Characterization of zero-noise limit measures for cellular automata

Webinar THERMOGAMAS

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20 may 2025

Problematic

Stochastic stability

Let (X, F) be a measurable dynamical system.

- Idea: study the long-term behavior of F (the invariant measures) assuming a small amount of errors.
- Denote by F_ϵ a random perturbation of F by a noise of "size ϵ" and $\mathcal{M}_\epsilon$ its invariant measures.

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Definition

Set of zero-noise limit measures:

$$\mathcal{M}_0^\ell = \underset{\epsilon \to 0}{\mathrm{Acc}}(\mathcal{M}_\epsilon) = \bigcap_n \overline{\bigcup_{0<\epsilon<\frac{1}{n}} \mathcal{M}_\epsilon}$$

Stochastic stability

Definition

$$\mu \in \mathcal{M}_0^\ell \iff \exists (\pi_{\epsilon_n})_{n\in\mathbb{N}} \text{ such that } \pi_{\epsilon_n} \in \mathcal{M}_{\epsilon_n} \text{ and } \epsilon_n \underset{n\to\infty}{\longrightarrow} 0 \text{ and } \pi_{\epsilon_n} \underset{n\to\infty}{\longrightarrow} \mu$$

$$\mu \textit{ stable } \iff \exists (\pi_\epsilon)_{\epsilon>0} \text{ such that } \pi_\epsilon \in \mathcal{M}_\epsilon \text{ and } \pi_\epsilon \underset{\epsilon\to 0}{\longrightarrow} \mu$$

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Some questions:

- It is possible to characterize for which dynamics $\mathcal{M}_0^\ell$ is a singleton?
- Which set $\mathcal{M}_0^\ell$ can be reached?
- There exists system without stable measure?
- $\mathcal{M}_0^\ell$ can be uniformly approached?

Definition

$\mathcal{M}_0^\ell$ is *uniformly approached* if for any choose of family $(\pi_\epsilon)_{\epsilon>0}$, one has

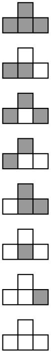
$$\mathcal{M}_0^\ell = \mathrm{Acc}(\pi_\epsilon)$$

Cellular Automata

Definition

- $\mathcal{A} = \{\square, \blacksquare\}$ finite *alphabet*
- $\mathcal{A}^{\mathbb{Z}}$ set of *configurations*
- $f : \mathcal{A}^{[-r,r]} \longrightarrow \mathcal{A}$ *local rules*

$F(x)_{\mathbf{i}} = f(x_{\mathbf{i}+[-r,r]})$ for all $x \in \mathcal{A}^{\mathbb{Z}}$



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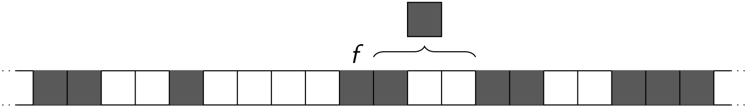

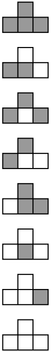

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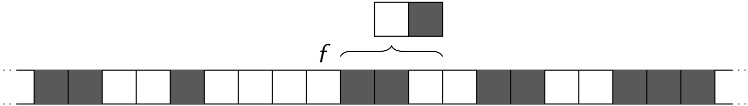

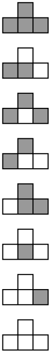

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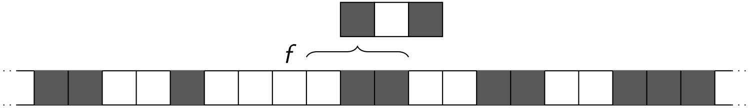

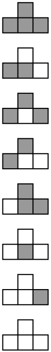

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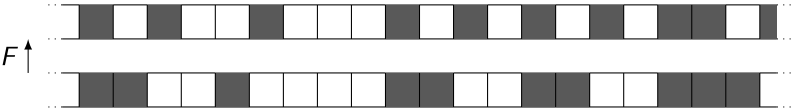

