Genericity in ergodic optimization

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Summary

- I. **The additive problem**
- II. The multiplicative problem

Notations

Consider a Tonelli Lagrangian $L : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ which satisfies

1. $v \mapsto \frac{\partial^2 L}{\partial v^2}(x,v)$ is strictly convex
2. $v \mapsto L(x,v)$ is super linear

$$\lim_{R\to+\infty} \inf_{x\in\mathbb{T}^d, \|v\|\geq R} \frac{L(x,v)}{\|v\|+1} = +\infty$$

The Euler -Lagrange flow is given by the solutions of the ODE

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial v}\Big) = \frac{\partial L}{\partial x}$$

Instead of working with an invariant measure with respect to the flow, a weaker notion is introduced : a probability measure μ on $\mathbb{T}^d \times \mathbb{R}^d$ is said to be holonomic if

$$\forall\, \phi : \mathbb{T}^d \to \mathbb{R},\ \int D\phi(x)\cdot v\, \mu(dx,dv) = 0.$$

Theorem [Mañé, 1996].
There exists a residual set of potentials $\phi : \mathbb{T}^d \to \mathbb{R}$ such that $L + \phi$ admits a unique minimizing measure μ_{min} : an holonomic probability measure that satisfies

$$\int \big(L(x,v) + \phi(x)\big)\, \mu_{min}(dx, dv)$$

$$\min \left\{ \int \big(L(x,v) + \phi(x)\big)\, \mu(dx, dv) : \mu \text{ holonomic } \right\}$$

Actually minimizing measures are invariant by the Euler-Lagrange flow.

Moreover for generic ϕ $\mathrm{supp}(\mu_{min})$ is uniquely ergodic.

Mañé conjecture : Is it true that for generic Lagrangian L, the set of minimizing measures is a periodic orbit ?

Notations
Consider an expanding map $T : X \to X$, for instance $X \subseteq \{1, \dots, r\}^{\mathbb{N}}$ is a subshift of finite type and T is the left shift. Consider a Lipschitz function $\phi : X \to \mathbb{R}$. A maximizing measure μ_{max} is a probability measure, invariant by T and satisfying

$$\bar{\phi} := \int \phi \, d\mu_{max} = \max \Big\{ \int \phi \, d\mu : \mu \text{ invariant by } T \Big\}.$$

Theorem [Contreras, 2016].
For generic (open and dense) ϕ there is a unique maximizing measure μ_{max} and μ_{max} is supported on a periodic orbit.

Main tool The existence of subactions $u : X \to \mathbb{R}$, a Lipschitz function satisfying

$$\phi \geq u \circ T - u + \bar{\phi}.$$

I. A different approach

Theorem [Wen Huang, Zeng Lian, Xiao Ma, Leiye Xu, and Yiwei Zhang, 2019].
The authors extend Contreras theorem for bilateral dynamical systems $T : X \to X$ satisfying the shadowing property and the existence of subactions (called in their paper Mañé-Conze-Guivarc'h-Bousch property).

In a second paper they also proved Mañé conjecture for Axiom A flows.

Summary

- I. The additive problem
- II. **The multiplicative problem**

Notations

We consider a one-sided dynamical system $T : X \to X$ and a matrix cocycle $A : X \times \mathbb{N} \to \mathrm{Mat}(d, \mathbb{R})$, that is a Lipschitz continuous map satisfying

$$A(x, m+n) = A(T^m(x), n) A(x, m).$$

Example : one-step cocycles

1. $X = \{1, \ldots, r\}^{\mathbb{N}}$ is the full shift,
2. $\mathcal{A} = \{M_1, \ldots, M_r\} \subset \mathrm{Mat}(d, \mathbb{R})$ is a finite subset of matrices,
3. the cocycle is defiend by

$$A(x, n) = M_{x_{n-1}} \cdots M_{x_1} M_{x_0}, \quad x = (x_0, x_1, \cdots).$$

We would like to understand the set of trajectories that realize the largest Lyapunov exponent.

