Uniform domination for nonautonomous linear difference equations in infinite dimension

Philippe Thieullen (Bordeaux)

joint work with A. Quas (Victoria) and M. Zarrabi (Bordeaux)

CMC conference: Nonlinear dynamics of many-body systems and related topics

August 21-24, 2017, Seoul National University

Outline

I. Different notions of hyperbolicity

- Hyperbolicity in the sense of Sacker-Sell
- Hyperbolicity in the sense of domination
- Hyperbolidity in the sense of singular value

Abstract framework

- We consider a nonautonomous linear differential equations

$$\dot{v} = L(t)v, \quad \forall t \in \mathbb{R}$$

where $v(t) \in X$, Banach space, $L(t) \in \mathcal{B}(X)$ bounded linear operator continuous in $t \in \mathbb{R}$ for the norm topology

- the fundamental solution $A(s,t) \in \mathcal{B}(X)$ solves

$$\begin{cases} \frac{\partial}{\partial t} A(s,t) = L(s+t)A(s,t), \quad \forall t \geq 0 \\ A(s,0) = \mathrm{Id} \end{cases}$$

- $A(s,t)$ is written as a cocycle

$$A(s,t+t') = A(s+t,t')A(s,t), \quad \forall s \in \mathbb{R}, \ \forall t,t' \geq 0$$

Question

- How can we extend Floquet theory for non periodic $L(t)$?
- Is it possible to define a notion of spectrum?

Hyperbolicity in the sense of Sacker-Sell

Assumption the resolvant is assumed to be invertible. The cocycle property is extended for all time by

$$A(s,-t) := A(s-t,t)^{-1}, \quad \forall t \geq 0$$

First definition We say that $\lambda \in \mathbb{R}$ belongs to the Sacker-Sell resolvant if there exist an equivariant family of projectors $(P_s)_{s\in\mathbb{R}}$ and constants $K \geq 1$ and $\epsilon > 0$ such that for every $t \geq 0$

- $A(s,t)P_s = P_{s+t}A(s,t)$
- $\|A(s,t)P_s\| \leq Ke^{(\lambda-\epsilon)t}$
- $\|A(s,-t)(\mathrm{Id} - P_s)\| \leq Ke^{-(\lambda+\epsilon)t}$

Second definition Let be

$$F_s := \mathrm{Im}(P(s)), \quad E_s := \ker(P(s))$$

We say that $\lambda \in \mathbb{R}$ belongs to the Sacker-Sell resolvant if there exist a uniform equivariant splitting $X = E_s \oplus F_s$, $\forall s \in \mathbb{R}$, and constants $K \geq 1$, $\epsilon > 0$ such that for every $t \geq 0$

- $A(s,t)E_s = E_{s+t}, \quad A(s,t)F_s \subset F_{s+t}$
- $\angle(E_s, F_s) \geq K^{-1}$
- $\forall v \in F_s, \quad \|A(s,t)v\| \leq K e^{(\lambda-\epsilon)t}\|v\|$
- $\forall v \in E_s, \quad \|A(s,t)v\| \geq K^{-1} e^{(\lambda+\epsilon)t}\|v\|$

Remark

- We don't assume any more $A(s,t)$ is invertible
- E_s is called the fast space, $A(s,t) : E_s \to E_{s+t}$ is invertible
- F_s is called the slow space, $\ker(A(s,t)) \subset F_s$

Hyperbolicity in the sense of domination

It is a weaker notion

Remark 1 Hyperbolicity in the sense of Sacker-Sell implies four results

- The existence of an equivariant splitting $X = E_s \oplus F_s$
- The splitting is uniform $\angle(E_s, F_s) \geq K^{-1}$
- E_s dominates F_s (solutions grow faster in E_s than in F_s)
- The existence an exponent λ of dichotomy for the growth of vectors

In the domination case, we just keep the first three properties

Remark 2 Cone equivariance implies readily domination and is easier to prove than splitting equivariance

