Zero temperature limite and the cell problem in Hamilton-Jacobi equation

Philippe Thieullen

Université Bordeaux 1, Institut de Mathématiques

Roma, Mars 2011

Objectives

The ultimate goal of the project:
To interprete the solution of the cell equation in Hamilton-Jacobi as a ground state, that is, a zero temperature limite of Gibbs measures

1. A short introduction on the semi-discretized cell problem
2. A dynamical interpretation by the Frenkel-Kontorova model
3. Gibbs measures and phase transition

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A semi-discretized cell equation

- **Notations:** We consider an Hamiltonian $H(x, p)$, continuous, periodic in x, convex in p and uniformly superlinear in p

$$\lim_{\|p\|\to+\infty} \inf_{x\in\mathbb{R}^d} \frac{H(x,p)}{\|p\|} = +\infty.$$

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- **The cell equation:** There exist a unique constant $\bar{H}(P)$ such that that, for any $P \in \mathbb{R}^d$ fixed,

$$H(x, Du(x) + P) = \bar{H}(P), \quad \forall\, x \in \mathbb{R}^d$$

admits a periodic solution $u(x) = u(x,P)$ in the viscosity sense.

A semi-discretized cell equation

- **An approximate cell problem:** (Hamilton-Jacobi-Belman) There exists a unique viscosity solution $u_\epsilon(x)$ of

$$\epsilon u_\epsilon(x) + H(x, Du_\epsilon(x) + P) = 0, \quad \forall\ x \in \mathbb{R}^d$$

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- **The value function:** (of an infinite horizon problem)

$$u_\epsilon(x) = u_\epsilon(x, P) = \inf \Big\{ \int_{-\infty}^{0} \Big[L(\gamma, \dot\gamma) - P \cdot \dot\gamma \Big] e^{-\epsilon|s|}\, ds \ : \\ \gamma \in W^{1,\infty}(]-\infty, 0], \mathbb{R}^d),\ \gamma(0) = x \Big\}$$

where $L(x, v) = \sup\{p \cdot v - H(x, p) \ : \ p \in \mathbb{R}^d\}$ is the associated Lagrangian

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- **LPV result:** For some subsequence of ϵ, uniformly in x

$$\epsilon u_\epsilon(x) \to -\bar H(P), \quad u_\epsilon(x) - \min u_\epsilon \to u(x)$$

A semi-discretized cell equation

- **The semi-discretized HJB equation:** Let $\tau > 0$ be a time step. Let $t_{-k} = -\tau k$, k=1,2,...

$$u_{\epsilon,\tau}(x) = \inf \left\{ \sum_{k=1}^{+\infty} \tau \Big[L(x_{-k}, v_{-k}) - P \cdot v_{-k} \Big] (1 - \epsilon\tau)^{k-1} \ : \right.$$
$$\left. v_{-k} \in \mathbb{R}^d, \ x_{-k+1} = x_{-k} + \tau v_{-k}, \ x_0 = x \right\}$$

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- **Results (Falcone, Rorro):** If in addition $H(x,p)$ is C^2-smooth and strictly convex in p, then, for some constant C,

$$\sup_{x \in \mathbb{R}^d} |u_{\epsilon,\tau}(x) - u_\epsilon(x)| \leq C \frac{\tau}{\epsilon}$$

A semi-discretized cell equation

- **The discrete dynamical programming principal:** The discrete value function $u(x) = u_{\epsilon,\tau}$ satisfies

$$\epsilon u(x) + \sup_{v \in \mathbb{R}^d} \left\{ (1 - \epsilon\tau) \frac{u(x) - u(x - \tau v)}{\tau} - L(x - \tau v, v) + P \cdot v \right\} = 0$$

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- **versus a semi-discrete cell equation:**

