A dynamical approach of some Hamilton-Jacobi equations

Philippe Thieullen (Bordeaux)

(Joint work with Xifeng Su, Beijing)

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Summary of the talk

I. Main results: first part

I.a. Objectives

Give a semi-discrete scheme of the cell equation or the discounted cell equation of some Hamilton-Jacobi equations. Semi-discrete is with respect to time, not to space.

Notations $H(x,p) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is C^2 Hamiltonian, periodic in x, strictly convex and super-linear in p

$$\Big[\frac{\partial^2 H(x,p)}{\partial p_i \partial p_j}\Big] \geq \alpha \big[\delta_{ij}\big]$$

$$\lim_{R\to+\infty} \inf_{x,\|p\|\geq R} \frac{H(x,p)}{\|p\|} = +\infty$$

$H(x,p)$ is called Tonelli. Notice that H is independent of the time.

I.a. Objectives

The cell equation

$$H(x, \nabla u(x)) = \bar{H}$$

u is C^0 periodic, ∇u is defined in the sense of viscosity, $\bar{H}$ is a constant

Viscosity approach u is said to be a sub-solution in the viscosity sense if

$$\forall x_0,\ \forall \phi : \mathbb{T}^d \to \mathbb{R},\ \text{such that } \phi(x_0) = u(x_0),\ \text{and } \phi(x) \geq u(x)$$
$$H(x_0, \nabla\phi(x_0)) \leq \bar{H}$$

The discounted cell equation

$$\delta u(x) + H(x, \nabla u(x)) = 0$$

For the cell equation, $\bar{H}$ is unique; for the discounted cell equation, u is unique

I.b. Results

Notations

- Take a Tonelli Hamiltonian: $H(x,p)$
- Define the Legendre transform of H: $L(x,v)$
  $L(x,v) = \sup_p\{v \cdot p - H(x,p)\}$
- Define the discrete action: $E_\tau(x,y) = \tau L(x, \frac{y-x}{\tau})$
- Define the discrete Lax-Oleinik operator
  $T_\tau[u](y) = \min_x\{u(x) + E_\tau(x,y)\}$, for C^0 periodic u
- Solve the equation: $T_\tau[u_\tau] = u_\tau + \bar{E}_\tau$, $u_\tau \in C^0(\mathbb{T}^d)$

Call u_τ a discrete weak KAM solution (not unique)
Call $\bar{E}_\tau$ the effective discrete action (unique)

I.b. Results

Theorem: first part $H(x,p)$ is Tonelli, $L(x,v)$ is the Legendre transform, $E_\tau = L(x, \frac{y-x}{\tau})$,

- $\exists\ u_\tau$ solution of $u_\tau(y) + \bar{E}_\tau = \min_x\{u_\tau(x) + E_\tau(x,y)\}$
- $\dfrac{\bar{E}_\tau}{\tau} \to -\bar{H}$ (the limit exists and the error is of order $O(\tau)$)
- $\mathrm{Lip}(u_\tau) \le C$ uniformly
- Take a convergent sub-sequence $u_{\tau_i} \to u$
- $H(x, \nabla u(x)) = \bar{H}$, in the viscosity sense

Main drawback: a sub-sequence need be taken for u.

A possible solution: solve the discounted discrete equation.

I.b. Results

An example The inverse pendulum

- $H(x,p) = \frac{1}{2}p^2 - V(x), \quad V(x) = \frac{K}{(2\pi)^2}\big(1 - \cos 2\pi x\big)$
- $L(x,v) = \frac{1}{2}v^2 + V(x)$
- $E_\tau(x,y) = \frac{1}{2\tau}(y-x)^2 + \tau V(x)$
- $u_\tau(y) + \bar{E}_\tau = \min_x\{u_\tau(x) + E_\tau(x,y)\}$
- $\bar{E}_\tau/\tau \to -\bar{H}$
- $u_{\tau_i} \to u$
- $\frac{1}{2}|\nabla u|^2 - V(x) = \bar{H}$

II. Heuristic of the discrete Lax-Equation equation

II.a. The characteristics method

Weierstrass-Tonelli We want to solve the evolutionary HJE

$$\frac{\partial u}{\partial t} + H(x, \nabla u(t,x)) = 0, \quad \forall x \in \mathbb{R}^d, \quad \forall t \geq 0$$
$$u(0,x) = u_0(x), \quad u_0 \in C^2(\mathbb{T}^d)$$