Definition

1. The maximizing Lyapunov exponent

$$\phi_n(x) := \ln \|A(x,n)\|_2, \quad \phi_{m+n}(x) \leq \phi_m(x) + \phi_n \circ T^m,$$
$$\beta(A) := \lim_{n\to+\infty} \max_{x\in X} \frac{1}{n} \ln \|A(x,n)\|_2).$$

For one-step cocycles, $\beta(A)$ is the log of the joint spectral radius

$$\beta(A) = \lim_{n\to+\infty} \max_{i,\ldots,i_n \in [\![1,r]\!]} \frac{1}{n} \ln \|M_{i_n} \cdots M_{i_1}\|_2.$$

2. The integrated Lyapunov exponent of an invariant measure μ

$$\lambda_1(\mu) := \lim_{n\to+\infty} \frac{1}{n} \int \|A(x,n)\|_2 \, d\mu(x).$$

Proposition [Morris, 2013].
The maximizing Lyapunov exponent admits several equivalent definitions

$$\begin{aligned}\beta(A) &= \lim_{n\to+\infty} \sup_{x\in X} \frac{1}{n} \ln \|A(x,n)\|_2 \\ &= \inf_{n\geq 1} \sup_{x\in X} \frac{1}{n} \ln \|A(x,n)\|_2 \\ &= \sup_{\mu\in\mathcal{M}(X,T)} \inf_{n\geq 1} \frac{1}{n} \int \ln \|A(x,n)\|_2 \, d\mu(x) \\ &= \sup_{\mu\in\mathcal{M}(X,T)} \lambda_1(\mu).\end{aligned}$$

A maximizing measure is an invariant probability measure that realizes the supremum in the last equality. Morris also proved that for generic A there exists a unique maximizing measure.

Remark

If we want to say more about the set of maximizing measure or the set of optimal trajectories we need to choose a more restrictive class of cocycles.

II. Mather set

Definition
The Mather set is the closed set

$$\mathrm{Mather}(A) = \bigcup \{\mathrm{supp}(\mu) : \mu \text{ is maximizing}\}.$$

Notice that the Mather set does not necessarily satisfies the subordination principle

Definition
The Mather set satisfies the subordination principle if

$$\mu \in \mathcal{M}(X, T) \text{ and } \mathrm{supp}(\mu) \subseteq \mathrm{Mather}(A) \quad \Rightarrow \quad \mu \text{ is maximizing.}$$

Theorem [Bochi, Rams, 2016].
Let $\mathcal{A} = \{M_1, \dots, M_r\}$ be a one-step cocycle in $SL(2,\mathbb{R})$ over the full shift. If $\mathcal{A}$ admits a forward non-overlapping multicone, then the Mather set satisfies the subordination principle and has zero entropy.

forward non-overlapping multicone The condition means there exist disjoint cones $C_1, \dots, C_k$ such that, if $C = \bigcup_{i=1}^k C_i$, then

1. $\forall\, j \in [\![1, r]\!],\ M_j(C_i) \subset \mathrm{int}(C)$,
2. $\forall\, j, j' \in [\![1, r]\!],\ j \neq j' \ \Rightarrow\ M_j(C) \bigcap A_{j'}(C) = \emptyset$.

II. Non defective

Definition
A cocycle $A(x,n)$ is said to be non-defective if there exists a constant $C \geq 1$ such that

$$\forall\, x \in X,\ \forall\, n \geq 0,\ \|A(x,n)\|_2 \leq Ce^{n\beta(A)}.$$

Obvious example For instance, for the one-step example $\mathcal{A} = \{M_1\}$ the cocycle is non defective if and only if M_1 has no Jordan bloc.

II. Extremal norm

Definition

Assume A is non defective.

