A long route in thermodynamic formalism: Two examples of Artur's contributions from complex dynamics to ergodic optimization

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Outline

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- Dynamics of endomorphisms on $\mathbb{P}^k(\mathbb{C})$

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- Ergodic optimization

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- Dynamics of endomorphisms on $\mathbb{P}^k(\mathbb{C})$
  - Large deviation of the maximal entropy measure
  - Dimension sprectrum for rational maps
  - Pressure and phase transition
  - Billiards and decay of correlation
  - Thermodynamic formalism for C^* algebras
  - Spectral analysis of time series of chaotic systems
- Ergodic optimization

Dynamics of endomorphisms on $\mathbb{P}^k(\mathbb{C})$

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where $P(z,w)$ and $Q(z,w)$ are homogeneous polynomials,

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- Question: Fixe some $a \in \mathbb{P}^1$, and consider the algebraic roots of

$$f^n(x) = a \quad \text{for some } n \geq 0 \text{ and } x \in \mathbb{P}^1$$
$$f^n = f \circ \cdots \circ f \quad n \text{ times.}$$

How do they distribute ?

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the dynamics $\{f^n\}_n$ belongs to a compact family of endomorphisms.

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- The Julia set: $\mathrm{Julia}(f) = \mathbb{P}^1 \setminus \mathrm{Fatou}(f)$

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- The exceptional set $\text{Excep}(f)$ is such that

$$\forall\, U \text{ open},\ U \cap \text{Julia} \neq \emptyset \quad \Rightarrow \quad f^n(U) = \mathbb{P}^1 \setminus \text{Excep}(f),\ n \text{ large}$$

Dynamics of endomorphisms on $\mathbb{P}^k(\mathbb{C})$

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- Uniformly in $x \in \mathbb{P}^1 \setminus \mathrm{Excep}(f)$, the following limit exists

$$\frac{1}{d^n} \sum_{p:\ f^n(p)=x} \delta_p \longrightarrow \mu_f$$

(counted with algebraic multiplicity)

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- μ_f is f-invariant, is supported on the Julia set and has constant Jacobian

$$\mathcal{L}_f^*[\mu_f] = d\ \mu_f, \quad \mathcal{L}_f[\phi](x) := \sum_{p: f(p)=x} \phi(p)$$

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- μ_f is the unique measure of maximal entropy

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- A potential theory approach: $F(z_0, z_1, \dots, z_k) : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ homogeneous of degree d and non-degenerate

$$\frac{1}{d^n} \ln \|F^n(x)\| \to U_F(x) \quad \text{exists} \quad \forall\, x = (z_0, \dots, z_k)$$

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- There exists a unique closed positive $(1,1)$-current ω_F called Green current

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- Let $\mu_f := \omega_f \wedge \ldots \wedge \omega_f$, k times: μ_f is a positive measure

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- A thermodynamic formalism for “algebraic” observables $\phi \neq 0$?

Ergodic optimization

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$$\bar{E}(\lambda) = \min\left\{ \int E_\lambda(x)\, d\mu(x) \;:\; \mu \text{ is } f\text{-inv.}\right\}, \quad E_\lambda(x) := \cos 2\pi(x-\lambda)$$

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- locking at $\mathbb{Q}$-frequencies: $\cup_{p/q\in\mathbb{Q}} \mathrm{int}\{\lambda \,:\, \omega_\lambda = \frac{p}{q}\}$ has full measure

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- Consider a Tonelli Lagrangian and the minimizing problem

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- There exists (semi-concave) correctors of the Lax-Oleinik equation

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$$H(x, Du_-(x)) = -\bar{L}(P) \quad \text{in the viscosity sense}$$

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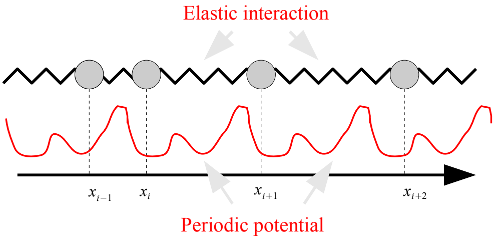
- The physical model: The model describes the set of configuration of a chain of atoms at equilibrium



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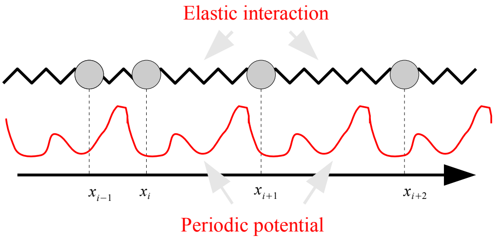

