The quasi-periodic Frenkel-Kontorova model

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Summary of the talk

– **I.** The general problem of minimizing configurations

– **II.** Almost periodic and quasi-crystalline models

– **III.** Some ideas of the proof
I. The general problem of minimizing configurations
I.1. General problem

**Notations**
- Consider a discrete path in $\mathbb{R}^d$: $(x_n)_{x \in \mathbb{Z}}$ (chain of atoms)
- Choose a discrete action $E(x, y): \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ (energy interaction between two consecutive atoms)
- Define the total action of the path

\[ "E_{tot} := \sum_{n \in \mathbb{Z}} E(x_n, x_{n+1})" \]

**Problem**
- Find a bi-infinite path which “minimizes” the total action
I.2. Three examples

The periodic 1D Frenkel-Kontorova model

\[ E(x, y) = \frac{1}{2} |y - x - \lambda|^2 + K(1 - \cos(2\pi x)) \]
\[ = W_\lambda (y - x) + V(x) \]

- \( \lambda \): distance at rest when there is no external interaction
- \( W_\lambda (x, y) = W_0 (x, y) - \lambda (y - x) + \text{cte} \): elastic internal interaction
- \( V(x) \): periodic external interaction

The almost periodic 1D Frenkel Kontorova

\[ E(x, y) = \frac{1}{2} |y - x - \lambda|^2 + K_1 (1 - \cos(2\pi x)) + K_2 (1 - \cos(2\pi x\sqrt{2})) \]

The quasicrystalline 1D model
- To be explained later
1.3. Minimizing configurations

In the weak sense A configuration \((x_n)_{n \in \mathbb{Z}}\) is said to be minimizing (in the weak sense) if
– for any finite sub-path \((x_m, x_{m+1}, \ldots, x_n)\)
– define the total action of the sub-path:

\[
E(x_m, \ldots, x_n) := \sum_{k=m+1}^{n} E(x_{k-1}, x_k)
\]

– by fixing the two endpoints \(x_m\) and \(x_n\), the total action of the sub-path can only increase: let \((y_m, y_{m+1}, \ldots, y_n)\) be another finite path, with \(y_m = x_m\) and \(y_n = x_n\), then

\[
E(x_m, \ldots, x_n) \leq E(y_m, \ldots, y_n)
\]
1.4. Calibrated configurations

Remark
– The notion of minimizing configurations is close to the notion of minimal geodesics
– $E(x, y)$ plays the role of the distance (or a cost)
– But there is no reason to ask $E(x, y) \geq 0$ and $E(x, y) = 0 \iff x = y$
– a normalizing factor $\bar{E} \in \mathbb{R}$ has to be introduce

Minimizing in the strong sense
A configuration $(x_n)_{n \in \mathbb{Z}}$ is minimizing
– if for any finite path $(x_m, x_{m+1}, \ldots, x_n)$
– by fixing the two endpoints $x_m$ and $x_n$
– if $(y_k, y_{k+1}, \ldots, y_l)$ is another path, possibly not having the same length, but with the same endpoints $y_k = x_m$ and $y_l = x_n$

$$E(x_m, \ldots, x_n) - (n - m) \bar{E} \leq E(y_k, \ldots, y_l) - (l - k) \bar{E}$$

A minimizing configuration in the strong sense will be called calibrated
I.5. Mañé potential

The effective action $\bar{E}$
(or Mañé critical value, or mean energy per site, or ground energy)

$$\bar{E} := \lim_{n \to +\infty} \inf_{x_0, \ldots, x_n \in \mathbb{R}^d} \frac{1}{n} E(x_0, \ldots, x_n)$$

The Mañé potential $S(x, y)$ Given two points $x, y \in \mathbb{R}^d$

$$S(x, y) = \inf_{n \geq 1} \inf_{x=x_0, \ldots, x_n=y} \left[ E(x_0, \ldots, x_n) - n \bar{E} \right]$$

