Zero-temperature Gibbs measures for some subshifts of finite type

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Warwick, 11-15 July 2011

Outline

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- μ_β : σ-invariant probability on Σ_G

$$\mu_\beta[C_n(x)] \asymp \exp\Big(-\beta\big[\sum_{k=0}^{n-1} H\circ\sigma^k(x) - n\bar{H}_\beta\big]\Big)$$

$$\forall\, x \in \Sigma_G^+, \quad \forall\, n \geq 1.$$

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- If not, how to characterize the set of accumulation points?

The locally finite case

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- μ_∞^i has maximal topological entropy on this SFT

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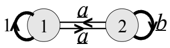
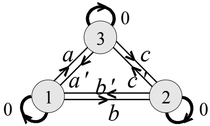


- Two symbols: $H(1,1) = 1$, $H(1,2) = a$
- The minimizing possible cycles gives $\bar{H} \in \{1, a, b\}$



- Three symbols: assume $a, a', b, b', c, c' > 0$
- The minimizing possible cycles gives $\bar{H} \in \{0, \frac{1}{2}(a+a'), \ldots, \frac{1}{3}(a+b+c), \ldots\}$
- With the above assumption, $\bar{H} = 0$

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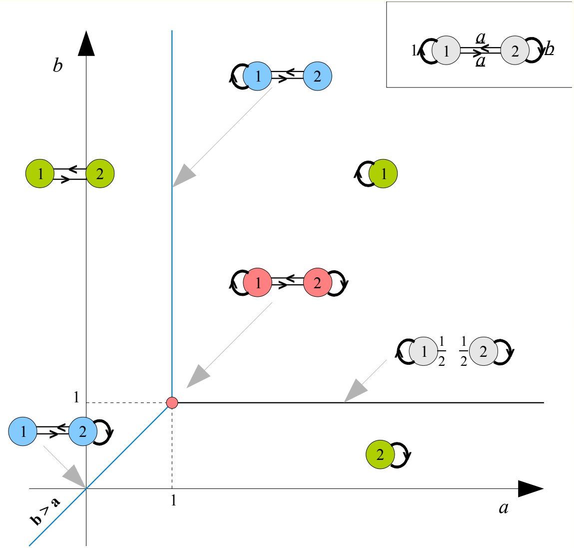
Solution A possible proof is to show that all quantities λ_β, $L_\beta(i)$, ... admit a Puiseux series expansion. Let $\epsilon = e^{-\beta}$

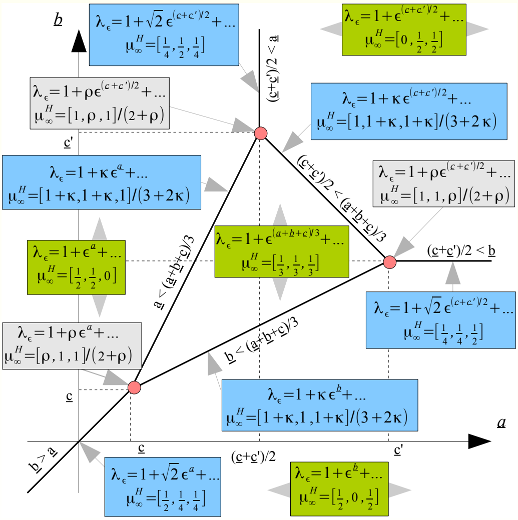
$$\lambda_\epsilon = \lambda_0 \epsilon^{a_0} + \lambda_1 \epsilon^{a_1} + \ldots$$
$$a_0 < a_1 < \ldots < a_n < a_n + 1 < \ldots$$



zero-temperature phase diagram for 2×2 matrix

$$M_\epsilon = \begin{bmatrix} \epsilon & \epsilon^a \\ \epsilon^a & \epsilon^b \end{bmatrix}$$

- Each phase is a convex polygon
- On $2D$-phase μ_∞ is a periodic orbit
- μ_∞ may have positive entropy
- μ_∞ may be a barycenter of two periodic orbits



zero-temperature phase diagram for 3×3 matrix

$$M_\epsilon = \begin{bmatrix} 1 & \epsilon^{a} & \epsilon^{b} \\ \epsilon^{a'} & 1 & \epsilon^{c} \\ \epsilon^{b'} & \epsilon^{c'} & 1 \end{bmatrix}$$

$$a, b, c, a', b,' c,' > 0$$

- For each phase μ_∞ is a barycenter of periodic orbits
- The coefficients of the barycenter may not be rational