For short time a C^2 solution exists: $\forall x, \ \forall t$ small

- $\exists! \ X_0 = X_0(t,x)$ such that, if $P_0 = \nabla u_0(X_0)$,
- if $(X(s), P(s))$ evolves according to the Hamiltonian flow

$$\dot X = \frac{\partial H}{\partial P}(X,P), \qquad X(0) = X_0$$
$$\dot P = -\frac{\partial H}{\partial X}(X,P), \qquad P(0) = P_0$$

  then $X(t) = x$
- the solution of the evolutionary HJE is given by

$$u(t,x) := u_0(X_0(t,x)) + \int_0^t L(X, \dot X)\, ds$$

II.a. The characteristics method

Remark The Weierstrass-Tonelli solution is a minimizer

$$u(t,x) = \inf_{\gamma(t)=x} \left\{ u_0(\gamma(0)) + \int_0^t L(\gamma, \dot\gamma)\, ds \right\}$$

(the infimum is taken over absolutely continuous path with ending at x)

Indeed
$$\begin{aligned} u(t,\gamma(t)) - u_0(\gamma(0)) &= \int_0^t \frac{d}{ds} u(s,\gamma(s))\, ds \\ &= \int_0^t \left[\frac{\partial u}{\partial t} + \nabla u \cdot \dot\gamma \right] ds \\ &\le \int_0^t \left[\frac{\partial u}{\partial t} + H(\gamma, \nabla u) + L(\gamma, \dot\gamma) \right] ds \\ &= \int_0^t L(\gamma, \dot\gamma)\, ds \end{aligned}$$

II.b. The Lax-Oleinik operator

Definition For every $u \in C^0(\mathbb{T}^d)$

$$T^t[u](x) := \inf_{\gamma(0)=x} \left\{ u(\gamma(-t)) + \int_{-t}^{0} L(\gamma, \dot{\gamma})\, ds \right\}$$

Weierstrass-Tonelli For every $u_0 \in C^2(\mathbb{T}^d)$,

$$u(t,x) := T^t[u_0](x) \quad \text{solve the HJE}$$

$$\frac{\partial u}{\partial t} + H(x, \nabla u(t,x)) = 0, \quad \forall x \in \mathbb{R}^d, \quad \text{for small time}$$

$$u(0,x) = u_0(x), \quad u_0 \in C^2(\mathbb{T}^d)$$

II.b. The Lax-Oleinik operator

The weak KAM theorem (Fathi)

- For every $u \in C^0(\mathbb{T}^d)$
  $u(t,x) := T^t[u_0](x)$ solve the HJE
  $\dfrac{\partial u}{\partial t} + H(x, \nabla u(t,x)) = 0, \quad \forall x \in \mathbb{R}^d, \quad \forall t \geq 0$, in the viscosity sense
- T^t is a semi group of operators
  $\exists \bar{u} \in C^0(\mathbb{T}^d), \ \exists \bar{H} \in \mathbb{R}$, s.t. $T^t[\bar{u}] = \bar{u} - t\bar{H}, \quad \forall t \geq 0$
  define $u(t,x) := T^t[\bar{u}](x)$, then
  $H(x, \nabla \bar{u}(x)) = \bar{H}$
  $\bar{u}$ is called a weak KAM solution
- for every $u \in C^0(\mathbb{T}^d)$
  $$T^t[u] + tu \to \bar{u},$$
  uniformly for some $\bar{u}$ solution of the cell equation

II.c. The discrete Lax-Oleinik operator

The continous Lax-Oleinik operator For every $u \in C^0(\mathbb{T}^d)$

$$T^t[u](x) := \inf_{\gamma(0)=x} \Big\{ u(\gamma(-t)) + \int_{-t}^{0} L(\gamma, \dot\gamma)\, ds \Big\}$$

The discrete Lax-Oleinik operator

$$T_\tau[u](y) = \min_x \Big\{ u(x) + \tau L\Big(x, \frac{y-x}{\tau}\Big) \Big\}$$