Minimizing in the strong sense $(x_n)_{n \in \mathbb{Z}}$ is said to be calibrated if

$$\forall m \leq n, \quad S(x_m, x_n) = E(x_m, \ldots, x_n) - (n - m) \bar{E}$$
1.6. Assumptions for 1D periodic models

**General hypotheses**

- $E(x, y) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is $C^2$
- $E(x, y)$ is translation periodic:

$$E(x + 1, y + 1) = E(x, y)$$

- $E(x, y)$ is superlinear:

$$\lim_{R \rightarrow +\infty} \inf_{|y - x| \geq R} \frac{E(x, y)}{|y - x|} = +\infty$$

- $E(x, y)$ is twist

$$\forall x, y, \quad \frac{E(x, y)}{\partial x \partial y}(x, \cdot) < 0 \quad \text{and} \quad \frac{E(x, y)}{\partial x \partial y}(\cdot, y) < 0 \quad \text{a.e.}$$

(check that if $E(x, y) = \frac{1}{2} (y - x)^2$ then $\frac{\partial E}{\partial x \partial y} = -1$)
1.7. Main question again

**Remark** Under the hypotheses of translation periodicity, superlinerarity and twist

\[ \bar{E} = \lim_{n \to +\infty} \inf_{x_0, \ldots, x_n \in \mathbb{R}^d} \frac{1}{n} E(x_0, \ldots, x_n) \text{ exists and is finite} \]

The Mañé potential

\[ S(x, y) = \inf_{n \geq 1} \inf_{x=x_0, \ldots, x_n=y} \left[ E(x_0, \ldots, x_n) - n\bar{E} \right] \]

is finite and satisfies

- \( S(x, z) \leq S(x, y) + S(y, z) \) (sub-cocycle)
- \( S(x, y) \leq E(x, y) - \bar{E} \)
- \( S(x, x) \geq 0 \)

**Question** Do there exist calibrated configurations \((x_n)_{n \in \mathbb{Z}}\)?

\[ \forall m \leq n \quad E(x_m, \ldots, x_n) - (n - m)\bar{E} = S(x_m, x_n) \]
1.8. Aubry theory for 1D periodic models

One-parameter family

\[ E_\lambda(x, y) := E(x, y) - \lambda(y - x) \]

Aubry theory (1983)
- \( \lambda \in \mathbb{R} \mapsto \bar{E}(\lambda) \) is \( C^1 \) concave
- for every \( \lambda \), there exist calibrated configurations for \( E_\lambda \); and all calibrated configurations have the same rotation number

\[ \rho = \lim_{n \to +\infty, \ m \to -\infty} \frac{x_n - x_m}{n - m} = -\frac{d\bar{E}}{d\lambda}(\lambda) \]

- conversely, every \( \rho \) is the rotation number of a calibrated configuration for some \( E_\lambda \)
I.9. The original 1D Frenkel-Kontorova

\[ E_{\lambda,K}(x, y) = \frac{1}{2}|y - x|^2 - \lambda(y - x) + \frac{K}{(2\pi)^2} \left(1 - \cos(2\pi x)\right) \]

\[ \rho = -\frac{\partial E}{\partial \lambda}(\lambda, K) \]

The system is “locked” at rational rotation number \( \rho \)
II. Almost periodic and quasi-crystalline models
II.1. Example of almost periodic models

The almost periodic 1D Frenkel-Kontorova

- \( E_\omega(x, y) = W(y - x) + V_\omega(x) \)
- \( W(y - x) = \frac{1}{2}|y - x - \lambda|^2 \)
- \( V_\omega(x) = K_1 \left( 1 - \cos 2\pi(\omega_1 + x) \right) + K_2 \left( 1 - \cos(\omega_2 + x\sqrt{2}) \right) \)

