Zero-temperature Gibbs measures for some sub-shifts of finite type

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Summary of the talk

– I. Motivations
– II. Thermodynamic formalism
– III. Selection principle
I. Motivations

**Cell equation** Consider $H(x, p)$ a Tonelli Hamiltonian: $C^2$, autonomous, periodic in $x$, super-linear in $p$, and definite positive. The cell equation is

$$H(x, d_x u) = \bar{H}, \quad u \text{ as a viscosity solution}$$

**Objective** The cell equation is very degenerate. There are two approaches:
- a PDE approach (L.P.V.),
- a dynamical approach using Fathi’s weak KAM theory.
- A third approach:

  $\bar{H}$ and $u$ are thermodynamic objects that can be obtained as a limit when the temperature of some the system goes to 0.
**Discretization in time** (Bardi, Capuzzo-Dolcetta, Falcone, 2008)
- solve the discounted cell equation

\[ \delta u_\delta + H(x, d_x u_\delta) = 0 \]

- use the representation formula

\[ u_\delta(x) = \inf_{\gamma} \int_{-\infty}^{0} e^{-\delta |t|} L(\gamma, \dot{\gamma}) \, dt \]

where \( \gamma: (-\infty, 0] \to \mathbb{R}^d \) is absolutely continuous and \( \gamma(0) = x \)

- apply the Lions-Papanicolaou-Varadhan theorem (1987)

\[ -\delta u_\delta \overset{C^0}{\to} \tilde{H}, \quad \left\| u_\delta + \frac{\tilde{H}}{\delta} \right\|_\infty \leq C, \quad \text{Lip}(u_\delta) \leq C. \]

Take a sub-sequence \( \delta_i \to 0 \) so that \( u_\delta + \frac{\tilde{H}}{\delta_i} \overset{C^0}{\to} u \)

**Theorem** (Davini, Fathi, Iturriaga, Zavidovique, 2015)

\[ u_\delta + \frac{\tilde{H}}{\delta} \overset{C^0}{\to} u, \quad \text{solution of the cell equation} \quad H(x, d_x u) = \tilde{H} \]

(A true limit as \( \delta \to 0 \))
Discretization in time

- discretize the representation formula

\[
    u_{\delta}(x) = \inf \gamma \int_{-\infty}^{0} e^{-\delta|t|} L(\gamma, \dot{\gamma}) \, dt
\]

- \( \tau \) time step

\[
    u_{\tau,\delta}(x) = \inf_{(v_{-k})_{k \geq 0}} \sum_{k=0}^{+\infty} (1 - \tau\delta)^k \tau L(x_{-k-1}, v_{-k-1})
\]

\[
    \exp(-\delta \tau k) \simeq (1 - \tau\delta)^k, \quad x_{-k} = x_{-k-1} + \tau v_{-k-1}
\]

- Dynamical programming principle

\[
    u_{\tau,\delta}(x_0) = \inf_{x_{-1}} \left\{ (1 - \tau\delta)u_{\tau,\delta}(x_{-1}) + \tau L(x_{-1}, \frac{x_0 - x_{-1}}{\tau}) \right\}
\]

- it is easy to see, for fixed \( \delta > 0 \), \( u_{\tau,\delta} \to u_\delta \) as \( \tau \to 0 \)

What happens if \( \tau \) is fixed and \( \delta \to 0 \)?
Theorem (Xifeng Su, Ph. Thieullen) $H(x, p)$ is a Tonelli Hamiltonian. Let $\tau, \delta > 0$ and $u_{\tau, \delta}$ be a solution of the discrete discounted cell equation

$$u_{\tau, \delta}(y) = \inf_{x \in \mathbb{R}^d} \left\{ (1 - \tau \delta) u_{\tau, \delta}(x) + \tau L\left(x, \frac{x - y}{\tau}\right) \right\}, \quad \forall y \in \mathbb{R}^d$$

