The quasi-periodic Frenkel-Kontorova model

Philippe Thieullen (Bordeaux)

joint work with
E. Garibaldi (Campinas) et S. Petite (Amiens)

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Summary of the talk

- **I.** The general problem of minimizing configurations
- **II.** Almost periodic and quasi-crystalline models
- **III.** Some ideas of the proof

I. The general problem of minimizing configurations

I.1. General problem

Notations

– Consider a discrete path in $\mathbb{R}^d$: $(x_n)_{x\in\mathbb{Z}}$
(chain of atoms)

– Choose a discrete action $E(x,y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$
(energy interaction between two consecutive atoms)

– Define the total action of the path

$$\text{“ } E_{tot} := \sum_{n\in\mathbb{Z}} E(x_n, x_{n+1}) \text{ ”}$$

Problem

– Find a bi-infinite path which “minimizes” the total action

I.2. Three examples

The periodic 1D Frenkel-Kontorova model

$$\begin{aligned} E(x,y) &= \frac{1}{2}|y-x-\lambda|^2 + K\big(1-\cos(2\pi x)\big) \\ &= W_\lambda(y-x) + V(x) \end{aligned}$$

- λ: distance at rest when there is no external interaction
- $W_\lambda(x,y) = W_0(x,y) - \lambda(y-x) + \text{cte}$: elastic internal interaction
- $V(x)$: periodic external interaction

The almost periodic 1D Frenkel Kontorova

$$E(x,y) = \frac{1}{2}|y-x-\lambda|^2 + K_1\big(1-\cos(2\pi x 1)\big) + K_2\big(1-\cos(2\pi x\sqrt{2})\big)$$

The quasicrystalline 1D model

- To be explained later

1.3. Minimizing configurations

In the weak sense A configuration $(x_n)_{n\in\mathbb{Z}}$ is said to be minimizing (in the weak sense) if

– for any finite sub-path $(x_m, x_{m+1}, \dots, x_n)$

– define the total action of the sub-path:

$$E(x_m, \dots, x_n) := \sum_{k=m+1}^{n} E(x_{k-1}, x_k)$$

– by fixing the two endpoints x_m and x_n, the total action of the sub-path can only increase: let $(y_m, y_{m+1}, \dots, y_n)$ be another finite path, with $y_m = x_m$ and $y_n = x_n$, then

$$E(x_m, \dots, x_n) \leq E(y_m, \dots, y_n)$$

1.4. Calibrated configurations

Remark

– The notion of minimizing configurations is close to the notion of minimal geodesics
– $E(x, y)$ plays the role of the distance (or a cost)
– But there is no reason to ask $E(x, y) \geq 0$ and $E(x, y) = 0 \;\Leftrightarrow\; x = y$
– a normalizing factor $\bar{E} \in \mathbb{R}$ has to be introduce

Minimizing in the strong sense A configuration $(x_n)_{n\in\mathbb{Z}}$ is minimizing
– if for any finite path $(x_m, x_{m+1}, \ldots, x_n)$
– by fixing the two endpoints x_m and x_n
– if $(y_k, y_{k+1}, \ldots, y_l)$ is another path, possibly not having the same length, but with the same endpoints $y_k = x_m$ and $y_l = x_n$

$$E(x_m, \ldots, x_n) - (n-m)\bar{E} \;\leq\; E(y_k, \ldots, y_l) - (l-k)\bar{E}$$

A minimizing configuration in the strong sense will be called calibrated

I.5. Mañé potential

The effective action $\bar{E}$
(or Mañé critical value, or mean energy per site, or ground energy)

$$\bar{E} := \lim_{n\to+\infty} \inf_{x_0,\ldots,x_n\in\mathbb{R}^d} \frac{1}{n} E(x_0,\ldots,x_n)$$

The Mañé potential $S(x,y)$ Given two points $x, y \in \mathbb{R}^d$

$$S(x,y) = \inf_{n\geq 1} \inf_{x=x_0,\ldots,x_n=y} \left[E(x_0,\ldots,x_n) - n\bar{E}\right]$$