**Remark** In the periodic case there is no need to introduce the space \( \Omega \) of all phases. In the almost periodic case, it is more appropriate to work with a full family of discrete actions

**Notations**

- \( \Omega = \mathbb{T}^2 = \{ (\omega_1, \omega_2) : \omega_i \mod 1 \} \) the set of all environments
- \( \tau_t : \Omega \to \Omega, \quad \tau_t(\omega_1, \omega_2) := (\omega_1 + t, \omega_2 + t\sqrt{2}) \) is minimal
- \( V_\omega(x) = V(\tau_x(\omega)) \) where \( V : \Omega \to \mathbb{R} \) and

\[ \forall t \in \mathbb{R}, \quad E_\omega(x+t, y+t) = E_{\tau_t(\omega)}(x, y) \] (translation invariance property)
II.2. A general almost periodic model

**Definition** An almost periodic model \((\Omega, \{\tau_t\}_{t \in \mathbb{R}^d}, L)\)

- \(\Omega\) a compact space
- \(\tau_t : \Omega \to \Omega\) a minimal flow \(\tau_{s+t} = \tau_s \circ \tau_t\)
- a family of translation-invariant discrete actions

\[
\forall x, y, t, \in \mathbb{R}^d \quad E_\omega(x + t, y + t) = E_{\tau_t(\omega)}(x, y)
\]

- we assume actually that \(E_\omega(x, y)\) has a Lagrangian form

\[
E_\omega(x, y) := L(\tau_x(\omega), y - x)
\]

- \(L(\omega, t) : \Omega \times \mathbb{R}^d \to \mathbb{R}\) is \(C^0\) in \((\omega, t)\) and superlinear in \(t\)

\[
\lim_{R \to +\infty} \inf_{\omega \in \Omega} \inf_{\|t\| \geq R} \frac{L(\omega, t)}{\|t\|} = +\infty
\]
II.3. A general quasicrystalline model

**Definition** A quasicrystalline model \((\Omega, \{\tau_t\}_{t \in \mathbb{R}}, L)\)

- \((\Omega, \{\tau_t\}_{t \in \mathbb{R}}, L)\) is an almost periodic model
- \((\Omega, \{\tau_t\})\) is minimal and **uniquely ergodic**
- \(d = 1\) and (to simplify) \(L(\omega, t) = W(t) + V(\omega)\)

\[
E_\omega(x, y) = W(y - x) + V_\omega(x) \quad \text{and} \quad V_\omega(x) = V(\tau_x(\omega))
\]

- \(V : \Omega \to \mathbb{R}\) is \(C^0\)
- \(W\) is **twist**: \(W\) is \(C^2\) and convex \(W'' > 0\) a.e.

\[
\forall x, y \in \mathbb{R} \quad \frac{\partial E_\omega}{\partial x \partial y}(x, \cdot) < 0 \quad \text{a.e.} \quad \text{and} \quad \frac{\partial E_\omega}{\partial x \partial y}(\cdot, y) < 0 \quad \text{a.e.}
\]

- \(W\) is superlinear: \(\lim_{|t| \to +\infty} \frac{W(t)}{|t|} = +\infty\)

**In addition to the above properties**

- \(V\) is locally transversally constant

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II.4. Locally transversally constant

The periodic case

1 tile $L$

1 function

$V_L : [0, L] \rightarrow \mathbb{R}$

The quasicrystalline case

2 tiles $L, S$

2 functions

$V_L : [0, L] \rightarrow \mathbb{R}$

$V_S : [0, S] \rightarrow \mathbb{R}$

$\omega = (q_n)_{n \in \mathbb{Z}}$ ordered along $\mathbb{R}$ with minimal complexity
II.5. Quasicrystals

**Delone set** An ordered subset \( \omega^* = (q_n^*)_{n \in \mathbb{Z}} \subset \mathbb{R} \) which is uniformly discrete and relatively dense