Then as $\delta \to 0$

- uniformly in $x$, $\tau \delta u_{\tau, \delta}(x) \to \bar{E}_\tau$
- uniformly in $x$, $u_{\tau, \delta}(x) - \frac{\bar{E}_\tau}{\tau \delta} \to u_\tau(x)$, $u_\tau$ is Lipschitz
- $u_\tau$ is a particular solution of the discrete cell equation

$$u_\tau(y) + \bar{E}_\tau = \inf_{x \in \mathbb{R}^d} \left\{ u_\tau(x) + E_\tau(x, y) \right\}, \quad \forall y \in \mathbb{R}^d$$

where $E_\tau(x, y) = \tau L\left(x, \frac{y - x}{\tau}\right)$ is the discrete Lagrangian

- uniformly in $\tau$, $\|u_\tau\|_\infty \leq C$, $\text{Lip}(u_\tau) \leq C$
- for some sub-sequence $\tau \to 0$,

$$u_{\tau} \to u \quad \frac{-\bar{E}_\tau}{\tau} \to \bar{H}$$

(some solution of $H(x, d_x u) = \bar{H}$)
Conclusion  Discretization in time of the cell equation
\[ T_-[u] = u + \bar{E}, \quad \text{with} \quad T_-[u](y) = \inf_{x \in \mathbb{R}^d} \{ u(x) + E(x, y) \} \]

\((T_-[u]\) is called (backward) Lax-Oleinik operator). \(E(x, y) = \tau L(x, \frac{y-x}{\tau})\)

Proposition  If \(E(x, y)\) is \(C^0\), coercive \(E(x, y) \to +\infty\) as \(\|y - x\| \to +\infty\) and periodic \(E(x + k, y + k) = E(x, y)\), then

- \(\bar{E}\) is the unique additive eigenvalue
- \(\exists u\) periodic, but \(u\) may not be unique.

Remark  The minimization can be taken on \([0, 1]^d\)

\[ \tilde{E}(x, y) = \min_{k \in \mathbb{Z}^d} E(x + k, y) \implies T^{-\tilde{E}}[u] = T^{-\bar{E}}[u] \]

Program of research  Identify the solution \((\bar{E}, u)\) as the ground level of some associated dynamical system. Simplify the problem by assuming, the space is discrete, \(\{x_i : i = 1...N\}\) a grid of \([0, 1]^d\), and the operator is discrete

\[ T_-[u](x_j) = \min_{1 \leq i \leq N} \{ u(x_i) + \tilde{E}(x_i, x_j) \} \]
II. Termodynamic formalism

Objective Give a precise meaning to the notion of configurations at equilibrium for some temperature $T > 0$.

configuration $\equiv (\cdots , x_{i-2}, x_{i-1} \mid x_i, x_{i+1}, \cdots ), \quad x_i \in \text{grid of } [0, 1]^d$

Notations
- $S = \{1, \cdots , N\}$, some finite state space
- $\Omega = S^\mathbb{Z}$, the space of all configurations,

$\omega = (\cdots , \omega_{-1} \mid \omega_0, \omega_1, \cdots ), \quad \omega_k \in S$

- $\sigma : \Omega \to \Omega$, the left shift on the indices

$\sigma(\omega) = (\cdots , \omega_{-1}, \omega_0 \mid \omega_1, \omega_2, \cdots )$

- $E : \Omega \to \mathbb{R}$, some Hölder function (long range interaction)

A short range interaction: $E(\omega) = E(\omega_0, \omega_1)$

Before we had $E(x_i, x_j) = \tau L(x_i, \frac{x_j-x_i}{\tau})$
Important notions

- stationary or invariant measure $\mu$: a probability measure such that if

$$[i_0, \cdots, i_n]_k := \{\omega \in \Omega : \omega_k = i_0, \omega_{k+1} = i_1, \cdots, \omega_{k+n} = i_n\}$$

then $\mu([i_0, \cdots, i_n]_k)$ is independent of $k$. For example

i.i.d. measure: $\mu([i_0, \cdots, i_n]) = \left(\frac{1}{N}\right)^{n+1}$

- the entropy of an invariant measure

$$\text{Ent}_n(\mu) := \sum_{i_0 \cdots i_{n-1}} -\mu([i_0, \cdots, i_{n-1}]) \ln \mu([i_0, \cdots, i_{n-1}])$$

$$\lim_{n \to +\infty} \frac{1}{n} \text{Ent}_n(\mu) = \text{exists} := \text{Ent}(\mu).$$