Minimizing in the strong sense $(x_n)_{n\in\mathbb{Z}}$ is said to be calibrated if

$$\forall m \leq n, \quad S(x_m,x_n) = E(x_m,\ldots,x_n) - (n-m)\bar{E}$$

I.6. Assumptions for 1D periodic models

General hypotheses

– $E(x,y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^2

– $E(x,y)$ is translation periodic:

$$E(x+1,y+1) = E(x,y)$$

– $E(x,y)$ is superlinear:

$$\lim_{R\to+\infty} \inf_{|y-x|\geq R} \frac{E(x,y)}{|y-x|} = +\infty$$

– $E(x,y)$ is twist

$$\forall x,y, \quad \frac{E(x,y)}{\partial x \partial y}(x,\cdot) < 0 \quad \text{and} \quad \frac{E(x,y)}{\partial x \partial y}(\cdot,y) < 0 \quad \text{a.e.}$$

(check that if $E(x,y) = \frac{1}{2}(y-x)^2$ then $\frac{\partial E}{\partial x \partial y} = -1$)

I.7. Main question again

Remark Under the hypotheses of translation periodicity, superlineraty and twist

$$\bar{E} = \lim_{n\to+\infty} \inf_{x_0,\ldots,x_n\in\mathbb{R}^d} \frac{1}{n} E(x_0,\ldots,x_n) \quad \text{exists and is finite}$$

The Mañé potential

$$S(x,y) = \inf_{n\geq 1} \inf_{x=x_0,\ldots,x_n=y} \left[E(x_0,\ldots,x_n) - n\bar{E}\right]$$

is finite and satisfies

– $S(x,z) \leq S(x,y) + S(y,z)$ (sub-cocycle)
– $S(x,y) \leq E(x,y) - \bar{E}$
– $S(x,x) \geq 0$

Question Do there exist calibrated configurations $(x_n)_{n\in\mathbb{Z}}$?

$$\forall m \leq n \quad E(x_m,\ldots,x_n) - (n-m)\bar{E} = S(x_m,x_n)$$

1.8. Aubry theory for 1D periodic models

One-parameter family

$$E_\lambda(x,y) := E(x,y) - \lambda(y-x)$$

Aubry theory (1983)

– $\lambda \in \mathbb{R} \mapsto \bar{E}(\lambda)$ is C^1 concave

– for every λ, there exist calibrated configurations for E_λ ; and all calibrated configurations have the same rotation number

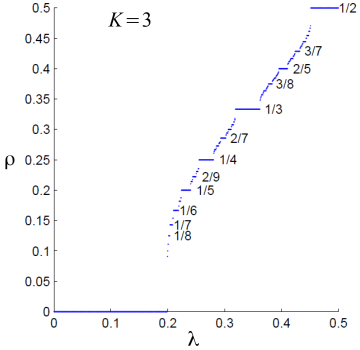
$$\rho = \lim_{n\to+\infty,\ m\to-\infty} \frac{x_n - x_m}{n-m} = -\frac{d\bar{E}}{d\lambda}(\lambda)$$

– conversely, every ρ is the rotation number of a calibrated configuration for some E_λ

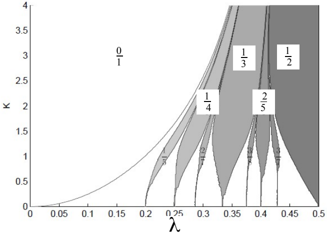
I.9. The original 1D Frenkel-Kontorova

$$E_{\lambda,K}(x,y) = \frac{1}{2}|y-x|^2 - \lambda(y-x) + \frac{K}{(2\pi)^2}\Big(1-\cos(2\pi x)\Big)$$

$$\rho = -\frac{\partial \bar{E}}{\partial \lambda}(\lambda, K)$$



The system is “locked” at rational rotation number ρ



II. Almost periodic and quasi-crystalline models

II.1. Example of almost periodic models

The almost periodic 1D Frenkel-Kontorova

- $E_\omega(x,y) = W(y-x) + V_\omega(x)$
- $W(y-x) = \frac{1}{2}|y-x-\lambda|^2$
- $V_\omega(x) = K_1\big(1-\cos 2\pi(\omega_1 + x)\big) + K_2\big(1-\cos(\omega_2 + x\sqrt{2})\big)$

Remark In the periodic case there is no need to introduce the space Ω of all phases. In the almost periodic case, it is more appropriate to work with a full family of discrete actions

Notations

- $\Omega = \mathbb{T}^2 = \{(\omega_1,\omega_2) \,:\, \omega_i \mod 1\}$ the set of all environments
- $\tau_t : \Omega \to \Omega, \quad \tau_t(\omega_1,\omega_2) := (\omega_1 + t, \omega_2 + t\sqrt{2})$ is minimal
- $V_\omega(x) = V(\tau_x(\omega))$ where $V : \Omega \to \mathbb{R}$ and