**Finite complexity** There is a finite number of jumps

\[
\forall n \in \mathbb{Z}, \quad \Delta_n := q_{n+1}^* - q_n^* \in \{L_1, \ldots, L_r\}
\]

**Repetitive** For instance assume there are 2 tiles \( \{L, S\} \). A pattern \( P = LSL \) is any finite sub-word. An occurrence of a pattern is some \( n \)

\[
\Delta_n = L, \quad \Delta_{n+1} = S, \quad \Delta_{n+2} = L
\]

repetitive = for every pattern the set of occurrences is relatively dense

**Uniformly distributed** For every pattern \( P \), uniformly in \( x \)

\[
\frac{1}{2T} \text{Card}\{\text{occurrences of } P \text{ in } [x - T, x + T]\} \rightarrow \nu(P) > 0
\]
II.6. A model of Sturmian type

**Construction** A model of type “cut and projection”

- let be $\alpha \notin \mathbb{Q}$ and $\mathcal{B} = \{(x, y) \in \mathbb{R}^2 : \alpha x - \alpha < y \leq \alpha x + 1\}$
- we construct a sequence of points $(q^*_n \overrightarrow{u})_{n \geq 0}$ on the line $\mathcal{D} = \mathbb{R} \overrightarrow{u}$, $\overrightarrow{u} = (1, \alpha)$, by projecting orthogonally on $\mathcal{D}$ the set $\mathbb{Z}^2 \cap \mathcal{B}$

$$\omega^* = (q^*_n)_{n \in \mathbb{Z}}$$ is a quasicrystal with 2 tiles

$$q^*_{n+1} - q^*_n \in \left\{ \frac{1}{\sqrt{1+\alpha^2}}, \frac{\alpha}{\sqrt{1+\alpha^2}} \right\} = \{L, S\}$$

**A particular case** For $\alpha = \frac{\sqrt{5}-1}{2}$, one obtains the Fibonacci substitution

$$S \rightarrow L \quad \text{et} \quad L \rightarrow LS$$
II.7. A quasicrystalline model

Definition (first definition)

- $\omega^*$ a quasicrystal (an ordered subset of finite complexity, which is repetitive, with uniformly distributed patterns)

- a discrete action $E_{\omega^*}(x, y) = W(y - x) + V_{\omega^*}(x)$

- $V_{\omega^*} : \mathbb{R} \rightarrow \mathbb{R}$ a $C^0$ transversally constant function (a juxtaposition of potentials according to the tiles)

- $W : \mathbb{R} \rightarrow \mathbb{R}$ is $C^2$ and satisfies $W'' > 0$ a.e.

- $W$ is superlinear $\lim_{|t| \rightarrow +\infty} \frac{W(t)}{|t|} = +\infty$

Main theorem In a quasicrystalline model there exist calibrated configurations $(x_n)_{n \in \mathbb{Z}}$ that is satisfying

$$\forall m \leq n \quad E_{\omega^*}(x_m, \ldots, x_n) - (n - m) \bar{E}_{\omega^*} = S_{\omega^*}(x_m, x_n)$$

$$\bar{E}_{\omega^*} := \lim_{n \rightarrow +\infty} \inf_{x_0, \ldots, x_n \in \mathbb{R}^d} \frac{1}{n} E_{\omega^*}(x_0, \ldots, x_n)$$
III. Some ideas of the proof
### III.1. Why locally transversally constant?

