Convergence of the Gibbs potential for infinite-range interaction energy in one-dimensional lattices

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– Outline –

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– The general setting –

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$$\bar{H}_\beta := -\frac{1}{\beta}\mathrm{Pres}(-\beta H) = \min\Big\{ \int H\, d\mu - \frac{1}{\beta}\mathrm{Ent}(\mu) \ : \ \mu \ \sigma\text{-inv.} \Big\}$$

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- $\mathrm{Ent}(\mu)$ is the entropy of μ:

$$\mathrm{Ent}(\mu) := \lim_{n \to +\infty} \frac{1}{n} \sum_{C_n:\ \text{cylinder}} -\mu[C_n] \ln \mu[C_n]$$

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$$\min_{p\text{: probability vector}} \left\{ \sum_n \left(H_n p_n + \frac{1}{\beta} p_n \ln p_n \right) \right\}$$

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- the Gibbs measure is unique:

$$\mu_\beta[C_n(x)] \asymp \exp\Big(-\beta\big[\sum_{k=0}^{n-1} H\circ\sigma^k(x) - n\bar{H}_\beta \big] \Big)$$

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- Extends to higher dimensional lattices and finite-range interactions
- Construct a thermodynamic formalism for the Frenkel-Kontorova model: $\mu_\beta(dx) := \phi_+(x)\phi_-(x)\,dx$, $\phi_\pm(x)$ C^0 1-periodic,

$$\int_0^1 \Big[\sum_{k\in\mathbb{Z}} e^{-\beta E(x,y+k)}\Big]\phi_+(y)dy = e^{-\beta \bar{E}_\beta}\phi_+(x)$$

$$\int_0^1 \phi_-(x)\Big[\sum_{k\in\mathbb{Z}} e^{-\beta E(x-k,y)}\Big]dx = e^{-\beta \bar{E}_\beta}\phi_-(y)$$

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For finite-range interaction $H(x) = H(x_0, x_1)$

- There exists a subshift $\Omega_{GS} \subset \Sigma_G$, $\Omega_{GS} = \Sigma_{G_{GS}}$ is a SFT,

$$x \overset{G_{GS}}{\to} y \iff x \overset{G}{\to} y \text{ belongs to a minimizing cycle of } G$$

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- $G_{GS} = G_1 \cup \ldots \cup G_r$ is a semi-irreducible subgraph of G

$$S = S_0 \cup S_1 \cup \ldots \cup S_r, \quad G_i \subset S_i \times S_i$$
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- μ_∞ is a barycenter of finitely many μ^i_∞

$$\mathrm{Ent}(\mu_i) = \mathrm{Ent}(\Omega_i) = \mathrm{Ent}(\Omega_{GS})$$

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$$e^{-\beta \bar{H}_\beta} = A_0 e^{-\beta a_0} + A_1 e^{-\beta a_1} + \dots \quad a_0 < a_1 < \dots \to +\infty$$

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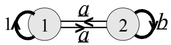
- In particular

$$x, y \in S_i \Rightarrow \frac{\mu_\beta[x]}{\mu_\beta[y]} \sim \frac{B_0(x)}{B_0(y)} = \frac{\pi_i(x)}{\pi_i(y)}$$

where (π, Q_i) is the Markov chain of maximal entropy of Ω_i

Two examples:



- Two symbols:
  $H(1,1) = 1$, $H(2,2) = b$
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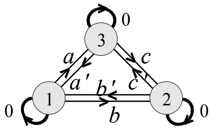
- Possible minimizing ergodic values
  $\bar{H} \in \{1, a, b\}$



- Three symbols:
  assume $a, a', b, b', c, c' > 0$
- Possible minimizing ergodic values
  $\bar{H} = 0$, cycles of order 1
  $\bar{H} \in \{\frac{1}{2}(a + a'), \frac{1}{2}(b + b'), \frac{1}{2}(c + c')\}$
  cycles of order 2
  $\bar{H} \in \{\frac{1}{3}(a + b' + c), \frac{1}{3}(a' + b + c')\}$
  cycles of order 3
- By assumption $\bar{H} = 0$