For example $\text{Ent}(\text{i.i.d.}) = \ln N$

- the free energy, let be $\beta^{-1} > 0$ called the temperature

$$\bar{E}_\beta := \inf_{\mu \text{ invariant}} \left\{ \int E \, d\mu - \beta^{-1} \text{Ent}(\mu) \right\}$$
Theorem (Bowen, Ruelle, ~1970) To simplify: $E(\omega) = E(\omega_0, \omega_1)$

- The infimum in the free energy

$$\bar{E}_\beta := \inf_{\mu \text{ invariant}} \left\{ \int E \, d\mu - \beta^{-1} \text{Ent}(\mu) \right\}$$

is reached by a unique invariant measure: called Gibbs measure at temperature $\beta^{-1}$ and denoted $\mu_\beta$

- There is an explicit formula

$$\mu_\beta([i_0, \cdots, i_n]) = \frac{\exp \left[ - \beta \sum_{k=0}^{n-1} E(i_k, i_{k+1}) \right]}{Z_\beta(i_0, \cdots, i_n)}$$

$$\frac{1}{Z_\beta(i_0, \cdots, i_n)} = \phi^-_\beta(i_0) \exp(\beta n \bar{E}_\beta) \phi^+_\beta(i_n)$$

- $\phi^\pm_\beta$ are backward and forward eigenfunctions of the transfer operator for the largest eigenvalue $\lambda_\beta = \exp[-\beta \bar{E}_\beta]$

$$\mathcal{L}^-_\beta[\phi^-_\beta](j) := \sum_{i=1}^{N} \phi^-_\beta(i) \exp[-\beta E(i, j)] = \exp[-\beta \bar{E}_\beta] \phi^-_\beta(j)$$

$$\mathcal{L}^+_\beta[\phi^+_\beta](i) := \sum_{j=1}^{N} \exp[-\beta E(i, j)] \phi^+_\beta(j) = \exp[-\beta \bar{E}_\beta] \phi^+_\beta(i)$$
Transfer operator and Lax-Oleinik

- The backward case

\[ \mathcal{L}_\beta^-(\phi)(j) := \sum_{i=1}^{N} \phi(i) \exp[-\beta E(i, j)] \]

- The Hopf technique:

\[ \phi(i) = \exp[-\beta u(i)], \quad \mathcal{L}_\beta^-[\exp(-\beta u)](j) = \exp[-\beta T_\beta^- u](j) \]

- let \( \beta \rightarrow +\infty \) in

\[ \exp[-\beta T_\beta^-[u](j)] = \sum_{i=1}^{N} \exp[-\beta (u(i) + E(i, j))] \]

- the zero-temperature of the transfer operator = Lax-Oleinik

\[ T_\beta^-[u](j) \rightarrow T^-[u](j) := \min_{1 \leq i \leq N} [u(i) + E(i, j)] \]
Important and simple facts

- The eigenvalue problem for transfer operator at temperature $\beta^{-1}$ and for the Lax-Oleinik operator

$$T_\beta^{-}[u_{\beta}] = u_{\beta} + \bar{E}_{\beta} \quad \text{versus} \quad T^{-}[u] = u + \bar{E}$$

- there exists $C > 0, \quad \|u_{\beta}\|_{\infty} \leq C, \quad \text{Lip}(u_{\beta}) \leq C$

- $\bar{E}_{\beta} = \inf_{\mu} \{ \int E \, d\mu - \beta^{-1} \text{Ent}(\mu) \} \to \inf_{\mu} \int E \, d\mu = \bar{E}$

- there exists $\beta_i \to +\infty, \quad \mu_{\beta_i} \to \mu_{min}$ and $u_{\beta_i} \to u$

$$\mu_{min} \in \text{arg min}_{\mu \text{ invariant}} \int E \, d\mu, \quad \text{Mather}(E) = \cup_{\mu_{min}} \text{supp}(\mu_{min})$$

Conclusion In the discrete case, both in time and in space

$$H(x, d_x u) = \bar{H} \iff \begin{cases} u \quad \text{“=} \quad \lim_{\beta \to +\infty} -\frac{1}{\beta} \ln \phi_{\beta} \\ -\bar{H} = \lim_{\beta \to +\infty} \bar{E}_{\beta} = \lim_{\beta \to +\infty} -\frac{1}{\beta} \ln \lambda_{\beta} \end{cases}$$

- problem: the limit has to be taken along a sub-sequence
III. Selection principle

**Question** By freezing the system, do the Gibbs measure select a particular configuration?

**Counter example** (Chazottes-Hochman, 2010) There exists an Hölder energy $E : \Omega \to \mathbb{R}$, (long range), such that $\mu_\beta$ does not converge.