Assume $V_{\omega^*}$ is given by the Fibonacci sequence:

$$\omega^* \simeq (\cdots, S, L \mid L, S, L, L, S, \cdots) \in \{L, S\}^\mathbb{Z}$$

$$E_{\omega^*}(x, y) = W(y - x) + V_{\omega^*}(x)$$

Denote $\sigma : \{L, S\}^\mathbb{Z} \rightarrow \{L, S\}^\mathbb{Z}$ the left shift. Then we obtain infinitely many similar Fibonacci sequences

$$\Sigma := \{\sigma^n(\omega^*) : n \in \mathbb{Z}\}$$
III.2. Why locally transversally constant?
There is a natural $\mathbb{R}$-action of the set of quasicrystals: $\omega \mapsto \omega - t$

$\Omega := \text{Hull}(\omega^*) = \{\tau_t(\omega^*) : t \in \mathbb{R}\}$

$\Omega \simeq$ suspension over $\Sigma$ of a locally constant return map
or $\Sigma$ can be seen as a global transverse section to the flow

$$V_\omega(t) = V(\tau_t \omega)) = \begin{cases} V_L(t) & \text{if } \omega \in \Sigma_L \\ V_S(t) & \text{if } \omega \in \Sigma_S \end{cases}$$
III.3. Why twist?

**Aubry lemma** The twist property

\[
\frac{\partial E(x, y)}{\partial x \partial y} (x, \cdot) < 0 \quad \text{and} \quad \frac{\partial E(x, y)}{\partial x \partial y} (\cdot, y) < 0 \quad \text{a.e.}
\]

implies

\[
E(x_0, x_1) + E(y_0, y_1) > E(x_0, y_1) + E(y_0, x_1)
\]

**Corollary** Consider a finite minimizing path \((x_k)_{k=0}^n\). Then the following configuration is forbidden.
III.4. Why twist?

Proof

\[ x_m \quad x_{m+1} \]

\[ \tilde{x}_m \quad \tilde{x}_{k-1} \quad \tilde{x}_k \quad \tilde{x}_{k+1} \]

\[ x_k \]

\[ \tilde{x}_m \quad \tilde{x}_{k-1} \quad \tilde{x}_k \quad \tilde{x}_{k+1} \]
III.5. Why twist?

**Corollary** Similarly the following configuration is forbidden

![Configuration Diagram]

**Consequence** If \( (x_k)_k^n = 0 \) is a finite minimizing path, for every pattern, say \( LSL \), the number of points \( x_k \) inside this pattern differs at most by 2
III.6. Why uniquely ergodic?

**Proposition** If \((x_{k}^{(n)})_{k=0}^{n} \) realizes the minimum of \(E_{\omega^*}(x_0^{(n)}, \ldots, x_n^{(n)})\) among all finite path of size \(n\), then

\[
|x_{k+1}^{(n)} - x_{k}^{(n)}| \leq R \quad \text{(is uniformly bounded)}
\]

\[
\lim_{n \to +\infty} \frac{|x_n^{(n)} - x_0^{(n)}|}{n} \simeq \frac{1}{\nu_L N_L + \nu_S N_S} \quad \text{(lim inf > 0 and lim sup < +\infty)}
\]

**Proof**

\[
T := x_n^{(n)} - x_0^{(n)}
\]

\[
\nu_L := (\text{nb of occurrences of } L )/T
\]

\[
N_L := \text{nb of } x_k \text{ inside each tile } L
\]

Then

\[
n = (T \nu_L) N_L + (T \nu_S) N_S
\]
III.7. Why a quasicrystalline model?

Beginning of the proof

- $\omega_*$ a Fibonacci quasicrystal
- $V_{\omega_*}$ a locally transversally constant potential
- $E_{\omega_*}$ a discrete action $E_{\omega_*}(x, y) = W(y - x) + V_{\omega_*}(x)$
- $(x^{(n)}_k)_{n=0}^{n} = \arg \min_{x_0, \ldots, x_n} E_{\omega_*}(x_0, \ldots, x_n)$ a finite minimizing path

Remark

There is no reason that this path is calibrated, that there exists a normalizing constant $\bar{E}_{\omega_*}$ such that

$$E_{\omega_*}(x^{(n)}_0, \ldots, x^{(n)}_n) - n \bar{E}_{\omega_*} = \bar{S}_{\omega_*}(x^{(n)}_0, x^{(n)}_n)$$

But we already know that

$$\sup_{n, k} |x^{(n)}_{k+1} - x^{(n)}_k| \leq R \quad \text{and} \quad \lim_{n \to +\infty} \frac{|x^{(n)}_n - x^{(n)}_0|}{n} > 0$$
III.8. Why a quasicrystalline model?