Extensions:

Theorem[T. Kempton 2007] The limite does exist and has maximal topological entropy in the case of a countable Markov chain with BIG propery and a uniformly locally finite intercation energy H with finite pressure

The Hölder case: known facts

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Theorem[Chazottes-Hochman 2010] There exists a compact invariant set $\Omega \subset \Sigma_{\{0,1\}}$ such that, for the specific interaction energy $H(x) = d(x, \Omega)$ (which is Hölder), μ_β admits at least 2 accumulation points, as $\beta \to +\infty$

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Definition μ_{min} is minimizing if

$$\int H\, d\mu_{min} = \min\Big\{ \int H\, d\mu \,:\, \mu \,:\, \sigma\text{-invariant}\Big\}$$

The minimizing ergodic value is

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Question How to characterize minimizing measures?

Proposition The support of a minimizing measure belongs to the set of ground-state configurations Ω_{GS}

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Definition The ground-state configuration is

$$\Omega_{GS} := \Big\{ x \in \Sigma_G : \forall\, \epsilon > 0,\ \exists\, n \geq 1,\ \exists\, z \in \Sigma_G \ \text{ s. t.}$$

$$d(x,z) < \epsilon,\ d(x, \sigma^n(z)) < \epsilon \text{ and } \Big| \sum_{k=0}^{n-1} [H \circ \sigma^k(z) - \bar{H}] \Big| < \epsilon \Big\}.$$

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Question Why is Ω_{GS} called the set of ground-state configurations?

Definition An effective potential (sub-coboundary), $V : \Sigma_G \to \mathbb{R}$

$$H(x) - V \circ \sigma(x) + V(x) - \bar{H} \geq 0, \quad \forall\, x \in \Sigma_G$$

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Proposition Ω_{GS} is the set of ground-state configurations in the sense

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Proposition [Mañé-Conze-Guivarc'h lemma] If H is Hölder, an effective potential does exist: a stronger version, called calibrated potential, may be proved

$$V(y) + \bar{H} = \min_{x \in \Sigma_G : \sigma(x) = y} \big[V(y) + H(y) \big]$$

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Theorem[Morris 2009] Extension to weakly expanding map $f : S^1 \to S^1$ of the form $f(x) = x + x^{1+\alpha} + \ldots$, for $\alpha \in]0,1[$. For H γ-Hölder, with $\alpha < \gamma$, there exists a calibrated potential V, $(\gamma - \alpha)$-Hölder. For some α-Hölder H, no continuous effective potential exists

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Proposition[Baraviera-Lopes-Thieullen 2006] If μ_{min} is unique, then μ_β satisfies a large deviation principle

$$\frac{1}{\beta} \ln \mu_\beta(C) \to -\inf_C I$$

- C is any cylinder
- $I(x) = \sum_{k \geq 0} [H - V \circ \sigma + V - \bar{H}] \circ \sigma^k(x)$ is l.s.c.
- V is any calibrated effective potential

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- The transfert operator equation

$$\sum_{x:\sigma(x)=y} \exp -\beta \big[H(x) - \bar{H}_\beta - V_\beta \circ (x) + V_\beta(x) \big] = 1, \quad \forall y \in \Sigma_G$$

- Let V_∞ any limite point of V_β, then V_∞ is calibrated

$$\min_{x:\sigma(x)=y} \big[H(x) - \bar{H} - V \circ \sigma(x) + V_\infty(x) \big] = 0, \quad \forall \ y \in \Sigma_G$$

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where

$$S_n^\epsilon(x,y) := \inf \Big\{ \sum_{k=0}^{n-1} (H - \bar{H}) \circ \sigma^k(z) \ : \ d(z,x) < \epsilon \ \text{ and } \ d(\sigma^n(z), y) < \epsilon \Big\}.$$

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Question Can we find p which minimizes above for all x? Is there a unique calibrated V up to the value $V(p_0)$ for some fixed $p_0 \in \Omega_{GS}$?

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New result[Garibaldi-Thieullen] If $\Omega_{GS} = \Omega_0 \cup \Omega_1 \cup \ldots \cup \Omega_r$ is a finite disjoint union of irreducible components so that Ω_0 has the largest topological entropy and all other Ω_i has a lower topological entropy, then for any fixed $p \in \Omega_0$

$$V_\beta - V_\beta(p) \to h(p, .), \quad \text{uniformly}$$