$\forall t \in \mathbb{R}, \quad E_\omega(x+t, y+t) = E_{\tau_t(\omega)}(x,y)$ (translation invariance property)

II.2.A general almost periodic model

Definition An almost periodic model $(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d}, L)$

– Ω a compact space

– $\tau_t : \Omega \to \Omega$ a minimal flow $\tau_{s+t} = \tau_s \circ \tau_t$

– a family of translation-invariant discrete actions

$$\forall x, y, t, \in \mathbb{R}^d \quad E_\omega(x+t, y+t) = E_{\tau_t(\omega)}(x, y)$$

–we assume actually that $E_\omega(x, y)$ has a Lagrangian form

$$E_\omega(x, y) := L(\tau_x(\omega), y - x)$$

– $L(\omega, t) : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is C^0 in (ω, t) and superlinear in t

$$\lim_{R\to+\infty} \inf_{\omega\in\Omega} \inf_{\|t\|\geq R} \frac{L(\omega, t)}{\|t\|} = +\infty$$

II.3. A general quasicrystalline model

Definition A quasicrystalline model $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$

– $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ is an almost periodic model

– $(\Omega, \{\tau_t\})$ is minimal and uniquely ergodic

– $d = 1$ and (to simplify) $L(\omega, t) = W(t) + V(\omega)$

$$E_\omega(x, y) = W(y - x) + V_\omega(x) \quad \text{and} \quad V_\omega(x) = V(\tau_x(\omega))$$

– $V : \Omega \to \mathbb{R}$ is C^0

– W is twist: W is C^2 and convex $W'' > 0$ a.e.

$$\forall x, y \in \mathbb{R} \quad \frac{\partial E_\omega}{\partial x \partial y}(x, \cdot) < 0 \quad \text{a.e.} \quad \text{and} \quad \frac{\partial E_\omega}{\partial x \partial y}(\cdot, y) < 0 \text{ a.e.}$$

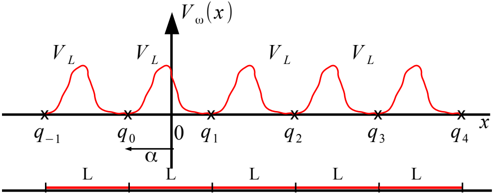
– W is superlinear: $\displaystyle\lim_{|t|\to+\infty} \frac{W(t)}{|t|} = +\infty$

In addition to the above properties

– V is locally transversally constant

II.4. Locally transversally constant

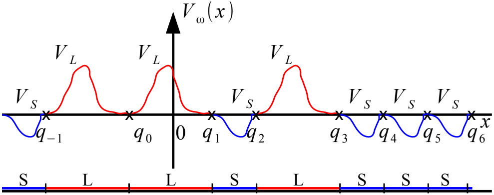
The periodic case



1 tile L
1 function

$V_L : [0, L] \to \mathbb{R}$

The quasicrystalline case



2 tiles L,S
2 functions

$V_L : [0, L] \to \mathbb{R}$

$V_S : [0, S] \to \mathbb{R}$

$\omega = (q_n)_{n \in \mathbb{Z}}$ ordered along $\mathbb{R}$ with minimal complexity

II.5. Quasicrystals

Delone set An ordered subset $\omega^* = (q_n^*)_{n\in\mathbb{Z}} \subset \mathbb{R}$ which is uniformly discrete and relatively dense

Finite complexity There is a finite number of jumps

$$\forall n \in \mathbb{Z}, \quad \Delta_n := q_{n+1}^* - q_n^* \in \{L_1, \ldots, L_r\}$$

Repetitive For instance assume there are 2 tiles $\{L, S\}$. A pattern $P = LSL$ is any finite sub-word. An occurrence of a pattern is some n

$$\Delta_n = L,\ \Delta_{n+1} = S,\ \Delta_{n+2} = L$$

repetitive = for every pattern the set of occurrences is relatively dense

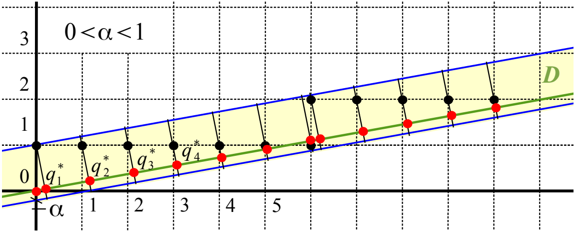
Uniformly distributed For every pattern P, uniformly in x

$$\frac{1}{2T}\mathrm{Card}\{\text{occurrences of } P \text{ in } [x-T, x+T]\} \longrightarrow \nu(P) > 0$$