Theorem: first part

- $\exists u_\tau \in C^0(\mathbb{T}^d)$ solution of $T_\tau[u_\tau] = u_\tau + \bar{E}_\tau$
  u_τ (not unique) is called a discrete weak KAM solution
- $\dfrac{\bar{E}_\tau}{\tau} \to -\bar{H}$ as $\tau \to 0$
- $u_\tau \to \bar{u}$, for some subsequence $\tau \to 0$
- $H(x, \nabla \bar{u}(x)) = \bar{H}$

III. Main results: second part

III.a. The discounted cell equation

The equation

$$\delta u_\delta(x) + H(x, \nabla u_\delta(x))) = 0, \quad \text{(in the viscosity sense)}$$

The solution There exists a unique solution given by

$$u_\delta(x) = \inf_{\gamma(0)=x} \int_{-\infty}^{0} e^{s\delta} L(\gamma(s), \dot\gamma(s))\, ds$$

The Dynamical Programming Principle

$$u_\delta(x) = \inf_{\gamma(0)=x} \left\{ e^{-t\delta} u_\delta(\gamma(-t)) + \int_{-t}^{0} e^{s\delta} L(\gamma(s), \dot\gamma(s))\, ds \right\}$$

The discounted discrete weak KAM solution

$$u_{\tau,\delta}(y) = \min_x \left\{ (1 - \tau\delta) u_{\tau,\delta}(x) + \tau L\Big(x, \frac{y-x}{\tau}\Big) \right\}$$

III.b. main results: second part

We recall A discounted discrete weak KAM solution

$$u_{\tau,\delta}(y) = \min_x \big\{(1-\tau\delta)u_{\tau,\delta}(x) + E_\tau(x,y)\big\}$$

$$E_\tau(x,y) = \tau L\Big(x, \frac{y-x}{\tau}\Big), \quad \text{(the discrete action)}$$

$$\bar{E}_\tau \quad \text{defined uniquely in} \quad T_\tau[u] = u + \bar{E}_\tau$$

Theorem: second part

- $u_{\tau,\delta}$ is unique $C^0(\mathbb{T}^d)$

$$u_{\tau,\delta}(x) = \inf_{(x_{-k}),\ x_0=x} \sum_{k=0}^{\infty} (1-\tau\delta)^k E_\tau(x_{-k-1}, x_{-k})$$

- For $\tau > 0$ fixed (as in the continuous case)

$$u_\delta + \frac{\bar{H}}{\delta} \to \bar{u} \qquad \text{(DFIZ theorem)}$$

$$u_{\tau,\delta} - \frac{\bar{E}_\tau}{\tau\delta} \to \bar{u}_\tau \quad \text{as } \delta \to 0$$

III.b. main results: second part

The limit $\bar{u}_\tau$ is characterized by dynamical notions

Holonomic measures A probability measure $\mu(dx, dv)$ on $\mathbb{T}^d \times \mathbb{R}^d$ is holonomic if for every test funtion $\phi(x)$

$$\int \phi(x)\,\mu(dx,dv) = \int \phi(x+\tau v)\,\mu(dx,dv)$$

Minimizing measure A probability measure is minimizing if

$$\mu \in \arg\min_{\mu} \int E_\tau(x, x+\tau v)\,\mu(dx,dv)$$

Mañé Potential Defined on $\mathbb{T}^d \times \mathbb{T}^d$

$$\Phi_\tau(x,y) = \inf_{n\geq 1} \inf_{x_0=x,\cdots,x_n=y} \sum_{k=0}^{n-1} \big[E_\tau(x_k, x_{k+1}) - \bar{E}_\tau\big]$$

III.b. main results: second part

We recall A discrete weak KAM solution w is any C^0 periodic solution of $T_\tau[w] = w + \bar{E}_\tau$, that is solution

$$w(y) + \bar{E}_\tau = \min_x \{w(x) + E_\tau(x, y)\}$$

Theorem: second part

- $u_{\tau,\delta} - \dfrac{\bar{E}_\tau}{\tau\delta} \to \bar{u}_\tau$ as $\delta \to 0$
- First characterization

$$\bar{u}(x) = \sup \Big\{ w(x) \ : \ w \text{ is a discrete weak KAM solution} \\ \text{such that } \int w \, d\mu \leq 0, \ \ \forall \mu \text{ minimizing} \Big\}$$

- Second characterization

$$\bar{u}_\tau(y) = \inf \Big\{ \int \Phi_\tau(x, y) \, d\mu \ : \ \mu \text{ minimizing} \Big\}$$