**Idea** Include this specific quasicrystal into a larger family

- \( \Omega = \text{Hull}(\omega^*) \) the Hull of \( \omega^* \)
- \( \tau_t : \Omega \to \Omega \) the flow acting on subsets by translation
- \( V : \Omega \to \mathbb{R} \) such that \( V_{\omega^*}(x) = V(\tau_x(\omega^*)) \)
- \( L(\omega, x) : \Omega \times \mathbb{R} \to \mathbb{R} \) the Lagrangian form of \( E \)

\[
L(\omega, x) = W(x) + V(\tau_x(\omega)) \quad \text{and} \quad E_{\omega}(x, y) = L(\tau_x(\omega), y - x)
\]

**Main observation** Define \( \mu_n := \frac{1}{n} \sum_{k=0}^{n-1} \delta_{\tau_{x_k}(\omega^*)} \) and \( \mu_\infty = \lim_{n \to +\infty} \mu_n \).

Then \( \mu_\infty \) gives positive mass to any flow box of sufficiently long length

**proof** Say \( \Xi = L S \)

\[
\mu_\infty([0, R] \times \Xi) = \lim_{n \to +\infty} \frac{1}{n} \text{nb occurrences of } L S
\]

\[
= \lim_{n \to +\infty} \frac{1}{n} \frac{|x_n - x_0|}{\text{nb occurrences of } L S} > 0
\]
III.8. Why a quasicrystalline model?

Corollary of the main observation

\( \forall \Xi \ \mu_\infty([0, R] \times \Xi) > 0 \implies \{ \text{supp}(\mu_\infty) \text{ meets every trajectory in time less than } R \} \)

- \( \mu_\infty \) appears to be the projection of a specific measure in \( \tilde{\Omega} := \Omega \times \mathbb{R} \)

\[ \mu_\infty = \text{pr}(\tilde{\mu}_\infty) \ \ \tilde{\mu}_\infty = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta(\tau_{x_k}(\omega^*), x_{k+1} - x_k) \]

Second main observation \( \tilde{\mu}_\infty \) satisfies the two following properties

- \( \tilde{\mu}_\infty \) is minimizing:

\[ \int L(\omega, t) \ d\tilde{\mu}_\infty(\omega, t) = \lim_{n \to +\infty} \frac{1}{n} \inf_{x_0, \ldots, x_n} E_{\omega^*}(x_0, \ldots, x_n) = \bar{E}_{\omega^*} \]

- \( \tilde{\mu}_\infty \) is holonomic

\[ \forall \phi \in C^0 \quad \int \phi(\omega) \ d\tilde{\mu}_\infty(\omega, t) = \int \phi(\tau_t(\omega)) \ d\tilde{\mu}_\infty(\omega, t) \]
III.9. Main result again

**Definition**
We say that a probability measure in $\Omega \times \mathbb{R}$ is minimizing if

$$\mu = \arg\min \left\{ \int L(\omega,t) \, d\nu : \nu \text{ is holonomic} \right\}$$

We call Mather set

$$\text{Mather}(L) := \bigcup \{ \text{supp}(\mu) : \mu \text{ is minimizing} \}$$

**Theorem**

- In the almost periodic case, the Mather set is a non empty compact set
- In the quasicrystaline case, the projected Mather set meets every trajectory
- $\bar{E}_\omega = \inf \{ \int L \, d\mu : \mu \text{ is minimizing} \}$ is independant of $\omega$
- for every $\omega \in \text{Mather}(L)$, there exists a calibrated configuration passing through $x_0 = 0$
Bibliographie I