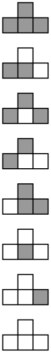

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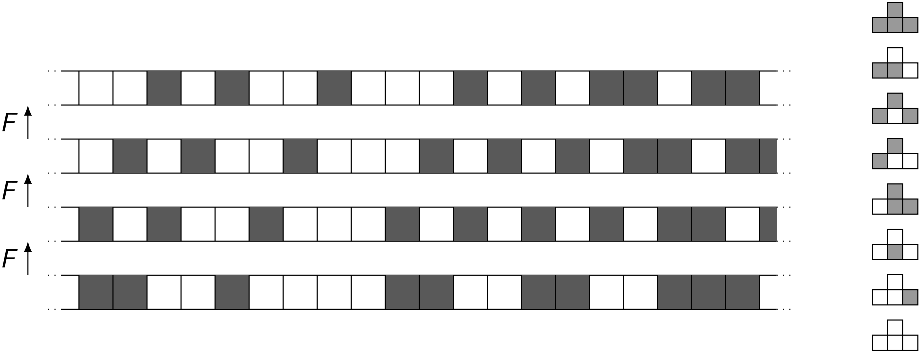

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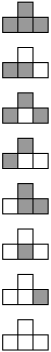
Theorem *(Hedlund-1969)*

$(\mathcal{A}^{\mathbb{Z}}, F)$ is a CA iff $F : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ is continuous and $F \circ \sigma = \sigma \circ F$.

$$\begin{array}{rccc} \sigma : & \mathcal{A}^{\mathbb{Z}} & \longrightarrow & \mathcal{A}^{\mathbb{Z}} \\ & (x_i)_{i \in \mathbb{Z}} & \longmapsto & (x_{i+1})_{i \in \mathbb{Z}}. \end{array}$$



Iteration of measures by a cellular automaton

$$F: \begin{array}{ccc} \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) & \longrightarrow & \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \\ \mu & \longmapsto & F\mu \end{array} \quad \textit{such that } \forall B \in \mathfrak{B} \quad F\mu(B) = \mu(F^{-1}(B)).$$

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Topology:

$\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$: *set of σ-invariant probability measures* with the weak* topology:

$$\mu_n \underset{n\to\infty}{\longrightarrow} \nu \quad \text{iff} \quad \forall u \in \mathcal{A}^{\mathbb{U}} \text{ one has } \mu_n([u]) \underset{n\to\infty}{\longrightarrow} \nu([u]).$$

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Exemple of measures:

- *Bernoulli measure* associated to $(p_a)_{a\in\mathcal{A}} \in [0;1]^{\mathcal{A}}$ such that $\sum_{a\in\mathcal{A}} p_a = 1$:

$$\lambda_{(p_a)_{a\in\mathcal{A}}}([u]) = p_{u_1}\dots p_{u_n} \text{ for } u = u_1\dots u_n \in \mathcal{A}^*.$$

- $\widehat{\delta_w}$ is the σ invariant supported by ${}^\infty w^\infty$.

Perturbations

Noisy cellular automata

Let F be a CA and $\epsilon \in [0,1]$, define the noisy CA $F_\epsilon : \mathcal{M}(\mathcal{A}^{\mathbb{Z}}) \longrightarrow \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$:

- apply the deterministic rule F,
- apply the transformation R_ϵ: for each cell, independently with probability ϵ, choose uniformly a symbol.

$$F_\epsilon = R_\epsilon \circ F$$

- **Deterministic version**

- **Small perturbations**

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Two approaches to study $\mathcal{M}_0^\ell = \mathrm{Acc}(\mathcal{M}_\epsilon) \subset \mathcal{M}(F)$

- Compute $\mathcal{M}_0^\ell$ for some classes of CA
- Given a set of measure $\mathcal{K}$, construct a CA such that $\mathcal{M}_0^\ell = \mathcal{K}$

Study of some classes of cellular automata

CA with spreading state

$\square$ is *spreading* if $(x_i = \square \text{ for } i \in \mathbb{U} \implies F(x)_i = \square)$.