- Assume there exists a (non equivariant) splitting $X = \tilde{E}_s \oplus \tilde{F}_s$
- Assume $(\mathrm{Id} - \tilde{P}_{s+t})A(s,t) : \tilde{E}_s \to \tilde{E}_{s+t}$ is invertible $\forall t \geq 0$
- Define the fast cone $\mathcal{C}_s(a) := \{u + v \in \tilde{E}_s \oplus \tilde{F}_s : \|v\| \leq a\|u\|\}$
- Assume there exists $T > 0$ s.t. $A(s,T)\mathcal{C}_s(1) \subset \mathcal{C}_{s+T}(\frac{1}{2})$
- Define $E_s := \{v \in X : \forall n \geq 0,\ A(s,nT)v \in \mathcal{C}_{s+nT}(a)\}$
- Define $F_s := \{v \in X : \forall n \geq 0,\ A(s,nT)v \notin \mathcal{C}_{s+nT}(a)\}$

Then

- $X = E_s \oplus F_s$ is equivariant (E_s and F_s are closed vector spaces)
- $\angle(E_s, F_s) \geq K^{-1}$
- E_s dominates F_s in the sense

$$\frac{\sup\{\|A(s,T)v\| : v \in F_s\}}{\inf\{\|A(s,T)v\| : v \in E_s\}} = \frac{\|A(s,T)|F_s\|}{\|(A(s,T)|E_s)^{-1}\|^{-1}} \leq \frac{1}{2}$$

Definition We say that the cocycle $A(s,t)$ is uniformly dominated if there exist a uniform equivariant splitting $X = E_s \oplus F_s$ and constants $K \geq 1$ and $T > 0$ such that

- $A(s,t)E_s = E_{s+t}, \quad A(s,t)F_s \subset F_{s+t}$
- $\angle(E_s, F_s) \geq K^{-1}$
- $\dfrac{\|A(s,T)|F_s\|}{\|(A(s,T)|E_s)^{-1}\|^{-1}} \leq \dfrac{1}{2}$

Remark

There is no reason to obtain an exponent λ of dichotomy, we just obtain a gap in the spectrum

$$\left[\liminf_{t\to+\infty} \frac{1}{t} \log \|(A(s,t)|E_s)^{-1}\|^{-1}\right] - \left[\limsup_{t\to+\infty} \frac{1}{t} \log \|A(s,t)|F_s\|\right] \geq \frac{\log 2}{T}$$

Hyperbolicity in the sense of singular value

It is weaker than uniform domination

Remark In the two previous notions of hyperbolicity, the existence of a uniform equivariant splitting is required

- $X = E_s \oplus F_s$
- $A(s,t)E_s = E_{s+t}, \;\; A(s,t)F_s \subset F_{s+t}$
- $\angle(E_s, F_s) \geq K^{-1}$

This is a very strong assumption! If $A(s,t) = e^{tB}$ then $E_s = E$ and $F_s = F$ are independent of s and correspond to eigenspaces.

Main goal Replace the domination property

$$\frac{\|A(s,T)|F_s\|}{\|(A(s,T)|E_s)^{-1}\|^{-1}} \leq \frac{1}{2}$$

(which requires the existence of a splitting) by another notion of spectral gap, using for instance, the singular values

Singular values Let A be a bounded operator on an Hilbert space X. We call singular values at index $r \geq 1$

$$\sigma_r(A) := \sup_{\dim(E)=r} \inf\{\|Av\| : v \in E,\ \|v\| = 1\}$$
$$= \inf_{\operatorname{codim}(F)=r-1} \sup\{\|Av\| : v \in F,\ \|v\| = 1\}$$

Remarks

- $\sigma_1(A) = \|A\|$,
- If A is compact, $\sigma_1(A) \geq \sigma_2(A) \geq \cdots$ are the eigenvalues of $\sqrt{A^*A}$
- If X is a Banach space, we choose the first definition
- If $X = E \oplus F$ and $\dim(E) = r$ then

$$\frac{\sigma_{r+1}(A)}{\sigma_r(A)} \leq \frac{\|A|F\|}{\|(A|E)^{-1}\|^{-1}}$$

Notation

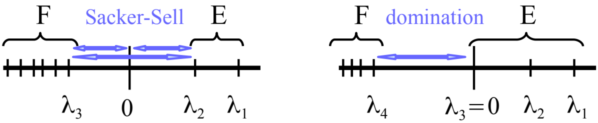
$$\sigma_r(s,t) = \text{the singular value of } A(s,t) \text{ at index } r$$