II.6. A model of Sturmian type

Construction A model of type "cut and projection"

– let be $\alpha \notin \mathbb{Q}$ and $\mathcal{B} = \{(x, y) \in \mathbb{R}^2 : \alpha x - \alpha < y \leq \alpha x + 1\}$
– we construct a sequence of points $(q_n^* \overrightarrow{u})_{n \geq 0}$ on the line $\mathcal{D} = \mathbb{R}\overrightarrow{u}$, $\overrightarrow{u} = (1, \alpha)$, by projecting orthogonally on $\mathcal{D}$ the set $\mathbb{Z}^2 \cap \mathcal{B}$



– $\omega^* = (q_n^*)_{n \in \mathbb{Z}}$ is a quasicrystal with 2 tiles

$$q_{n+1}^* - q_n^* \in \left\{1/\sqrt{1+\alpha^2}, \alpha/\sqrt{1+\alpha^2}\right\} = \{L, S\}$$

A particular case For $\alpha = \frac{\sqrt{5}-1}{2}$, one obtains the Fibonacci substitution

$$S \longrightarrow L \quad \text{et} \quad L \longrightarrow LS$$

II.7. A quasicrystalline model

Definition (first definition)

– ω^* a quasicrystal (an ordered subset of finite complexity, which is repetitive, with uniformly distributed patterns)

– a discrete action $E_{\omega^*}(x,y) = W(y-x) + V_{\omega^*}(x)$

– $V_{\omega^*} : \mathbb{R} \to \mathbb{R}$ a C^0 transversally constant function (a juxtaposition of of potentials according to the tiles)

– $W : \mathbb{R} \to \mathbb{R}$ is C^2 and satisfies $W'' > 0$ a.e.

– W is superlinear $\displaystyle\lim_{|t|\to+\infty} \frac{W(t)}{|t|} = +\infty$

Main theorem In a quasicrystalline model there exist calibrated configurations $(x_n)_{n\in\mathbb{Z}}$ that is satisfying

$$\forall m \leq n \quad E_{\omega^*}(x_m, \ldots, x_n) - (n-m)\bar{E}_{\omega^*} = S_{\omega^*}(x_m, x_n)$$

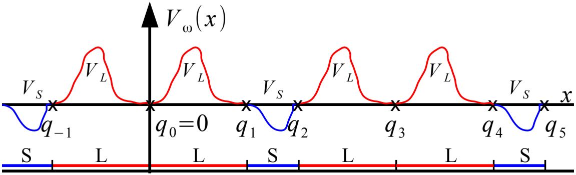
$$\bar{E}_{\omega^*} := \lim_{n\to+\infty} \inf_{x_0,\ldots,x_n\in\mathbb{R}^d} \frac{1}{n} E_{\omega^*}(x_0, \ldots, x_n)$$

III. Some ideas of the proof

III.1. Why locally transversally constant?

Assume V_{ω^*} is given by the Fibonacci sequence:

$$\omega^* \simeq (\cdots, S, L \mid L, S, L, L, S, \cdots) \in \{L, S\}^{\mathbb{Z}}$$



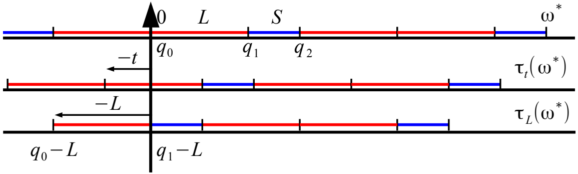
$$E_{\omega^*}(x, y) = W(y - x) + V_{\omega^*(x)}$$

Denote $\sigma : \{L, S\}^{\mathbb{Z}} \to \{L, S\}^{\mathbb{Z}}$ the left shift. Then we obtain infinitely many similar Fibonacci sequences

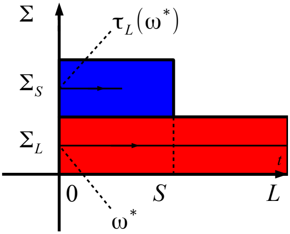
$$\Sigma := \overline{\{\sigma^n(\omega^*) \ : \ n \in \mathbb{Z}\}}$$

III.2. Why locally transversally constant?