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- **or a discrtet Lax-Oleinik equation**

$$u(x) + \bar{L}(P) = \inf_{x' \in \mathbb{R}^d} \left\{ u(x') + \tau L\left(x', \frac{x - x'}{\tau} \right) - \tau P \cdot (x - x') \right\}$$

$$\bar{L}(P) = -\bar{H}(P)$$

A semi-discretized cell equation

Conclusion:

- The cell equation admits “approximate” solutions satisfing an equation of the form

$$\begin{cases} u(x) + \bar{E} = \min_{x'}\{u(x') + E(x', x)\}, \\ u(x) \text{ is 1-periodic in } x, \\ E(x', x) = E(x'+1, x+1), \\ E(x', x) \text{ is strictly convex, superlinear in } \|x - x'\| \end{cases}$$

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- The nul sets

$$\{(x', x) \,:\, x' = \operatorname{argmin}_{x'}\{u(x') - u(x) + E(x', x) - \bar{E}\}\}$$

give the set of equations describing the ground configurations of the Frenkel-Kontorova model associated to the interaction energy $E(x', x)$.

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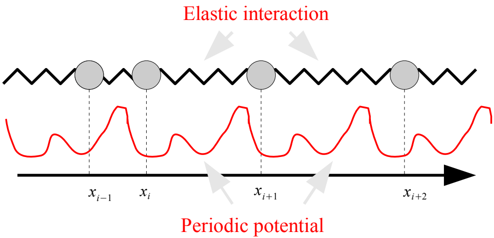
The Frenkel-Kontorova model

- **The physical model:** The model describes the set of configuration of a chain of atoms at equilibrium



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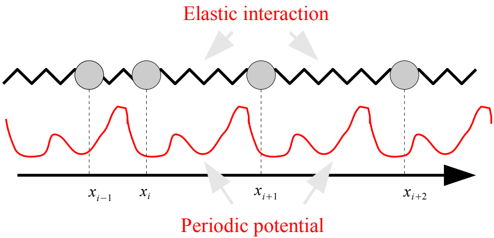

- **The original 1D-FK:**

$$E(x,y) = W(x,y) + V(x),$$

$$W(x,y) = \frac{1}{2}|y-x|^2, V(x) = \frac{K}{(2\pi)^2}\Big(1-\cos(2\pi x)\Big)$$

$$E_\lambda(x,y) = E_0(x,y) - \lambda(y-x).$$

The Frenkel-Kontorova model

- **Question:** How to define a notion of configuration $\underline{x} := (x_k)_{k\in\mathbb{Z}}$ with the smallest total energy

$$E_{tot}(\underline{x}) := \sum_{k=-\infty}^{+\infty} E(x_k, x_{k+1}) \leq E_{tot}(\underline{y}), \quad \forall\ \underline{y}$$

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- **Minimizing configuration:** $\underline{x} := (\dots, x_{-1}, x_0, x_1, \dots)$ such that

$$\begin{cases} E(x_m, x_{m+1}, \dots, x_n) := \sum_{k=m}^{n-1} E(x_k, x_{k+1}) \\ E(x_m, \dots, x_n) \leq E(y_m, \dots, y_n) \\ \forall\ \underline{y} \text{ configuration} \quad \text{s.t.} \quad y_m = x_m \text{ and } y_n = x_n \end{cases}$$

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- **Remark:** Let $E_\lambda(x, y) := E(x, y) - \lambda \cdot (y - x)$ then

$$(x_k)_{k\in\mathbb{Z}} \text{ is minimizing for } E \iff (x_k)_{k\in\mathbb{Z}} \text{ is minimizing for } E_\lambda$$

The Frenkel-Kontorova model

- **Effective potential (Chou-Griffiths):** A periodic C^0 function $u(x)$

$$\begin{cases} u(y) + \bar{E} = \min_x \{u(x) + E(x,y)\}, & \forall\ y \quad \text{(backward)} \\ u(x) + \bar{E} = \max_y \{u(y) - E(x,y)\}, & \forall\ x \quad \text{(forward)} \end{cases}$$