**Question** For which energy $E$ do $\bar{E}_\beta$ and $u_\beta$ converge?

\[
\begin{align*}
    u_\beta &= -\frac{1}{\beta} \ln \phi_\beta \\
    \bar{E}_\beta &= -\frac{1}{\beta} \ln \lambda_\beta
\end{align*}
\]

\[
\begin{align*}
    \mathcal{L}_\beta[\phi_\beta] &= \lambda_\beta \phi_\beta \\
    \mathcal{L}_\beta(i, j) &= \exp \left[ -\beta E(i, j) \right], \text{ (short range)}
\end{align*}
\]

A simpler question: for which $E$, do the Gibbs measure $\mu_\beta$ converge?

**Theorem** (Brémont, 2003) For short-range energy $E(i, j)$, $\mu_\beta \to \mu_\infty$ selects a particular minimizing invariant measure ($\mu_\infty \in \arg\min_\mu \int E \, d\mu$).
A simple example  Consider $\Omega = \{1, 2\}^\mathbb{Z}$ and $E(i, j) = \begin{bmatrix} 0 & 100 \\ 1 & 0 \end{bmatrix}$.

The energy is short range, null at the two fixed points, and positive along any other periodic cycle

$$E(1^\mathbb{Z}) = 0 = E(2^\mathbb{Z}), \quad i_0 = i_n \Rightarrow \sum_{k=0}^{n-1} E(i_k, i_{k+1}) > 0$$

As $\mu_\beta([i_0, \cdots, i_n]) = \exp[ -\beta \sum_{k=0}^{n-1} E(i_k, i_{k+1}) ] / Z_\beta(i_0, \cdots, i_n)$ the Gibbs measure chooses the configurations with the least energy

$$\mu_\beta \to \frac{1}{2} \delta_{1^\mathbb{Z}} + \frac{1}{2} \delta_{2^\mathbb{Z}}$$

For example $E(i, j) = \begin{bmatrix} 0.1 & 100 \\ 1 & 0 \end{bmatrix}$, $\mu_\beta \to \delta_{2^\mathbb{Z}}$

**Question** What can we say about the explicit convergence of $\mu_\beta$ when

$$\mathcal{L}_\beta(i, j) = \begin{bmatrix} 1 & e^{-111\beta} & e^{-45\beta} \\ e^{-\beta} & 1 & e^{-63\beta} \\ e^{-\beta} & e^{-\beta} & 1 \end{bmatrix}, \quad \text{or} \quad M_\epsilon(i, j) = \begin{bmatrix} 1 & \epsilon^a & \epsilon^b \\ \epsilon^{a'} & 1 & \epsilon^c \\ \epsilon^{b'} & \epsilon^{c'} & 1 \end{bmatrix}$$
**Algorithm** (E. Garibaldi, Ph. Thieullen, 2012) By induction on the dimension of the matrix $M_\epsilon(i,j)$. The framework needs to be extended

$$M_\epsilon(i,j) = A(i,j)\epsilon^{a(i,j)} + o(\epsilon^{a(i,j)})$$

$$o(\epsilon^{a(i,j)}) = A_1(i,j)\epsilon^{a_1(i,j)} + A_2(i,j)\epsilon^{a_2(i,j)} + \cdots = \text{a Puiseux series}$$

$$a(i,i) < a_1(i,j) < a_2(i,j) < \cdots \quad \text{(but} a_k(i,j) \text{are not rational)}$$

Let $\lambda_\epsilon, \phi_\epsilon^\pm$ be the eigenvalue and eigenvectors of $M_\epsilon$

$$\sum_i \phi_\epsilon^-(i)M_\epsilon(i,j) = \lambda_\epsilon \phi_\epsilon^-(j), \quad \sum_j M_\epsilon(i,j)\phi_\epsilon^+(j) = \lambda_\epsilon \phi_\epsilon^+(i)$$