- The original 1D-FK: $E(x,y) = W(x,y) + V(x)$

$$W(x,y) = \frac{1}{2}|y-x|^2, \quad V(x) = \frac{K}{(2\pi)^2}\Big(1-\cos(2\pi x)\Big)$$

$$E_\lambda(x,y) = E_0(x,y) - \lambda(y-x).$$

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- A configuration $\underline{x} := (x_k)_{k\in\mathbb{Z}}$ with the smallest total energy

$$E_{tot}(\underline{x}) := \sum_{k=-\infty}^{+\infty} E(x_k, x_{k+1}) \leq E_{tot}(\underline{y}), \quad \forall\ \underline{y}$$

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  $\forall\ \underline{y}$ configuration s.t. $y_m = x_m$ and $y_n = x_n$
- Remark: Let $E_\lambda(x,y) := E(x,y) - \lambda \cdot (y-x)$ then

$$(x_k)_{k\in\mathbb{Z}} \text{ is minimizing for } E \iff (x_k)_{k\in\mathbb{Z}} \text{ is minimizing for } E_\lambda$$

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D fundamental domain ($D = \mathbb{T} \times \mathbb{R}$), $\pi(D) = 1$ is normalized, $\mathrm{pr}^1_*(\pi) = \mathrm{pr}^2_*(\pi)$ has same marginals

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- A stronger notion of minimizing configurations which differentiates E_λ

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Effective potential and calibrated configurations:

Ergodic optimization

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- Effective potential (Chou-Griffiths): A periodic C^0 function $u(x)$

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- If $\omega \notin \mathbb{Q}$, $\Lambda(\omega)$ is reduced to a point: a unique calibrating λ

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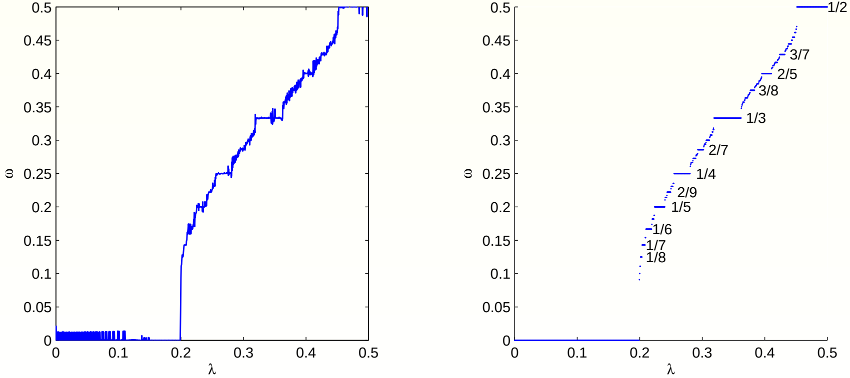

Ergodic optimization

Phase transition in parameters space $(\lambda, \boldsymbol{K})$: $\omega = -\dfrac{\partial \bar{E}(\lambda, K)}{\partial \lambda}$

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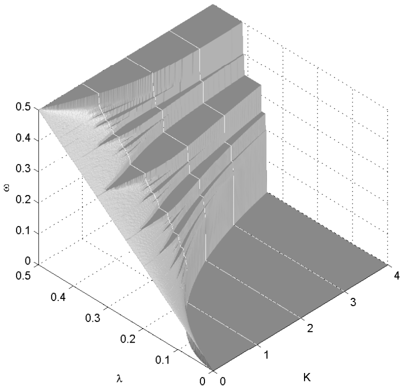

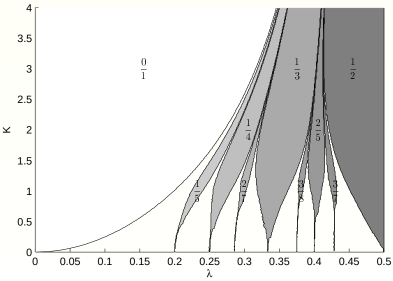

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Conclusion: How are the three problems related?

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- Homogenezation problem: τ, a time discretization

$$-\frac{1}{\tau^2}\bar{E}(\tau P, \tau^2 C) \longrightarrow \bar{H}(P,C)$$
$$\frac{1}{\tau}u_{\tau P,\tau^2 C}(x) \longrightarrow u_{P,C}(x)$$

many thanks to Renaud and Eduardo
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for all the mathematics I have learnt