(effectif potential = discrete viscosity solution = calibrated solution = corrector)

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- **Ground configuration:** A configuration $(x_k)_{k\in\mathbb{Z}}$ such that

$$\exists\ u(x) \text{ effective potential s.t.}$$
$$u(x_{k+1}) + \bar{E} = u(x_k) + E(x_k, x_{k+1}), \quad \forall\ k \in \mathbb{Z}$$

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- **Easy results:**
There exist effective potential ($\bar{E}$ is unique)
All ground configuration are minimizing

The Frenkel-Kontorova model

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- "recurrent" minimizing configurations admit rotation vectors

$$\omega := \lim_{n-m\to+\infty} \frac{x_n - x_m}{n-m} = -\frac{d\bar{E}}{d\lambda}(\lambda)$$

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- If $\omega \in \mathbb{Q}$ then

$$\Lambda(\omega) := \{\lambda \text{ s.t. } \omega = -\frac{d\bar{E}}{d\lambda}(\lambda)\}$$

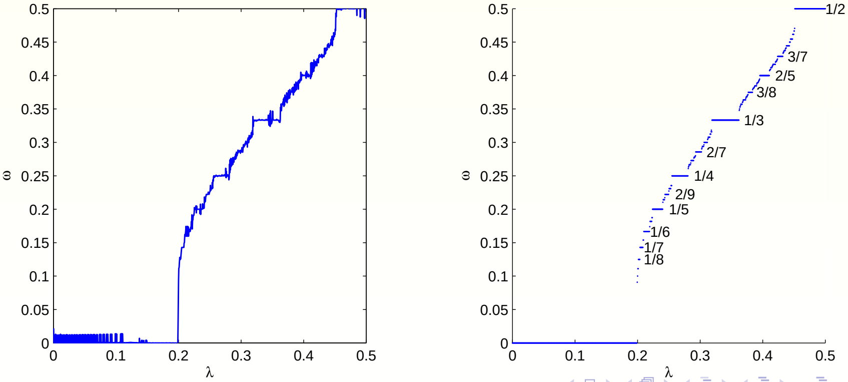
has non empty interior. If $\omega \notin \mathbb{Q}$, $\mathrm{int}(\Lambda(\omega)) = \emptyset$.

The Frenkel-Kontorova model

Phase locking at rational rotation number:

$$E_{\lambda,K}(x,y) = \frac{1}{2}|y-x|^2 - \lambda(y-x) + \frac{K}{(2\pi)^2}\big(1-\cos(2\pi x)\big)$$

$$K = 3, \quad \omega = -\frac{\partial \bar{E}}{\partial \lambda}(\lambda, K)$$



The Frenkel-Kontorova model

Phase transition with respect to K:

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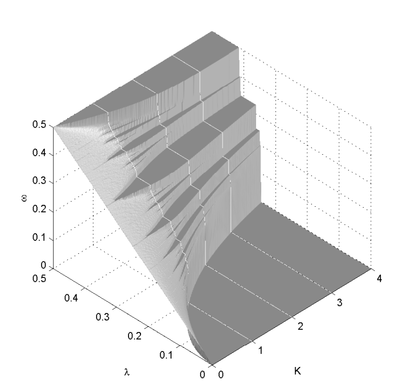

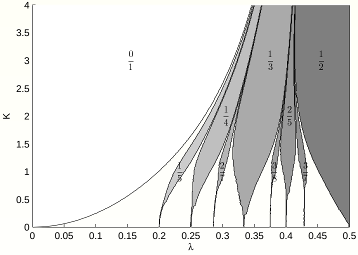

The Frenkel-Kontorova model

Conclusion:

- The effective Hamiltonian $\bar{H}(P)$ and the ergodic value $\bar{E}(\lambda)$ are related by

$$\bar{H}(P,K) \simeq -\frac{1}{\tau^2}\bar{E}(\tau P, \tau^2 K)$$

$$H_{P,K}(x,p) = \frac{1}{2}|p+P|^2 - \frac{K}{(2\pi)^2}(1-\cos(2\pi x))$$

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- $\bar{H}$ can be considered as the ground energy of some configurations in the Frenkel-Kontorova model. The ground energy is also interpreted as the energy of the system at zero temperature.