**Objective:** find a Puiseux series expansion of $\lambda_\epsilon, \phi_\epsilon, \mu_\epsilon$

**Observation 1** (Special case for short-range $E$) if $\mu_\beta \to \mu_\infty$, for some sub-sequence, then $\mu_\infty$ is a barycenter of measures supported on minimizing cycles

**Observation 2** (In general) The Mather set $\cup_{\mu_{\text{min}}} \text{supp}(\mu_{\text{min}})$ may have several components; $\mu_\infty$ chooses the one with the largest topological entropy
Proposition \[ M_\varepsilon(i, j) = A(i, j)\varepsilon^{a(i,j)} + o(\varepsilon^{a(i,j)}) \implies \lambda_\varepsilon \sim \bar{\alpha} \varepsilon^{\bar{a}} \]

- Remember \( \bar{E} = \inf_\mu \int E \, d\mu = \lim_{\beta \to +\infty} \bar{E}_\beta = \lim_{\varepsilon \to 0} \frac{\ln \lambda_\varepsilon}{\ln \varepsilon} = \bar{a} \)

Definitions
- \( \bar{a} = \inf_{\text{cycles}} \frac{1}{n} \sum_{k=0}^{n-1} a(i_k, i_{k+1}), \quad \text{cycle} = (i_0, \cdots, i_n) \text{ with } i_n = i_0 \)

For example

\[
M_\varepsilon \sim \begin{bmatrix} \varepsilon^a & \varepsilon^b \\ \varepsilon^c & \varepsilon^d \end{bmatrix}, \quad \implies \bar{a} = \min \left( a, d, \frac{b+c}{2} \right)
\]

- \( G_{min} \subset S \times S \) the sub-graph of minimizing cycles.
- \( A_{min} = [A(i, j)]_{(i,j) \in G_{min}} \) the restriction of \( A \) to \( G_{min} \)

For example

\[
\frac{b+c}{2} < \min(a, d) \implies A_{min} = \begin{bmatrix} 0 & A(1, 2) \\ A(2, 1) & 0 \end{bmatrix}
\]

- \( \bar{\alpha} = \text{the spectral radius of } A_{min} \)
Reduction I Let $M_{\epsilon}(i, j) = A(i, j)e^{a(i, j)} + o(e^{a(i, j)})$

Then solve the discrete Lax-Oleinik equation for $a(i, j)$, that is find $u(i)$

\[
\begin{align*}
\begin{cases}
  a(i, j) &\geq u(j) - u(i) + \bar{\alpha} &\forall i, j \\
  a(i, j) &= u(j) - u(i) + \bar{\alpha} &\iff (i, j) \in G_{min}
\end{cases}
\end{align*}
\]

- Let $\Delta_{\epsilon} = \text{diag}(e^{u(1)}, \ldots, e^{u(N)})$, then

\[
M_{\epsilon}^1 := \Delta_{\epsilon} M_{\epsilon} \Delta_{\epsilon}^{-1} / \epsilon \bar{\alpha} = \begin{bmatrix}
    A'_{\text{min}} & 0 & 0 \\
    0 & A''_{\text{min}} & 0 \\
    0 & 0 & 0
\end{bmatrix} + N_{\epsilon}, \quad N_{\epsilon} = o(\text{Id})
\]

with $\rho(A''_{\text{min}}) < \rho(A'_{\text{min}}) = \bar{\alpha}$

\[
M_{\epsilon}^1 = \begin{bmatrix}
    A'_{\text{min}} & 0 \\
    0 & D
\end{bmatrix} + N_{\epsilon}, \quad A'_{\text{min}} = \text{diag}(A^1_{\text{min}}, \ldots, A^r_{\text{min}})
\]

with $\rho(A^1_{\text{min}}) = \cdots = \rho(A^r_{\text{min}}) = \bar{\alpha}$

From now on $M_{\epsilon}$ is reduced to its normal form
Reduction II \[ M_\epsilon = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + N_\epsilon, \quad A = \text{diag}(A^1, \ldots, A^r) \] with \[ \rho(D) < \rho(A^1) = \cdots = \rho(A^r) = \bar{\alpha}. \] We already know \( \lambda_\epsilon \sim \bar{\alpha} \)