There is a natural $\mathbb{R}$-action of the set of quasicrystals: $\omega \mapsto \omega - t$



$$\Omega := \textbf{Hull}(\omega^*) = \overline{\{\tau_t(\omega^*)\ :\ t \in \mathbb{R}\}}^{\text{compact-Hausdorff}}$$



$\Omega \simeq$ suspension over Σ of a locally constant return map
or Σ can be seen as a global transverse section to the flow

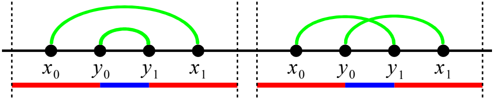
$$V_\omega(t) = V(\tau_t\omega)) = \begin{cases} V_L(t) & \text{if} \ \ \omega \in \Sigma_L \\ V_S(t) & \text{if} \ \ \omega \in \Sigma_S \end{cases}$$

III.3. Why twist?

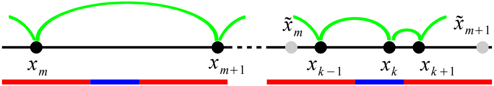
Aubry lemma The twist property

$$\frac{\partial E(x,y)}{\partial x \partial y}(x,\cdot) < 0 \quad \text{and} \quad \frac{\partial E(x,y)}{\partial x \partial y}(\cdot,y) < 0 \quad \text{a.e.}$$

implies $\quad E(x_0,x_1)+E(y_0,y_1) \quad > \quad E(x_0,y_1)+E(y_0,x_1)$

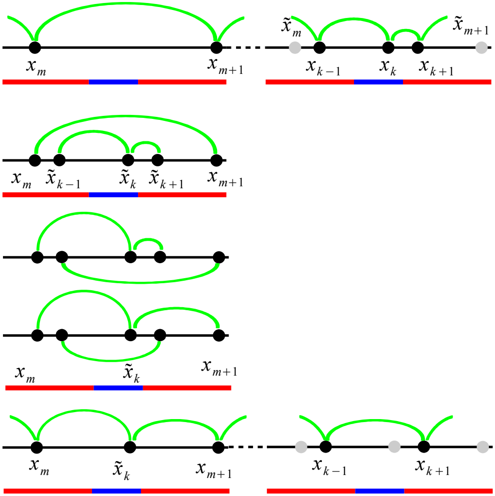


Corollary Consider a finite minimizing path $(x_k)_{k=0}^n$. Then the following configuration is forbidden.



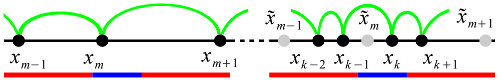
III.4. Why twist?

Proof



III.5. Why twist?

Corollary Similarly the following configuration is forbidden



Consequence If $(x_k)_{k=0}^n$ is a finite minimizing path, for every pattern , say LSL, the number of points x_k inside this pattern differs at most by 2

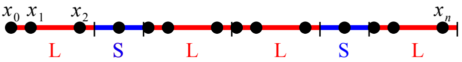
III.6. Why uniquely ergodic?

Proposition If $(x_k^{(n)})_{k=0}^n$ realizes the minimum of $E_{\omega^*}(x_0^{(n)}, \ldots, x_n^{(n)})$ among all finite path of size n, then

$$|x_{k+1}^{(n)} - x_k^{(n)}| \leq R \qquad \text{(is uniformly bounded)}$$

$$\lim_{n \to +\infty} \frac{|x_n^{(n)} - x_0^{(n)}|}{n} \simeq \frac{1}{\nu_L N_L + \nu_S N_S} \qquad (\liminf > 0 \text{ and } \limsup < +\infty)$$

Proof



$T := x_n^{(n)} - x_0^{(n)}$
$\nu_L :=$ (nb of occurrences of L)$/T$
$N_L :=$ nb of x_k inside each tile L
Then

$$n = (T\nu_L)N_L + (T\nu_S)N_S$$

III.7. Why a quasicrystalline model?