Definition We say that the cocycle $A(s,t)$ admits a gap at index $r \geq 1$ in the singular-value spectrum, if there exist constants $D \geq 1$ and $\tau > 0$ such that for every $s \in \mathbb{R}$

$$\frac{\sigma_{r+1}(s,t)}{\sigma_r(s,t)} \leq De^{-\tau t}, \quad \forall t \geq 0$$

Conclusion

- Sacker-Sell hyperbolicity $\Longrightarrow$ domination



- Domination $\Longrightarrow$ gap in the singular value spectrum
- But there is no reason that

  gap in the singular value spectrum $\stackrel{?}{\Longrightarrow}$ existence of a splitting

II. Main results

Problem We consider a simplified problem where time is discrete. Consider a nonautonomous (or switched) linear difference equation

$$v_{k+1} = A_k v_k, \quad v_k \in X, \quad \forall k \in \mathbb{Z}$$

where $A_k : X \to X$ is a bounded linear operator on a Banach space X. Define the cocycle

$$A(k,n) := A_{k+n-1} \cdots A_{k+1} A_k$$

Does the gap in the singular-value spectrum imply the existence of a uniform dominated equivariant splitting?:

- $X = E_k \oplus F_k$ (splitting)
- $\dim(E_k) = r$, $A_k|E_k$ is injective (invertibility in the fast direction)
- $A_k E_k = E_{k+1}, \quad A_k F_k \subset F_{k+1}$ (equivariance)
- $\angle(E_k, F_k) \geq K^{-1}$ (uniform "minimal angle")
- $\dfrac{\|A(k,n)|F_k\|}{\|(A(k,n)|E_k)^{-1}\|^{-1}} \leq K e^{-n\tau}, \ \forall n \geq 0$ (domination property)

Bochi-Gourmelon result (2009) $X = \mathbb{R}^d$. Let be $r \geq 1$, $\tau > 0$, $D \geq 1$. Assume

- $(\mathrm{SVG})_{\mathrm{weak}} \quad \dfrac{\sigma_{r+1}(k,n)}{\sigma_r(k,n)} \leq De^{-n\tau}, \quad \forall k \in \mathbb{Z},\ \forall n \geq 0$
- The closure of $\{A_k : k \in \mathbb{Z}\}$ is a compact set of $\mathrm{GL}(d,\mathbb{R})$

Then there exits a uniform dominated equivariant splitting

- $\mathbb{R}^d = E_k \oplus F_k,\ \dim(E_k) = r,$
- $A_k E_k = E_{k+1},\ A_k F_k = F_{k+1}$
- $\angle(E_k, F_k) \geq K^{-1}$
- $\dfrac{\|A(k,n)|F_k\|}{\|(A(k,n)|E_k)^{-1}\|^{-1}} \leq Ke^{-n\tau}$

Remarks on Bochi-Gourmelon

- The proof is done using ergodic theory, by introducing a topological dynamical system (M, T) and a continuous map $A : M \to \mathrm{GL}(d, \mathbb{R})$
- A new cocycle is introduced

$$A(x, n) := A(T^{n-1}(x)) \cdots A(T(x))A(x)$$

- Invertiblity of $A(x)$ is a fundamental assumption of the proof
- Compactness of M, continuity of T, are fundamental assumptions
- $\Rightarrow$ existence of a continuous dominated equivariant splitting
  - $\mathbb{R}^d = E(x) \oplus F(x)$
  - $A(x)E(x) = E(T(x)),\ \ A(x)F(x) = F(T(x))$
- The main difficult part is to prove $E(x) \cap F(x) = \{0\}$
- The proof uses strongly Oseledets theorem for each ergodic measure and some techniques of ergodic optimization

Can we avoid the use of ergodic theory?