- \( R_\epsilon, L_\epsilon \) the right and left eigenvectors of \( M_\epsilon \) for \( \lambda_\epsilon \)

\[
\begin{align*}
M_\epsilon^{AA}R_\epsilon^A + M_\epsilon^{AD}R_\epsilon^D &= \lambda_\epsilon R_\epsilon^A \\
M_\epsilon^{DA}R_\epsilon^A + M_\epsilon^{DD}R_\epsilon^D &= \lambda_\epsilon R_\epsilon^D
\end{align*}
\]

Extract \( R_\epsilon^D = (\lambda_\epsilon - M_\epsilon^{DD})^{-1}M_\epsilon^{DA}R_\epsilon^A \) and substitute

- We are left to study \( M_\epsilon \) of the form

\[
M_\epsilon = \begin{bmatrix} A^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A^r \end{bmatrix} + N_\epsilon, \quad \rho(A^1) = \cdots \rho(A^r) = \bar{\alpha}
\]

- \( R^i_\epsilon, L^i_\epsilon \) the restriction of \( R_\epsilon, L_\epsilon \) to the indices of \( A^i \)

- \( R^i, L^i \) the right and left eigenvectors of \( A^i \) for \( \bar{\alpha} \)

Proposition

\[
\frac{R^i_\epsilon(x)}{R^i_\epsilon(y)} \sim \frac{R^i(x)}{R^i(y)}, \quad \forall x \sim y \quad \text{but} \quad \frac{R^i_\epsilon(x)}{R^j_\epsilon(y)} \rightarrow ?? \quad \text{for} \ i \neq j
\]

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Reduction II $M_\epsilon = \text{diag}(A^1, \cdots, A^r) + N_\epsilon, \ \rho(A^1) = \cdots = \rho(A^r) = \bar{\alpha}$

Write a system of $r$ equations by blocs

$$(A^i + N_\epsilon^{ii})R_\epsilon^i + \sum_{j \neq i} N_\epsilon^{ij} R_\epsilon^j = \lambda_\epsilon R_\epsilon^i$$

Take the scalar product with the left fixed eigenvector $L^i$ of $A^i$

$$\left( \frac{L^i N_\epsilon^{ii} R_\epsilon^i}{L^i R_\epsilon^i} \right) L^i R_\epsilon^i + \sum_{j \neq i} \left( \frac{L^i N_\epsilon^{ij} R_\epsilon^j}{L^i R_\epsilon^j} \right) L^i R_\epsilon^j = (\lambda_\epsilon - \bar{\alpha})L^i R_\epsilon^i$$

$$M_\epsilon^{II}(i, j) := \left[ \frac{L^i N_\epsilon^{ij} R_\epsilon^j}{L^i R_\epsilon^j} \right], \quad M_\epsilon^{II} R_\epsilon^i = (\lambda_\epsilon - \bar{\alpha}) R_\epsilon^i$$

$$M_\epsilon^{II}(i, j) = A^{II}(i, j)\epsilon^{a^{II}(i, j)} + o(\epsilon^{a^{II}(i, j)}),$$

$$a^{II}(i, j) > 0, \quad \lambda_\epsilon - \bar{\alpha} \sim \bar{\alpha}^{(2)}\epsilon^{\bar{\alpha}^{(2)}} \quad \text{(by induction)}$$

The missing case  There is a problem if $r = N$

$$M_\epsilon = \bar{\alpha}\text{Id} + N_\epsilon, \quad (M_\epsilon - \bar{\alpha}\text{Id}) = N_\epsilon = B(i, j)\epsilon^{b(i, j)} + o(\epsilon^{b(i, j)})$$

$$\text{diag}(N_\epsilon) = \text{diag}(B^1)\epsilon^{b^1} + \cdots + \text{diag}(B^s)\epsilon^{b^s} + N_\epsilon^{II}$$

the minimizing sub-graph of $N_\epsilon^{II}$ either contains a cycle of order at least 2, or the number of cycles of order 1 est less than the dimension of $N_\epsilon^{II}$
An example \( N = 3, \) \( M_\epsilon(i, j) = \exp[-\beta E(i, j)]. \) Reduction I gives