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Main lemma: All the quantities λ_β, $R_\beta(x)/R_\beta(y)$, $L_\beta(x)/L_\beta(y)$ are equivalent to an expression of the form

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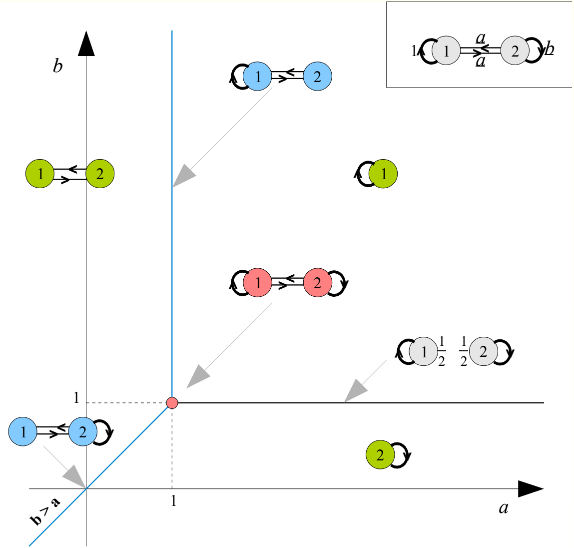
We prove actually with E. Garibaldi that all the quantities $A_\beta = \lambda_\beta$, $L_\beta(x)$, $R_\beta(x)$ admit a Puiseux series expansion. Let $\epsilon = e^{-\beta}$, then

$$A_\epsilon = A_0 \epsilon^{a_0} + A_1 \epsilon^{a_1} + \ldots + A_n \epsilon^{a_n} + A_{n+1} \epsilon^{a_n+1} + \ldots$$
$$a_0 < a_1 < \ldots < a_n, \quad A_i \neq 0$$

Case of 2 symbols:

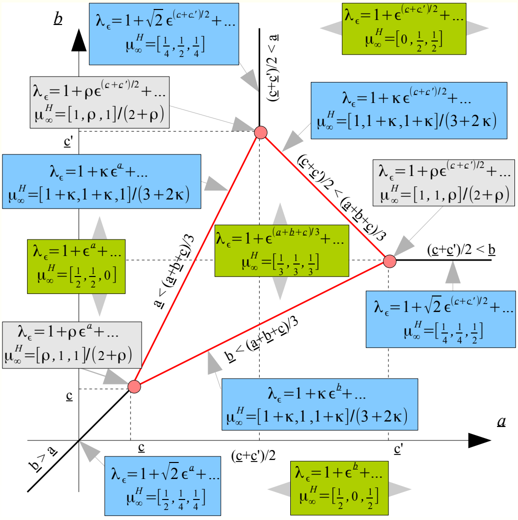


zero-temperature phase diagram for 2×2 matrix

$$M_\epsilon = \begin{bmatrix} \epsilon & \epsilon^a \\ \epsilon^a & \epsilon^b \end{bmatrix}$$

- Each phase is a convex polygon
- On $2D$-phase μ_∞ is a periodic orbit
- μ_∞ may have positive entropy
- μ_∞ may be a barycenter of two periodic orbits

Case of 3 symbols:



zero-temperature phase diagram for 3×3 matrix

$$M_\epsilon = \begin{bmatrix} 1 & \epsilon^{a} & \epsilon^{b} \\ \epsilon^{a'} & 1 & \epsilon^{c} \\ \epsilon^{b'} & \epsilon^{c'} & 1 \end{bmatrix}$$

$$a, b, c, a', b,' c,' > 0$$

- For each phase μ_∞ is a barycenter of periodic orbits
- The coefficients of the barycenter may not be rational
- $\rho^2 - \rho - 1 = 0$
- $\kappa^3 - \kappa - 1 = 0$