Blumenthal-Morris result (preprint) X is a Banach space. Let be $r \geq 1$, $\tau > 0$, $D \geq 1$. Assume

- $(\mathrm{SVG})_{\mathrm{strong}} \begin{cases} \dfrac{\sigma_{r+1}(k,n)\|A_{k+n}\|}{\sigma_r(k,n+1)} \leq De^{-n\tau} \\ \\ \dfrac{\|A_{k-n-1}\|\sigma_{r+1}(k-n,n)}{\sigma_r(k-n-1,n+1)} \leq De^{-n\tau} \end{cases}$
- The closure for the norm topology of $\{A_k : k \in \mathbb{Z}\}$ is a compact set made of injective operators

Then there exists a uniform dominated equivariant splitting:

- $X = E_k \oplus F_k, \quad \dim(E_k) = r$
- $A_k E_k = E_{k+1}, \quad A_k F_k \subset F_{k+1}$
- domination

Remark

- The crucial part is to show $E_k \cap F_k = \{0\}$
- injectivity and compactness are again fundamental assumptions for the proof

Questions In both theorems,

- Can we get rid off the invertibility assumption? Can we prove Bochi-Gourmelon result for endomorphisms in finite dimension?
- Can we get rid off the compactness of $(A_k)_{k\in\mathbb{Z}}$? Compactness for the norm topology is certainly a too strong condition, can we only assume compactness for the SOT?
- Can we avoid the use of ergodic theory and Oseledets theorem?

The central technical problem

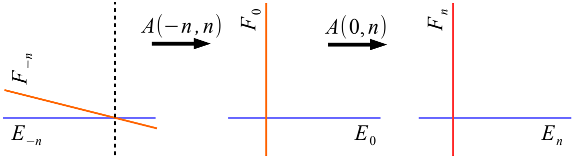
Can we obtain an effective estimate of a bound from below of the angle between the fast and slow spaces obtained solely from the constants which characterize the sequence $(A_k)_{k\in\mathbb{Z}}$?

An easy counter example

- define a sequence $(A_k)_{k\in\mathbb{Z}} \in \mathrm{GL}(2,\mathbb{R})$,

$$A_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\tau} \end{bmatrix}, \ \forall k \geq 0, \qquad A_k = \begin{bmatrix} \frac{1}{|k|} & 1 \\ 0 & \frac{e^{-\tau}}{|k|} \end{bmatrix}, \ \forall k < 0$$

- $\overline{\{A_k : k \in \mathbb{Z}\}}$ is compact in $\mathrm{End}(2,\mathbb{R})$,
- $(\mathrm{SVG})_{\mathrm{strong}}$ is satisfied
- but there is no uniform equivariant splitting: $\mathbb{R}^2 = E_k \oplus F_k$ with minimal angle uniformly bounded from below



$\Longrightarrow \begin{cases} \{A_k : k \in \mathbb{Z}\} \text{ is not compact in } \mathrm{GL}(2,\mathbb{R} \\ \text{A notion of partial invertibility is needed} \end{cases}$

Notations

- X is an Banach space
- $(A_k)_{k\in\mathbb{Z}}$ is a two-sided sequence of bounded operators $A_k \in \mathcal{B}(X)$, which may have a kernel
- we call abstract cocycle: $A(k,n) := A_{k+n-1}\cdots A_{k+1}A_k$

$$A(k,m+n) = A(k+m,n)A(k,m)$$

- We call singular values $\sigma_1(k,n) \geq \sigma_2(k,n) \geq \cdots$

$$\sigma_r(k,n) := \sup_{\dim(E)=r} \inf\{\|A(k,n)u\| : u \in E,\ \|u\| = 1\}$$

- We call minimal angle of a splitting $X = E \oplus F$

$$\gamma(E,F) := \inf\{\mathrm{dist}(u,F) : u \in E, \|u\| = 1\}$$

$\gamma(E,F) = 1 \Leftrightarrow E \perp F$ (Hilbert case),

$\gamma(E,F)$ is called minimal gap (Kato, Gohberg-Krein, ...)