- A unique dominant irreducible component: \( \bar{\alpha} = 1, \)

\[
M^1_\epsilon = \begin{bmatrix}
\epsilon^a & 1 & \epsilon^{b'} \\
\epsilon^{c'} & \epsilon^b & 1 \\
1 & \epsilon^{a'} & \epsilon^c
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
\epsilon^a & 1 & \epsilon^d \\
1 & \epsilon^b & \epsilon^e \\
\epsilon^{d'} & \epsilon^{e'} & \epsilon^c
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
1 & \epsilon^a & \epsilon^c \\
\epsilon^{a'} & \epsilon^b & \epsilon^d \\
\epsilon^{c'} & \epsilon^{d'} & \epsilon^e
\end{bmatrix}
\]

- Two irreducible components with equal dominant spectral radius

\( \bar{\alpha} = 1, \)

\[
M^1_\epsilon = \begin{bmatrix}
1 & \epsilon^a & \epsilon^{b'} \\
\epsilon^{a'} & \epsilon^c & 1 \\
\epsilon^{b'} & 1 & \epsilon^{d'}
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
1 & \epsilon^a & \epsilon^b \\
\epsilon^{a'} & 1 & \epsilon^c \\
\epsilon^{b'} & \epsilon^{c'} & \epsilon^d
\end{bmatrix}
\]

- Three irreducible components with equal dominant spectral radius:

\( \bar{\alpha} = 1, \)

\[
M^1_\epsilon = \begin{bmatrix}
1 & \epsilon^a & \epsilon^b \\
\epsilon^{a'} & 1 & \epsilon^c \\
\epsilon^{b'} & \epsilon^{c'} & 1
\end{bmatrix}
\]

- \( \mu_\beta \to \) a barycenter of the periodic minimizing cycles

\[
\mu_\beta \to \mu^H_{\text{min}} := c_1 \delta_1^z + c_2 \delta_2^z + c_3 \delta_3^z
\]
Example A $3 \times 3$ matrix with three irreducible components

\begin{align*}
\lambda_\epsilon &= 1 + \sqrt{2} \epsilon^{(c+c')/2} + \ldots \\
\mu_H^{min} &= \left[ \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \right] \\
\end{align*}

\begin{align*}
\lambda_\epsilon &= 1 + \rho \epsilon^{(c+c')/2} + \ldots \\
\mu_H^{min} &= \left[ 1 + \rho, 1, 1 \right] / (2 + \rho) \\
\end{align*}

\begin{align*}
\lambda_\epsilon &= 1 + \kappa \epsilon^{(c+c')/2} + \ldots \\
\mu_H^{min} &= \left[ 1 + 1 + \kappa, 1 + 1 + \kappa \right] / (3 + 2 \kappa) \\
\end{align*}

\begin{align*}
\lambda_\epsilon &= 1 + \epsilon^{(a+b+c)/3} + \ldots \\
\mu_H^{min} &= \left[ \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right] \\
\end{align*}

\begin{align*}
\lambda_\epsilon &= 1 + \sqrt{2} \epsilon^{(c+c')/2} + \ldots \\
\mu_H^{min} &= \left[ \frac{1}{2}, 0, \frac{1}{2} \right] \\
\end{align*}

\begin{align*}
\lambda_\epsilon &= 1 + \kappa \epsilon^{b} + \ldots \\
\mu_H^{min} &= \left[ 1 + \kappa, 1 + \kappa \right] / (3 + 2 \kappa) \\
\end{align*}

\begin{align*}
\lambda_\epsilon &= 1 + \epsilon^{b} + \ldots \\
\mu_H^{min} &= \left[ \frac{1}{4}, \frac{1}{4}, \frac{1}{2} \right] \\
\end{align*}

\begin{align*}
\lambda_\epsilon &= 1 + \epsilon^{a} + \ldots \\
\mu_H^{min} &= \left[ \frac{1}{2}, \frac{1}{2}, 0 \right] \\
\end{align*}
Bibliography


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