1. A vectorial norm $\|\cdot\|$ is said to be extremal if

$$\forall\, x \in X,\ \|A(x)\| \leq e^{\beta(A)}.$$

2. In the case of the full shift over $\{1, \dots, r\}$ and one-step cocycles $\mathcal{A} = \{M_1, \dots, M_r\}$, a Barabanov norm $\|\cdot\|$ is a vectorial norm satisfying

$$\forall\, v \in \mathbb{R}^d,\ \exists\, i \in [\![1, r]\!],\ \|M_i v\| = e^{\beta(A)}\|v\|.$$

A Barabanov norm is obviously an extremal norm.

Proposition [Jungers, 2009].
In the case of the full shift and one-step cocycles $\mathcal{A} = \{M_1, \ldots, M_r\}$,

1. A is non defective if and only if A admits an extremal norm,
2. if A is irreducible (no strict vector space invariant by each M_i) then A admits a Barabanov norm.

II. Extremal norm

Theorem [Bochi, Garibaldi, 2019].
If (X,T) is a transitive "Axiom A" and $A : X \times \mathbb{N} \to \mathrm{GL}(d,\mathbb{R})$ is a strongly bunched cocycle : the condition number

$$\sup_{x\in X} \|A(x)\| \, \|A(x)^{-1}\|$$

is sufficiently small,

1. if the cocycle is irreducible (no continuous invariant sub-bundle), then A is spannable (the $u-s$ holonomies span $\mathbb{R}^d$),
2. if the cocycle is spannable, then it admits an extremal norm.

II. Mañé conjecture

What about Mañé conjecture ?

The conjecture has another name : finiteness conjecture

II. Finiteness conjecture

Definition In the case of the full shift and one-step cocycles, the finiteness property says (using the vocabulary of ergodic optimization) the Mather set contains a periodic orbit $\{x, T(x), \cdots, T^{\tau-1}(x)\}$.

The maximizing Lyapunov exponent is computed explicitely by

$$\beta(A) = \frac{1}{\tau} \ln \rho(A(x,\tau)).$$

($\rho(A)$ is the spectral radius of A).

II. Counter example

[Bousch, Mairesse, 2002]. There are counter example of one-step cocycles for the finiteness conjecture. For instance if the cocycle is

$$\mathcal{A} = \left\{ \begin{bmatrix} e^{\kappa h_0}+1 & 0 \\ e^{\kappa} & 1 \end{bmatrix}, \begin{bmatrix} 1 & e^{\kappa} \\ 0 & e^{\kappa h_1}+1 \end{bmatrix} \right\}, \quad \kappa > 0,\ h_0, h_1 > 0,\ h_0 + h_1 < 2,$$

then there exists uncountably many parameters κ, h_0, h_1 for which the Mather set is uniquely ergodic and conjugated to the subshift generated by a Sturmian sequence.

II. Counter example

Theorem [Jenkinson, Pollicott, 2018]. Another set of counter examples to the finiteness conjecture. If

$$\mathcal{A} = \left\{ A_0 = \begin{bmatrix} 1 & b \\ c & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & c \\ b & 1 \end{bmatrix} \right\}, \quad bc < 1,\ b, c \geq 0,$$

then there are uncountably many parameters $t \in \mathbb{R}$ such that $\mathcal{A}_t = \{A_0, tA_1\}$ admits a uniquely ergodic Mather set conjugated to the subshift generated by a Sturmian sequence.

A preliminary result :

Proposition [Mohammadpour, T].
Assume $\mathcal{A} = \{M_1, \ldots, M_r\} \subset \mathrm{SL}(2, \mathbb{R})$ is irreducible, none of the matrices M_i is a rotation, and $\beta(A) > 0$. Let $A : X \to \mathrm{SL}(2, \mathbb{R})$ be the associated one-step cocycle.

Then for every $\epsilon > 0$ there exists a cocycle (a priori not one-step) $B_\epsilon : X \to \mathrm{SL}(2, \mathbb{R})$ such that

1. $\sup_{x \in X} \|A(x) - B_\epsilon(x)\| < \epsilon$,
2. $\mathrm{Mather}(B_\epsilon)$ is a periodic orbit.