– The Hölder case: known facts –

A counter example

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A counter example

Theorem[Chazottes-Hochman 2010] There exists a compact invariant set $\Omega \subset \Sigma_{\{0,1\}}$ such that, for the specific interaction energy $H(x) = d(x, \Omega)$ (which is Hölder), μ_β admits at least 2 accumulation points, as $\beta \to +\infty$

Easy facts: We consider here (Σ_G, σ) an irreducible SFT anf $H : \Sigma_G \to \mathbb{R}$ a Hölder continuous function. We recall the μ_β is the unique σ-inv. measure minimizing

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Question: Is it possible to characterize the set Ω_{GS} containing the support of all minimizing measures?

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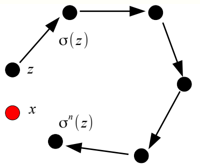
$$\Omega_{GS} := \Big\{ x \in \Sigma_G : \forall\, \epsilon > 0,\ \exists\, n \geq 1,\ \exists\, z \in \Sigma_G \text{ s. t.}$$
$$d(x,z) < \epsilon,\ d(x,\sigma^n(z)) < \epsilon \text{ and } \Big| \sum_{k=0}^{n-1} [H \circ \sigma^k(z) - \bar{H}] \Big| < \epsilon \Big\}.$$



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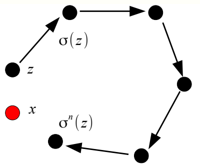

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Question: Why do we call Ω_{GS}, the set of ground-state configurations?

Definition: An effective potential (sub-action), $V : \Sigma_G \to \mathbb{R}$

$$H(x) - V \circ \sigma(x) + V(x) - \bar{H} \geq 0, \quad \forall\, x \in \Sigma_G$$

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Proposition: [Mañé-Conze-Guivarc'h lemma] If H is Hölder, an effective potential does exist. The existence of a stronger version, called calibrated potential, may be proved

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Theorem:[Morris 2009] Extension to weakly expanding map $f : S^1 \to S^1$ of the form $f(x) = x + x^{1+\alpha} + \ldots$, for $\alpha \in]0,1[$. For H γ-Hölder, with $\alpha < \gamma$, there exists a calibrated potential V, $(\gamma - \alpha)$-Hölder. For some α-Hölder H, no continuous effective potential exists

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Corollary: If Ω_{GS} has a unique measure μ_{min} of maximal entropy, then $\mu_\beta \to \mu_{min}$ exists

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Proposition[Baraviera-Lopes-Thieullen 2006] If μ_{min} is unique, then μ_β satisfies a large deviation principle

$$\frac{1}{\beta} \ln \mu_\beta(C) \to -\inf_C I$$

- C is any cylinder
- $I(x) = \sum_{k\geq 0}[H - V \circ \sigma + V - \bar{H}] \circ \sigma^k(x)$ is l.s.c.
- V is any calibrated effective potential

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How to characterize V_∞? We have seen that any such a V_∞ is calibrated:

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Definition: [Mather-Peierls barrier] Let $x, y \in \Sigma_G$

$$h(x,y) := \lim_{\epsilon \to 0} \liminf_{n \to +\infty} S_n^\epsilon(x,y),$$

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Question Can we find a minimizer p_0 independent of x? Is there a unique calibrated V up to the value $V(p_0)$ for some fixed $p_0 \in \Omega_{GS}$?

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Proposition If Ω_{GS} is irreducible and V is calibrated, then V is unique in a projective sense

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New result[Garibaldi-Thieullen] If $\Omega_{GS} = \Omega_0 \cup \Omega_1 \cup \ldots \cup \Omega_r$ is a finite disjoint union of irreducible components so that Ω_0 has the largest topological entropy and all other Ω_i has a lower topological entropy, then for any fixed $p \in \Omega_0$

$$V_\beta(x) - V_\beta(p) \to h(p,x) \quad \text{uniformly in } x$$