Robustness to Perturbations of the Gibbs Potential

A Brief Overview

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Thermodynamic Formalism

Potentials & Gibbs Measures

- $\Omega_{\mathcal{A}} := \mathcal{A}^{\mathbb{Z}^d}$ the configuration space on a finite alphabet $\mathcal{A}$
- $\varphi : \Omega_{\mathcal{A}} \to \mathbb{R}$ a potential
- $p_\beta : \mu \mapsto h(\mu) - \beta \int \varphi \mathrm{d}\mu$ the pressure function at inverse temperature β
- $\mathcal{G}(\beta)$ the set of Gibbs measures, that maximise the pressure p_β

Classes of Potentials

Continuous

$\cup$

α-Hölder, $0 < \alpha < 1$

$\cup$

Lipschitz

$\cup$

Exponentially-decreasing interactions

$\cup$

Finite-range (locally constant)

Low-Temperature Limit Behaviours

Let $\mathcal{G}(\infty) := \mathrm{Acc}_{\beta\to\infty}\mathcal{G}(\beta)$ the set of all zero-temperature accumulation points.

Lemma

Assume $\varphi \geq 0$ *and* $X := \left\{\omega \in \Omega_{\mathcal{A}}, \forall x \in \mathbb{Z}^d, \varphi\left(\sigma_x(\omega)\right) = 0\right\} \neq \emptyset$.

Then $\mathcal{G}(\infty) \subset \mathcal{M}_\sigma(X)$ *and these measures have maximal entropy* h *in* $\mathcal{M}_\sigma(X)$.

A model is *stable* if $\mathcal{G}(\infty)$ is a singleton.

Robustness

A property of φ is *robust* (in some class) if there is a neighbourhood U_φ, such that the property holds for all the potentials in U_φ.

Genericity

Ergodic Optimisation

Let $\mathcal{M}_\sigma(\varphi)$ the invariant measures that minimise $\mu \mapsto \int \varphi \mathrm{d}\mu$.

We have the inclusion $\mathcal{G}(\infty) \subset \mathcal{M}_\sigma(\varphi)$.

Generic Properties for Continuous Potentials

- Jenkinson, 2006: Ergodic Optimization
- Brémont, 2008: Entropy and Maximizing Measures of Generic Continuous Functions
- Morris, 2010: Ergodic Optimization for Generic Continuous Functions

Uniqueness of $\mu \in \mathcal{M}_\sigma(\varphi)$ ($\Rightarrow$ Gibbs stability),
and μ is non-mixing, has full support and zero entropy.

- van Enter & Miękisz, 2020: Typical Ground States for Large Sets of Interactions

The measure μ is weakly mixing, has a singular diffraction spectrum,
and does *not* maximise the pressure for any other continuous potential
(*i.e.* it is not a Gibbs measure).

Other Typicality Results

- Contreras, 2015: Ground States are Generically a Periodic Orbit

If $T : X \to X$ is expanding (*e.g.* for $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$, but *not* for $\Omega_{\mathcal{A}}$), then having a periodic support is generic for Lipschitz potentials.

- Shinoda, 2018: Uncountably Many Maximizing Measures for a Dense Subset of Continuous Functions

This holds for a dense but *not generic* set of continuous potentials.

Examples of Robustness

Robinson Tiling

- Gonschorowski, Quas & Siefken, 2019:
  Support Stability of Maximizing Measures for Shifts of Finite Type

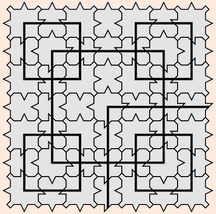

Figure 1: Robinson 3-macro-tile

Finite-range potential, minimal measure supported by the Robinson tiling.

With finite-range perturbations, we can eliminate the quasi-periodicity, beyond a finite scale of macro-tiles.

Decaying One-Dimensional Interactions

- Głodkowski & Miękisz, 2024:
  On Non-Stability of One-Dimensional Non-Periodic Ground States

Without the strict boundary condition,
if the interaction decay is of order $\frac{1}{r^\alpha}$ with $\alpha > 2$,
then the support is not robust to finite-range perturbations.

Hard Square

- Oguri & Shinoda, 2025:
  On the Stability of the Penalty Function for the $\mathbb{Z}^2$-Hard Square Shift

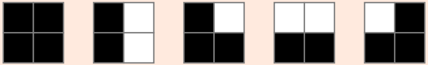

Figure 2: Forbidden patterns for the hard square shift

Finite-range potential, support of the minimal measures robust to Lipschitz perturbations.

Class of Non-Robust Zero-Temperature Limits

- Gayral & Sablik, 2025:
  Non-Robustness of the Zero-Temperature-Limit Gibbs Measures
  to Perturbations of the Potential.

Theorem

Let X a connected Π_2-computable compact set.

There is a "universal" potential φ_X s.t. $G(\infty) = X$ and, for any likewise Π_2 set Y, there is ψ_Y s.t. any perturbation $\varphi_X + \varepsilon\psi_Y$ $(\varepsilon > 0)$ induces Y as the accumulation set.

Notably, the potentials φ_X are not robust to finite-range perturbations.

Open Questions

- Existence of stable/chaotic models robust to finite-range perturbations?
- What about genericity?
- Study other well-known tilings (Kari...).

THE END OF PRESENTATION

ONE MORE SLIDE:

Thank you.