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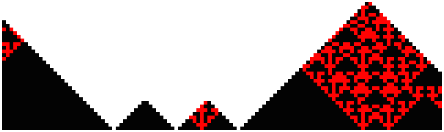

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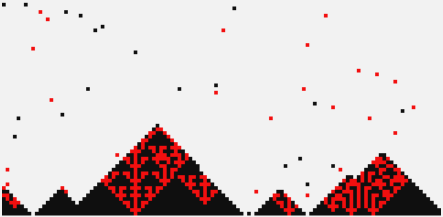

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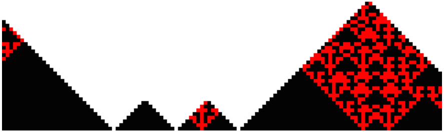

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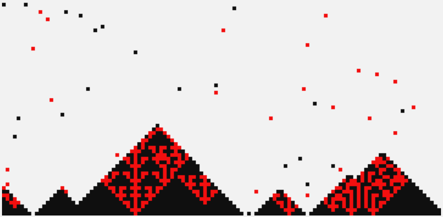

Proposition

$$\mathcal{M}(F) = \left\{\widehat{\delta_\square}\right\} \cup \mathcal{M}_F\left(\left(\mathcal{A} \setminus \{\square\}\right)^{\mathbb{Z}}\right)$$

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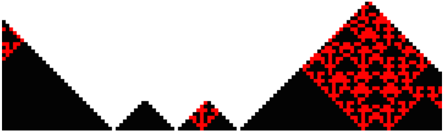

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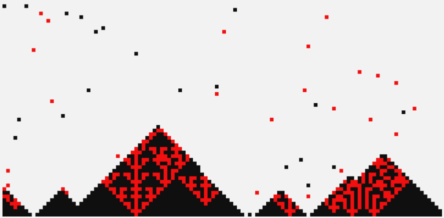

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Proposition *(Marsan-23)*

$$\mathcal{M}_0^\ell = \{\widehat{\delta_\square}\}$$

Surjective CA

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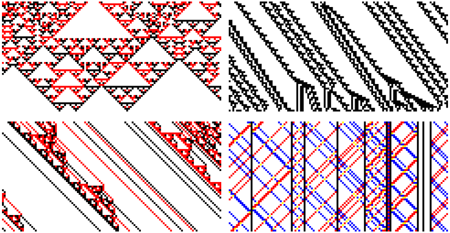

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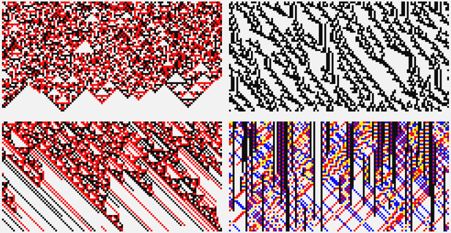

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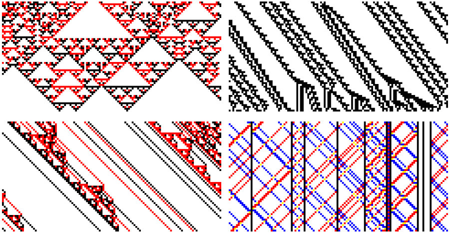

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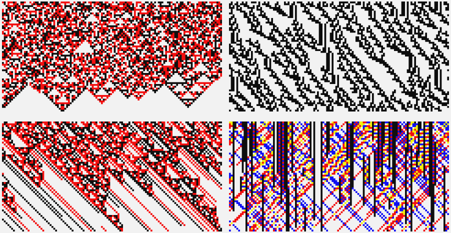

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If F is surjective, λ is F-invariant.

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Which additional constraints make λ attractive?

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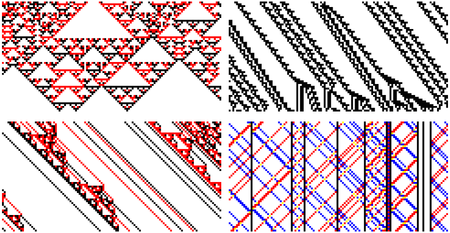

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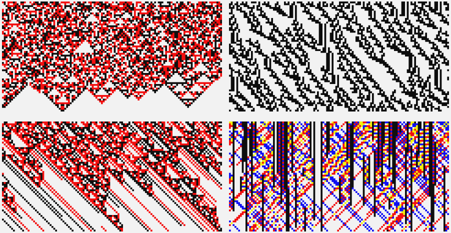

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Theorem *(Marcovici-Taati-S-19)*

For all $\epsilon \in]0,1]$ one has:

$$\mathcal{M}_\epsilon = \{\lambda\}$$

So

$$\mathcal{M}_0^\ell = \{\lambda\}$$

Which set can be realized as $\mathcal{M}_0^\ell$?