III.b. main results: second part

Theorem (Davini, Fathi, Iturriaga, Zavidovique) The unique solution of

$$\delta u_\delta(x) + H(x, \nabla u(x)) = 0$$

normalized by a constant, converges in C^0 to a weak KAM solution

$$u_\delta + \frac{\bar{H}}{\delta} \to \bar{u} \qquad \text{(solution of } H(x, \nabla \bar{u}(x)) = \bar{H}\text{)}$$

Theorem: second part Compared to the DFIZ solution, we obtain an error term

$$\Big\| \Big[u_{\tau,\delta} - \frac{\bar{E}_\tau}{\tau\delta} \Big] - \Big[u_\delta + \frac{\bar{H}}{\delta} \Big] \Big\| = O\Big(\frac{\tau}{\delta}\Big)$$

Provided $\tau \to 0$, $\delta \to 0$, and $\frac{\tau}{\delta} \to 0$

$$u_{\tau,\delta} - \frac{\bar{E}_\tau}{\tau\delta} \to \bar{u} \qquad \text{solution of the cell equation}$$

IV. Ideas of the proof

IV. a. Short-range actions

Two special actions

- $\mathcal{L}_\tau(x,y) = \tau L\Big(x, \dfrac{y-x}{\tau}\Big)$ the discrete action
- $\mathcal{E}_\tau(x,y) = \inf\limits_{\gamma(0)=x,\ \gamma(\tau)=y} \displaystyle\int_0^\tau L(\gamma,\dot\gamma)\,ds$ the minimal action

Short-range actions $E_\tau(x,y) \in C^0(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying

(H1) Translation periodic: $E_\tau(x+k, y+k) = E_\tau(x,y)$

(H2) Uniformly super-linear: $\inf\limits_{\|y-x\|\geq \tau R} \dfrac{E_\tau(x,y)}{\|y-x\|} \to +\infty$ as $R \to +\infty$

(H3) Uniformly Lipschitz: $\forall R>0,\ \exists C(R)>0$

$$\|y-x\| \leq \tau R, \text{ and } \|z-x\| \leq \tau R \implies |E_\tau(x,z) - E_\tau(x,y)| \leq C(R)\|z-y\|$$

IV. a. Short-range actions

Lemma The discrete action $\mathcal{L}_\tau(x,y)$ and the minimal action $\mathcal{E}_\tau(x,y)$ are short range.

A priori compactness lemma Every minimizer of

$$\inf_{\gamma(0)=x,\ \gamma(\tau)=y} \int_0^\tau L(\gamma,\dot\gamma)\,ds$$

satisfies $\|\dot\gamma\| \leq C$ and $\|\ddot\gamma\| \leq C$ for some constant C

Corollary $\forall R>0,\ \exists C(R)>0$

$$\|y-x\| \leq \tau R \quad \Longrightarrow \quad \big|\mathcal{L}_\tau(x,y) - \mathcal{E}_\tau(x,y)\big| \leq C(R)\tau^2$$

IV.b. Main new technical result

Main problem A discrete weak KAM solution $u_\tau \in C^0(\mathbb{T}^d)$ solve

$$u_\tau(y) + \bar{E}_\tau = \min_x \{u_\tau(x) + E_\tau(x,y)\}$$

$$\Big(\text{an example:} \quad u_\tau(y) + \bar{E}_\tau = \min_x \Big\{ u_\tau(x) + \frac{1}{2\tau}|y-x|^2 + \tau V(x) \Big\} \Big)$$

for some constant $\bar{E}_\tau$ to be founded. If the point x which attained the minimum is at a distance from y bounded from below as $\tau \to 0$, then u_τ has no reason to be bounded

IV.b. Main new technical result

Technical lemma Assume $E_\tau(x,y)$ is short range, that is satisfies

(H1) C^0 translation periodic: $E_\tau(x+k, y+k) = E_\tau(x,y)$

(H2) Uniformly super-linear:

$E_\tau(x,y)/\|y-x\| \to +\infty$ as $\|y-x\| \geq \tau R$ and $R \to +\infty$

(H3) Uniformly Lipschitz: $\forall\ \|z-x\| \leq \tau R$ and $\|y-x\| \leq \tau R$

$|E_\tau(x,z) - E_\tau(x,y)| \leq C(R)\|z-y\|$

Then for every discrete weak KAM solution u_τ $\quad (T_\tau[u_\tau] = u_\tau + \bar{E}_\tau)$