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Gibbs measures and phase transition

- **Full discrete Frenkel-Kontorova:**
  S is a finite set (discretization in space)
  $x = (x_k)_{k\in\mathbb{Z}} \in S^{\mathbb{Z}}$ configuration
  $\Sigma := S^{\mathbb{Z}}$ full shift
  $E(x) = E(x_0, x_1) : \Sigma \to \mathbb{R}$ short range interaction
  $E(x_0, \ldots, x_n) = \sum_{k=0}^{n-1} E \circ \sigma^k(x)$ ergodic sum
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- **Sub shift of finite type SFT:**
  G an irreducible directed graph on S
  $\Sigma_G^+ = \{(x_k)_{k \geq 0} \in S^{\mathbb{Z}} \ : \ x_k \to x_{k+1} \text{ is admissible } \}$
  $E : \Sigma_G^+ \to \mathbb{R}$ a Hölder function depending only on $(x_0, x_1, \ldots)$

Gibbs measures and phase transition

- **Gibbs measure at temperature β^{-1}:** Let $C_n = [x_0, \ldots, x_n]$ be a cylinder of size n, then a Gibbs measure μ_β is a probability measure such that

$$\mu_\beta \text{ is } \sigma\text{-invariant}, \quad \sigma_*(\mu_\beta) = \mu_\beta$$

$$\mu_\beta(C_n) \simeq \exp\Big(-\beta \sum_{k=0}^{n-1} [E(x_k, x_{k+1}) - \bar{E}_\beta] \Big), \quad \forall\, x \in C_n$$

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- **Correctors and Ground configurations:**

$$u(x) + \bar{E} = \min_{x' \,:\, \sigma(x') = x} \{u(x') + E(x')\}$$

$$x \text{ is a ground configuration } \iff \sigma^k(x) \in \{E = u \circ \sigma - u + \bar{E}\}, \ \forall\, k$$

Gibbs measures and phase transition

Results for general Hölder $E : \Sigma_G^+ \to \mathbb{R}$

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- $\bar{E}_\beta \to \bar{E} = \min\{\int E\,d\mu\ :\ \mu \text{ is } \sigma\text{-inv }\}$, $\beta(\bar{E}_\beta - \bar{E}) \to -\mathrm{Ent}(\Sigma_G^+)$

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- any accumulation point of μ_β is a minimizing measure μ_{min} of maximal topological entropy

$$\bar{E} = \int E\,d\mu_{min}, \quad \mathrm{Ent}(\mu_{min}) = \mathrm{Ent}(\Sigma_G^+), \quad \mathrm{supp}(\mu_{min}) \subset \mathrm{GC}$$

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- $\exists!$ additive eigenfunction $u_\beta(x)$ of

$$\exp\Big(-\beta[u_\beta(x) + \bar{E}_\beta]\Big) = \sum_{x' \,:\, \sigma(x')=x} \exp\Big(-\beta[u_\beta(x') + E(x')]\Big)$$

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Results for general Hölder $E : \Sigma_G^+ \to \mathbb{R}$