Assumptions Let be $r \geq 1,\ D \geq 1,\ \tau > 0,\ \mu > 0$. Assume $\forall k \in \mathbb{Z}$

- $(\mathrm{SVG})_{\mathrm{strong}} \quad \forall n \geq 1, \quad \begin{cases} \dfrac{\|A_k\|\sigma_{r+1}(k+1,n)}{\sigma_r(k,n+1)} \leq De^{-n\tau} \\ \dfrac{\sigma_{r+1}(k,n)\|A_{k+n}\|}{\sigma_r(k,n+1)} \leq De^{-n\tau} \end{cases}$
- $(\mathrm{FI}) \quad \forall n \geq 0, \quad \displaystyle\prod_{i=1}^{r} \frac{\sigma_i(k,n+1)}{\sigma_i(k,1)\sigma_i(k+1,n)} \geq e^{-\mu}$

Remark 1

- The (FI) condition is necessary and sufficient to obtain a uniform dominated equivariant splitting
- For $r = 1$ the (FI) means

$$\forall k \in \mathbb{Z},\ \forall n \geq 0, \quad \frac{\|A(k,n+1)\|}{\|A_k\|\|A(k+1,n)\|} \geq e^{-\mu}$$

  the norm of $A(k,n)$ is almost multiplicative
- 4 constants characterize the cocycle: (r, D, τ, μ)

Remark 2

- For uniformly invertible cocycles, the (FI) condition is automatically satisfied. If

$$M^* := \sup_{k\in\mathbb{Z}} \|A_k\|,\ M_* := \inf_{k\in\mathbb{Z}} \|A_k^{-1}\|^{-1}$$

  then

$$\begin{cases} (\mathrm{SVG})_{\mathrm{strong}} \Leftrightarrow (\mathrm{SVG})_{\mathrm{weak}} \\ (\mathrm{FI}) \text{ is always true with } \mu = r\log\left(\frac{M^*}{M_*}\right) \end{cases}$$

- The (FI) is equivalent to a (stronger) form

$$\forall k\in\mathbb{Z},\ \forall m,n\geq 0,\quad \prod_{i=1}^{r}\frac{\sigma_i(k,n+m)}{\sigma_i(k,m)\sigma_i(k+m,n)} \geq K^{-1}e^{-m\mu}$$

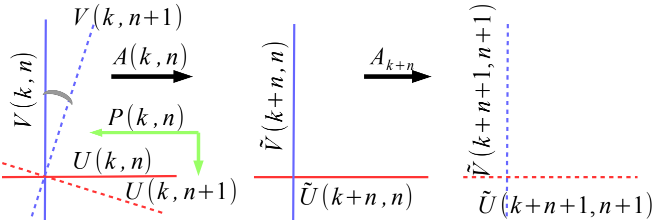
  for some constant $K\geq 1$

Theorem (QTZ) Assume X is Hilbert or Banach and $(A_k)_{k\in\mathbb{Z}}$ satisfies $(\mathrm{SVG})_{\mathrm{strong}}$ and (FI) for the constants (r, D, τ, μ). Then then cocyle admits a uniform dominated equivariant splitting

- $X = E_k \oplus F_k,\ \dim(E_k) = r$
- $A_k E_k = E_{k+1},\ A_k F_k \subset F_{k+1}$
- $\gamma(E_k, F_k) \geq \dfrac{1}{5K_r}\Big[\dfrac{1}{2K_r(3r+7)^2}\Big(\dfrac{1-e^{-\tau}}{De^{\tau}}\Big)\Big]^{\mu(\mu+4\tau)/(2\tau^2)}$
- $\forall n \geq 1, \quad \dfrac{\|A(k,n)|F_k\|}{\|(A(k,n)|E_k)^{-1}\|^{-1}} \leq \dfrac{5K_d}{\inf_k \gamma(E_k,F_k)} \dfrac{\sigma_{d+1}(k,n)}{\sigma_d(k,n)}$
- $K_r(X) = 1$ in the Hilbert case
- $(K_r(l^p(\mathbb{R})) = r^{|\frac{1}{2}-\frac{1}{p}|} \to 1$ as $p \to 2)$