- $\exists R \quad x \in \arg\min_x \{u_\tau(x) + E_\tau(x,y)\} \quad \Rightarrow \quad \|y-x\| \leq \tau R$
- $\exists C \quad \mathrm{Lip}(u_\tau) \leq C$
- $\exists C' \quad \|T_\tau[u] - u\| \leq \tau C'$ for every Lipschitz $\mathrm{Lip}(u) \leq C$

IV.b. Main new technical result

Remark The inf-convolution operator is a particular case of the Lax-Oleinik operator

- The inf-convolution

$$u_\epsilon(y) := \min_x \Big\{ u(x) + \frac{1}{2\epsilon}|y-x|^2 \Big\}$$

the optimal point satisfies the estimate

$$x \in \arg\min_x \Big\{ u(x) + \frac{1}{2\epsilon}|y-x|^2 \Big\} \quad \Longrightarrow \quad \|y-x\| \leq \sqrt{\|u\|}\sqrt{\epsilon}$$

- for a weak KAM solution (a particular example)

$$u_\tau(y) + \bar{E}_\tau = \min_x \Big\{ u_\tau(x) + \frac{1}{2\tau}|y-x|^2 + \tau V(x) \Big\}$$

the optimal point satisfies

$$x \in \arg\min_x \Big\{ u_\tau(x) + \frac{1}{2\tau}|y-x|^2 + \tau V(x) \Big\} \quad \Rightarrow \quad \|y-x\| \leq \tau R$$

IV.c. Ideas of the proof

We recall

- The discrete action: $\mathcal{L}_\tau(x,y) = \tau L\Big(x, \dfrac{y-x}{\tau}\Big)$
- The minimal action: $\mathcal{E}_\tau(x,y) = \inf\limits_{\gamma(0)=x,\ \gamma(\tau)=y} \displaystyle\int_0^\tau L(\gamma,\dot\gamma)\,ds$

Denote by T_τ and by T^τ the two Lax-Oleinik operators

We recall The a priori compactness estimate

$$\|y-x\| \le \tau R \quad \Longrightarrow \quad |\mathcal{L}_\tau(x,y) - \mathcal{E}_\tau(x,y)| \le C(R)\tau^2$$

Proof

- Let u_τ be a discrete weak KAM solution

$$u_\tau(y) + \bar{\mathcal{L}}_\tau = \min_x \{u_\tau(x) + \mathcal{L}_\tau(x,y)\}$$

- We want to prove $\frac{\bar{\mathcal{L}}_\tau}{\tau} \to -\bar{H}$, $u_\tau \to \bar{u}$, and $H(x, \nabla\bar{u}(x)) = \bar{H}$

IV.c. Ideas of the proof

- The a priori compactness $\Rightarrow \quad \| T_\tau[u] - T^\tau[u] \| = O(\tau^2)$
- u_τ is a fixed point of T_τ $\quad T_\tau[u_\tau] = u_\tau + \bar{\mathcal{L}}_\tau$
- $\mathcal{E}_\tau(x,y)$ is a short-range action satisfying the additional property

$$\mathcal{E}_\tau \otimes \mathcal{E}_{\tau'}(x,y) = \min_z \{\mathcal{E}_\tau(x,z) + \mathcal{E}_{\tau'}(z,y)\}$$

$\Longrightarrow \quad T^\tau$ is a semi-group of operators: $\quad T^{\tau+\tau'} = T^\tau T^{\tau'}$

- Every weak KAM solution for the minimal action satisfies

$T^\tau[u] = u + \tau \bar{\mathcal{E}}_1, \quad \forall \tau > 0,$ (the effective action is linear in τ)

$\bar{\mathcal{E}}_1 = -\bar{H}$ (by taking $\tau \to +\infty$)

- Then $\dfrac{\bar{\mathcal{L}}_\tau}{\tau} \to \bar{\mathcal{E}}_1$ as $\tau \to 0$
- every accumulation function $u_{\tau_i} \to \bar{u}$ is a weak KAM solution for the minimal action

$$T^\tau[\bar{u}] = \bar{u} - \tau \bar{H}, \quad \forall \tau > 0 \quad \Longleftrightarrow \quad H(x, \nabla \bar{u}(x)) = \bar{H}$$

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