- $\mathrm{Ent}(\mu) := \lim_{n\to+\infty} \sum_{C_n} -\frac{1}{n}\mu(C_n) \ln \mu(C_n)$: the entropy of μ
- $\mathrm{Ent}(\Sigma_G^+) := \max\{\mathrm{Ent}(\mu) \,:\, \mu \text{ is } \sigma\text{-inv}\}$: topological entropy
- $\bar{E}_\beta = \min\{\int E\, d\mu - \beta^{-1}\mathrm{Ent}(\mu) \,:\, \mu \text{ is } \sigma\text{-inv }\}$
- $\bar{E}_\beta \to \bar{E} = \min\{\int E\, d\mu \,:\, \mu \text{ is } \sigma\text{-inv }\}$, $\beta(\bar{E}_\beta - \bar{E}) \to -\mathrm{Ent}(\Sigma_G^+)$
- any accumulation point of μ_β is a minimizing measure μ_{min} of maximal topological entropy

$$\bar{E} = \int E\, d\mu_{min}, \quad \mathrm{Ent}(\mu_{min}) = \mathrm{Ent}(\Sigma_G^+), \quad \mathrm{supp}(\mu_{min}) \subset \mathrm{GC}$$

- $\exists!$ additive eigenfunction $u_\beta(x)$ of

$$\exp\Big(-\beta[u_\beta(x) + \bar{E}_\beta]\Big) = \sum_{x' \,:\, \sigma(x')=x} \exp\Big(-\beta[u_\beta(x') + E(x')]\Big)$$

- any accumulation point of u_β is an effective potential

Gibbs measures and phase transition

Results for short range interactions $E(x) = E(x_0, x_1)$:

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Gibbs measures and phase transition

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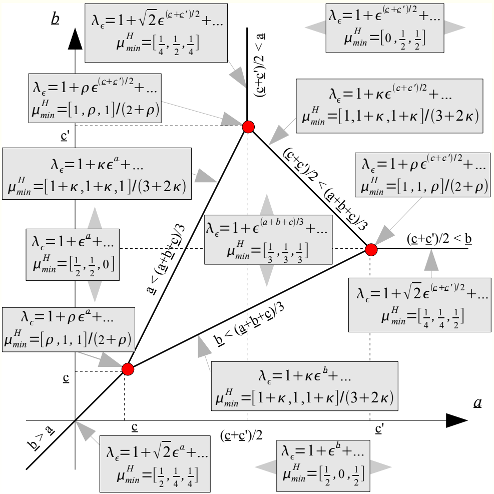
- The existence of $\bar{E}$ and μ_{min} are related to a problem of singular matrices: let

$$M_\beta := \left[e^{-\beta E(x,y)} \right]_{x,y \in S}$$

$$L_\beta M_\beta = \lambda_\beta L_\beta, \quad M_\beta R_\beta = \lambda_\beta R_\beta, \quad \sum_x L_\beta(x) R_\beta(x) = 1$$

then $\lambda_\beta = e^{-\beta \bar{E}_\beta} \sim \lambda_0 e^{-\beta \bar{E}}$ and $L_\beta(x) R_\beta(x) \to \mu_{min}(x)$

Gibbs measures and phase transition



$$S = \{1,2,3\}$$

$$M_\beta = \begin{bmatrix} 1 & e^{-\beta a} & e^{-\beta b} \\ e^{-\beta a} & 1 & e^{-\beta c} \\ e^{-\beta b} & e^{-\beta c'} & 1 \end{bmatrix}$$

$$a, b, c, c' > 0 \quad c > c'$$

$$\mu_{min} = c_1\delta_1 + c_2\delta_2 + c_3\delta_3$$

$$\delta_i = \text{ Dirac measure at}$$

$$(\dots, i, i, i, \dots)$$

$$\rho^2 - \rho - 1 = 0$$

$$\kappa^3 - \kappa - 1 = 0$$

Gibbs measures and phase transition

Conclusion: A possible research program

- Develop a thermodynamical formalism for the the cell problem

Gibbs measures and phase transition

Conclusion: A possible research program

- Develop a thermodynamical formalism for the the cell problem
- Use the fast technics developped in MCMC (Markov chain Monte Carlo) method to obtain the effective Hamiltonian $\bar{H}(P)$ and the viscosity solutions $u(x)$ as limits of quantities at equilibrium when the system is freezing