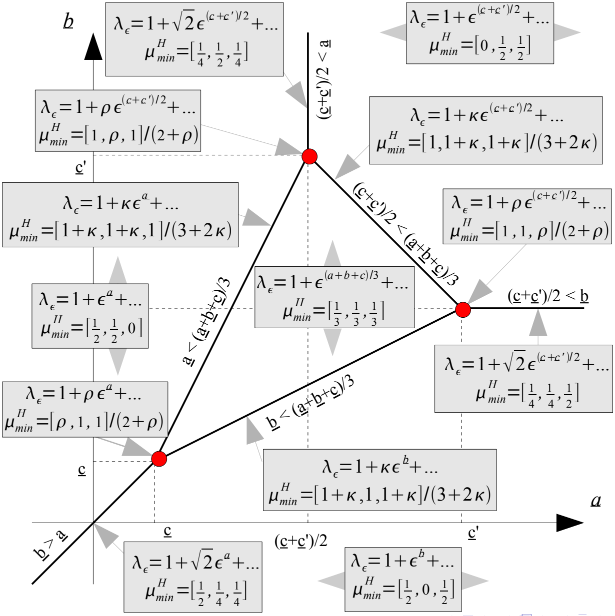
- Three irreducible components with equal dominant spectral radius:

$$\bar\alpha = 1, \quad M_\epsilon^{\mathsf{I}} = \begin{bmatrix} 1 & \epsilon^a & \epsilon^b \\ \epsilon^{a'} & 1 & \epsilon^c \\ \epsilon^{b'} & \epsilon^{c'} & 1 \end{bmatrix}.$$

- $\mu_\beta \to$ a barycenter of the periodic minimizing cycles

$$\mu_\beta \to \mu_{min}^H := c_1\delta_{1^{\mathbb{Z}}} + c_2\delta_{2^{\mathbb{Z}}} + c_3\delta_{3^{\mathbb{Z}}}$$

Example A 3×3 matrix with three irreducible components



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