III. Some elements of proof

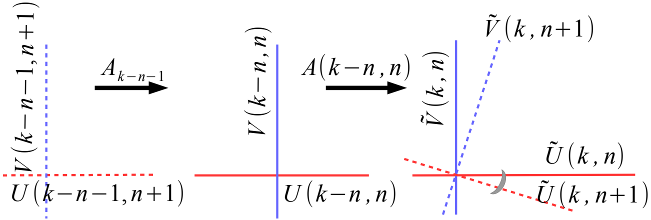
III.a. Sequences of 2×2 matrices



Raghunathan estimates for the slow space If $P(k,n)$ is the orthogonal projector onto $V(k,n)$

$$\|P(k,n) - P(k,n+1)\| \leq \frac{\sigma_2(k,n)\|A_{k+n}\|}{\sigma_1(k,n+1)}$$

Corollary $V(k,n) \to F_k$ exponentially fast



Raghunathan estimates for the fast space If $Q(k,n)$ is the orthogonal projector onto $\tilde U(k,n)$

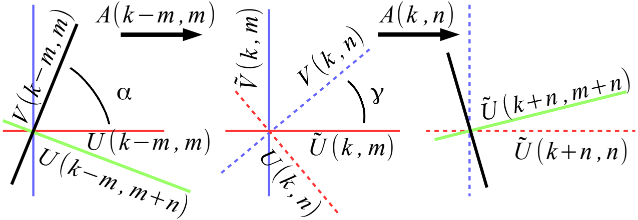
$$\|Q(k,n) - Q(k,n+1)\| \le \frac{\|A_{k-n-1}\|\sigma_2(k-n,n)}{\sigma_1(k-n-1,n+1)}$$

Corollary $\tilde U(k,n) \to E_k$ exponentially fast

Remark So far only $(\mathrm{SVG})_{\mathrm{strong}}$ has been used. There is no reason that $E_k \cap F_k = \{0\}$. Previous proofs use ergodic theory to conclude that E_k and F_k are complemented.

The role of (FI)

(FI) $$\Sigma_{m,n} := \frac{\sigma_1(k-m,m+n)}{\sigma_1(k-m)\sigma_1(k,n)}, \qquad \inf_{n\geq 1} \Sigma_{m,n} \geq e^{-m\mu}, \quad \forall m \geq 1$$



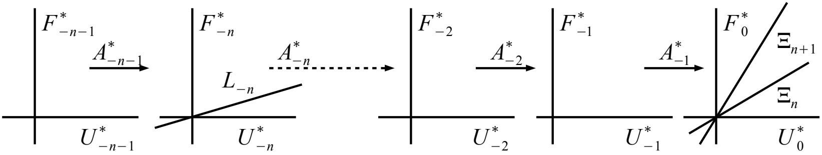
$$\Sigma_{m,n} \geq \gamma(\tilde{U}(k,m), V(k,n)) \geq \left[\left(\alpha_{m,n}^2 \Sigma_{m,n}^2 - \frac{\sigma_2(k,n)^2}{\sigma_1(k,n)^2}\right)^+\right]^{1/2}$$

$$\alpha_{m,n} := \gamma(U(k-m,m), V(k-m,m+n))$$

Corollary

$$\inf_{n\geq 1} \Sigma_{m,n} \geq \gamma(\tilde{U}(k,m), F_k) \geq \gamma(U(k-m,m), F_{k-m}) \inf_{n\geq 1} \Sigma_{m,n}$$