Beginning of the proof

- ω_* a Fibonacci quasicrystal
- V_{ω^*} a locally transversally constant potential
- E_{ω^*} a discrete action $E_{\omega^*}(x,y) = W(y-x) + V_{\omega^*}(x)$
- $(x_k^{(n)})_{k=0}^n = \arg\min_{x_0,\ldots,x_n} E_{\omega^*}(x_0,\ldots,x_n)$ a finite minimizing path

Remark

There is no reason that this path is calibrated, that there exists a normalizing constant $\bar{E}_{\omega^*}$ such that

$$E_{\omega^*}(x_0^{(n)},\ldots,x_n^{(n)}) - n\bar{E}_{\omega^*} = S_{\omega^*}(x_0^{(n)}, x_n^{(n)})$$

But we already know that

$$\sup_{n,k} |x_{k+1}^{(n)} - x_k^{(n)}| \le R \quad \text{and} \quad \lim_{n\to+\infty} \frac{|x_n^{(n)} - x_0^{(n)}|}{n} > 0$$

III.8. Why a quasicrystalline model?

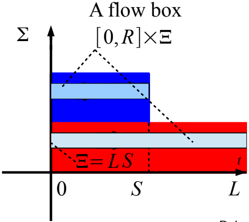
Idea Include this specific quasicrystal into a larger family

- $\Omega = \mathrm{Hull}(\omega^*)$ the Hull of ω_*
- $\tau_t : \Omega \to \Omega$ the flow acting on subsets by translation
- $V : \Omega \to \mathbb{R}$ such that $V_{\omega^*}(x) = V(\tau_x(\omega^*))$
- $L(\omega, x) : \Omega \times \mathbb{R} \to \mathbb{R}$ the Lagrangian form of E

$$L(\omega, x) = W(x) + V(\tau_x(\omega)) \quad \text{and} \quad E_\omega(x, y) = L(\tau_x(\omega), y - x)$$

Main observation Define $\mu_n := \frac{1}{n}\sum_{k=0}^{n-1} \delta_{\tau_{x_k}(\omega^*)}$ and $\mu_\infty = \lim_{n\to+\infty} \mu_n$

Then μ_∞ gives positive mass to any flow box of sufficiently long length



proof Say $\Xi = LS$

$$\mu_\infty([0, R] \times \Xi) = \lim_{n\to+\infty} \frac{1}{n} \text{nb occurrences of } LS$$

$$= \lim_{n\to+\infty} \frac{|x_n - x_0|}{n} \frac{\text{nb occurrences of } LS}{|x_n - x_0|} > 0$$

III.8. Why a quasicrystalline model?

Corollary of the main observation

$$\forall \Xi \quad \mu_\infty([0,R]\times\Xi) > 0 \quad \Longrightarrow \quad \left\{ \begin{array}{l} \mathrm{supp}(\mu_\infty) \text{ meets every trajectory} \\ \text{in time less than } R \end{array} \right.$$

– μ_∞ appears to be the projection of a specific measure in $\tilde{\Omega} := \Omega \times \mathbb{R}$

$$\mu_\infty = pr(\tilde{\mu}_\infty) \quad \tilde{\mu}_\infty = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{(\tau_{x_k}(\omega^*), x_{k+1}-x_k)}$$

Second main observation $\tilde{\mu}_\infty$ satisfies the two following properties

– $\tilde{\mu}_\infty$ is minimizing:

$$\int L(\omega,t)\, d\tilde{\mu}_\infty(\omega,t) = \lim_{n\to+\infty} \frac{1}{n} \inf_{x_0,\ldots,x_n} E_{\omega^*}(x_0,\ldots,x_n) = \bar{E}_{\omega^*}$$

– $\tilde{\mu}_\infty$ is holonomic

$$\forall \phi \in C^0 \qquad \int \phi(\omega)\, d\tilde{\mu}_\infty(\omega,t) = \int \phi(\tau_t(\omega))\, d\tilde{\mu}_\infty(\omega,t)$$

III.9. Main result again

Definition We say that a probability measure in $\Omega \times \mathbb{R}$ is minimizing if

$$\mu = \arg\min \left\{ \int L(\omega, t)\, d\nu \ : \ \nu \text{ is holonomic } \right\}$$

We call Mather set

$$\mathrm{Mather}(L) := \bigcup \{\mathrm{supp}(\mu) \ : \ \mu \text{ is minimizing } \}$$

Theorem

– In the almost periodic case, the Mather set is a non empty compact set

– In the quasicrystaline case, the projected Mather set meets every trajectory

– $\bar{E}_\omega = \inf\{\int L d\mu \ : \ \mu \text{ is minimizing } \}$ is independant of ω

– for every $\omega \in \mathrm{Mather}(L)$, there exists a calibrated configuration passing through $x_0 = 0$

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