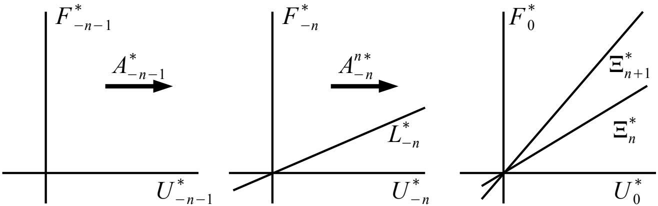
A reduced problem



- Choose N_* large enough, set k
- $A^*_{-n} := A(k - nN_*, N_*)$
- $F^*_{-n} := F_{k-nN_*}$ (the equivariant slow space)
- $U^*_{-n} := U(k - nN_*, nN_*)$ (the approximated fast space)
- $A^{n*}_{-n} := A^*_{-1}A^*_{-2} \cdots A^*_{-n}$
- $A^{n*}_{-n}U^*_{-n} = \mathrm{Graph}(\Xi_n)$ for some $\Xi_n : U^*_0 \to F^*_0$
- $A^*_{-n-1}U^*_{-n-1} = \mathrm{Graph}(L_{-n})$ for some $L_{-n} : U^*_{-n} \to F^*_{-n}$

Bootstrapping argument (FI) $\Longrightarrow L_{-n}$ is uniformly bounded

Bound from below of the angle



- $A^{n*}_{-n} = \begin{bmatrix} a^n_{\ n} & 0 \\ c^n_{-n} & d^n_{-n} \end{bmatrix}$
- $A^*_{-n-1} U^*_{-n-1} = \mathrm{Graph}(L^*_{-n}), \qquad L^*_{-n} = c_{-n-1}(a_{-n-1})^{-1}$
- $\Xi^*_{n+1} = \Xi^*_n + d^n_{-n} L^*_{-n} (a^n_{-n})^{-1}$
- $\|d_{-n}\| \le \delta_{-n}\ \sigma_1(k - nN^*, nN_*), \quad \delta_{-n} \to 0$ exponentially (SVG)
- $\|(a^n_{-n})^{-1}\|^{-1} \ge \sigma_1(k - nN_*, nN_*)/\|\mathrm{Id} \oplus \Xi^*_n\|$

$$\|\mathrm{Id} \oplus \Xi^*_{n+1}\| \le \|\mathrm{Id} \oplus \Xi^*_n\| \big[1 + \delta_{-n} \|L^*_{-n}\|\big]$$

$$\gamma(E^*_0, F^*_0) \ge \frac{1}{4}\Big(\frac{1}{10}\frac{1 - e^{-\tau}}{D_* e^{\tau}}\Big)^{\mu(\mu+4\tau)/(2\tau^2)}$$

III.b. Extension to Banach spaces

General strategy

- Extend the case of 2×2 matrices to the codimension 1 setting

$$(\mathrm{SVG})_{\mathrm{weak}} \quad \frac{\sigma_2(k,n)}{\sigma_1(k,n)} \leq De^{-n\tau}, \quad (\mathrm{FI}) \quad \frac{\sigma_1(k,n+1)}{\sigma_1(k,1)\sigma_1(k+1,n)} \geq e^{-\mu}$$

- In the general case, use the exterior product $\wedge^r X$ and notice that

$$\sigma_1(\wedge^r A) = \prod_{i=1}^{r} \sigma_i(A), \quad \sigma_2(\wedge^r A) = \Big[\prod_{i=1}^{r-1} \sigma_i(A)\Big]\sigma_{r+1}(A)$$

- Use $(\mathrm{SVG})_{\mathrm{strong}}$ instead of $(\mathrm{SVG})_{\mathrm{weak}}$ and the new

$$(\mathrm{FI}) \quad \frac{\sigma_1(\wedge^r A(k,n+1))}{\sigma_1(\wedge^r A_k)\sigma_1(\wedge^r A(k,n))} = \prod_{i=1}^{r} \frac{\sigma_i(k,n+1)}{\sigma_i(k,1)\sigma_i(k+1,n)} \geq e^{-\mu}$$

How far from an Hlibert space is a Banach space?

- $C \geq 1$, a basis of vectors $(e_1, \dots, e_r)$ is C-Auerbach if

$$\|e_i\| \leq C, \quad \mathrm{dist}(e_i, \mathrm{span}(e_j : j \neq i)) \geq C^{-1}$$

  (1-Auerbach basis exists)

- volumic distortion

$$\Delta_r(X) = \sup\left\{ \frac{\| \sum_{i=1}^r \lambda_i e_i \|}{\left[\sum_{i=1}^r |\lambda_i|^2\right]^{1/2}} : (\lambda_i) \neq 0,\ (e_i) \text{ 1-Auerbach} \right\}$$

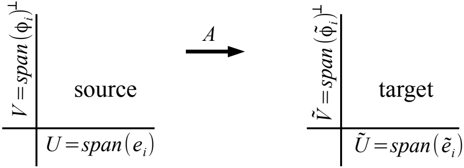
- example $X = l^p(\mathbb{Z}, \mathbb{R})$, $\quad \Delta_r(X) = r^{|\frac{1}{p} - \frac{1}{2}|}$,
  (Hilbert norm $\Delta_d(X) = 1$, $\quad$ sup-norm $\Delta_r(X) = \sqrt{r}$: worst case)
- the constant K_r in the main result is a polynomial function of $\Delta_r(X), \Delta_r(X^*), \Delta_r(X^{**})$

Approximated singular value decomposition X is Hilbert or Banach, $A \in \mathcal{B}(X)$, $\epsilon > 0$, $r \geq 1$. Assume $\sigma_r(A) > 0$. Then

- $X = U \oplus V = \tilde{U} \oplus \tilde{V}$, $\dim(U) = \dim(\tilde{U}) = d$
- $AU = \tilde{U}$, $AV \subset \tilde{V}$, $A^*\tilde{V}^\perp = V^\perp$, $A^*\tilde{U}^\perp \subset U^\perp$
- there exist $(1+\epsilon)K_r$-Auerbases $(e_1, \ldots, e_r)$, $(\tilde{e}_1, \ldots, \tilde{e}_r)$ and dual $(1+\epsilon)K_r$-Auerbases $(\phi_1, \ldots, \phi_r)$, $(\tilde{\phi}_1, \ldots, \tilde{\phi}_r)$, $\langle \phi_i | e_j \rangle = \delta_{i,j}$

$$U = \mathrm{span}(e_i), \ \tilde{U} = \mathrm{span}(\tilde{e}_i), \ V = \mathrm{span}(\phi_i)^\perp, \ \tilde{V} = \mathrm{span}(\tilde{\phi}_i)^\perp$$

- $Ae_i = \sigma_i(A)\tilde{e}_i$, $A^*\tilde{\phi}_i = \sigma_i(A)\phi_i$
- $K_r^{-1}(1+\epsilon)^{-1}\sigma_i(A) \leq \sigma_i(A|U) \leq \sigma_i(A)$ (idem for $A^*|\tilde{V}^\perp$)
- $K_r^{-1}(1+\epsilon)^{-1}\sigma_{r+1}(A) \leq \|A|V\| \leq \sigma_{r+1}(A)$ (idem for $A^*|\tilde{U}^\perp$)
- $\gamma(U,V) \geq K_r^{-1}(1+\epsilon)^{-1}$, $\gamma(\tilde{U},\tilde{V}) \geq K_r^{-1}(1+\epsilon)^{-1}$



$$K_r := \bar{\Delta}_r(X)^{6r^2+15r+4}\bar{\Delta}_2(X)^{3r^2+4r+4}$$

IV. Conclusion

- I. Different notions of hyperbolicity
- II. Main results
- III. Some elements of proof
- IV. Conclusion

Summary

- We have introduced a weak form of hyperbolicity
- It still implies the existence of a uniform equivariant splitting
- The stability of the splitting is controlled in an effective way. The bound from below of the angle is explicitly given by 4 constants

What is missing?

- Concrete examples which are hyperbolic in the sense of domination but not hyperbolic in the sense of Sacker-Sell. Such examples could be founded as a perturbation of an hyperbolic system with a neutral direction of dimension 1 coming from the vector field
- A "truly effective criteria" that is a criteria checkable in finite time. Both $(\mathrm{SVG})_{\mathrm{strong}}$ and (FI) are obtained as a limit as $n \to +\infty$
- The possibility to apply this theory for some dissipative systems:
  - for reaction-diffusion systems which admit a compact attractor,
  - for transfer operators which are positive operators and are used to find the density of stationary measures