Contraints for $\mathcal{M}_0^\ell$

Topological contraints:

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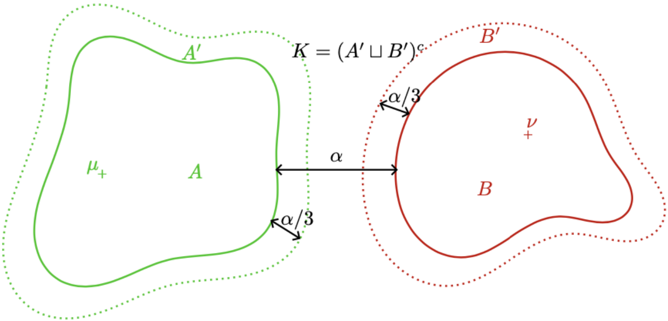

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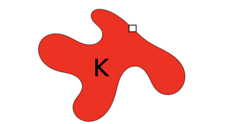
Definition

$\mathcal{K} \subset \mathcal{M}_\sigma$ is Π_k*-computable* if $\exists\ f : \mathcal{A}^* \times \mathbb{Q} \times \mathbb{N}^k \to \{0,1\}$ computable such that:



$$\mathcal{K} \cap \overline{B(\widehat{\delta_w}, r)} \neq \emptyset \iff \underbrace{\forall y_1, \exists y_2, \forall y_3, \ldots}_{k \text{ alternating quantifiers}} f(w, r, y_1, \ldots, y_k) = 1$$

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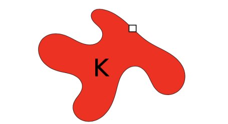

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- $\mathcal{M}_0^\ell$ is Π_3-computable.

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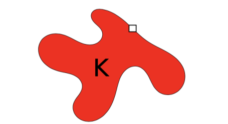
Definition

$\mathcal{K} \subset \mathcal{M}_\sigma$ is *Π_k-computable* if $\exists\ f : \mathcal{A}^* \times \mathbb{Q} \times \mathbb{N}^k \to \{0,1\}$ computable such that:



$$\mathcal{K} \cap \overline{B(\widehat{\delta_w}, r)} \neq \varnothing \iff \underbrace{\forall y_1, \exists y_2, \forall y_3, \ldots}_{k \text{ alternating quantifiers}}\ f(w, r, y_1, \ldots, y_k) = 1$$

- $\mathcal{M}_0^\ell$ is Π_3-computable.
- $\mathcal{M}_0^\ell$ is Π_2-computable if $\mathcal{M}_0^\ell$ is uniformly approached (i.e.$\mathcal{M}_0^\ell = \mathrm{Acc}(\pi_\epsilon)$ for any choose of family $(\pi_\epsilon)_{\epsilon>0}$)

Theorem *(Marsan-S-24)*

Let $\mathcal{K} \subset \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ be a Π_2-computable connected compact.
There exists F a CA on $\mathcal{B}^{\mathbb{Z}}$ with $\mathcal{A} \subset \mathcal{B}$ such that $\mathcal{M}_0^\ell = \mathcal{K}$.
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- Recursive sequence $(w_n)_{n\in\mathbb{N}}$ such that $\mathcal{K} = \mathrm{Acc}(w_n)$ and $d(\widehat{\delta_{w_n}}, \widehat{\delta_{w_{n+1}}}) \to 0$.

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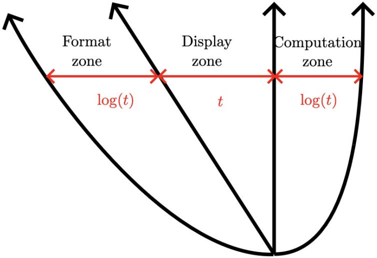
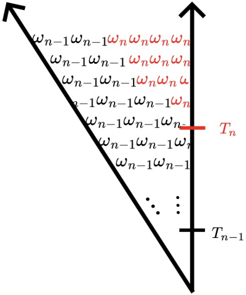
- $\mathcal{B}$ contains a special symbol $\star$ which generates the following things:





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- Recursive sequence $(w_n)_{n\in\mathbb{N}}$ such that $\mathcal{K} = \mathrm{Acc}(w_n)$ and $d(\widehat{\delta_{w_n}}, \widehat{\delta_{w_{n+1}}}) \to 0$.
- $\mathcal{B}$ contains a special symbol $\star$
- With high probability, the last initialization symbol appeared around $\frac{1}{\sqrt{\epsilon}}$ steps before. The word w_{n_ϵ} is produced and π_ϵ is close to $\mathrm{Conv}(\widehat{\delta_{w_{n-1}}}, \widehat{\delta_{w_n}}, \widehat{\delta_{w_{n+1}}})$

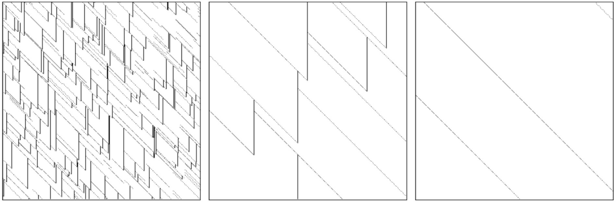
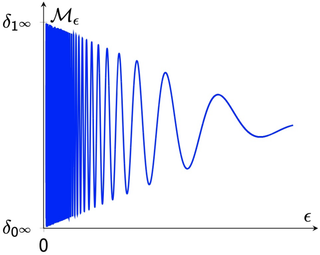


Figure: $\epsilon = 10^{-3}$, 10^{-5} and 10^{-6}

Some applications

- Strange behaviors when $\epsilon \to 0$.



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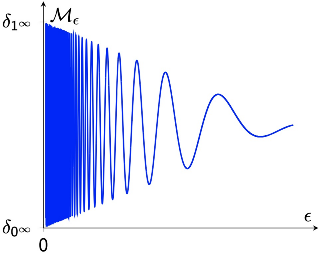

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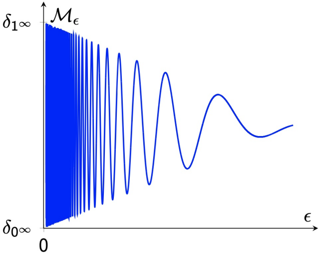

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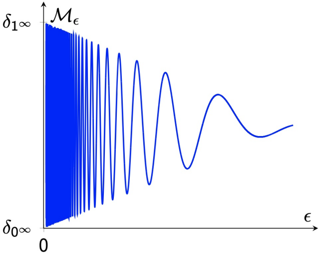

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Theorem *(Marsan-S-25)*

There exists a CA on $\mathcal{B} \supset \mathcal{A}$ such that for any bias α, for any connected compact set $\mathcal{K} \subset \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$, for any $\delta > 0$, there exists a bias α' such that

$$|\alpha - \alpha'| \leq \delta \text{ and } \mathcal{M}_{\alpha',0}^\ell = \mathcal{K} \times \{\lambda_{\alpha'}\}$$

Some perspectives

Other ways to select invariant measures

Asymptotic measures

Given a measure λ, one consider $\mathcal{M}^{\mathrm{Asym}}(F,\lambda) = \mathrm{Acc}(F^n\mu)$.

Theorem *(Hellouin-S-18)*

Let $\mathcal{K} \subset \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ be a Π_2-computable connected compact. There exists F a CA on $\mathcal{B}^{\mathbb{Z}}$ with $\mathcal{A} \subset \mathcal{B}$ such that $\mathcal{M}^{\mathrm{Asym}}(F,\lambda) = \mathcal{K}$.

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Theorem *(PhD-Marsan-25)*

Let $\mathcal{K},\mathcal{K}' \subset \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ be two Π_2-computable connected compact. There exists F a CA on $\mathcal{B}^{\mathbb{Z}}$ with $\mathcal{A} \subset \mathcal{B}$ such that $\mathcal{M}^{\mathrm{Asym}}(F,\lambda) = \mathcal{K}$ and $\mathcal{M}_0^\ell = \mathcal{K}'$.

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Thermodynamic Formalism

Given $\varphi : \mathcal{A}^Z \to \mathbb{R}$

$$\mathcal{M}^{\mathrm{Therm}}(\beta\varphi) = \left\{\mu \text{ which maximises } \nu \mapsto h_F(\nu) - \beta \int \varphi \mathrm{d}\nu\right\}$$

Mores definition in "*Thermodynamic Formalism for a family of cellular automata and duality with the shift*" *(Lopes-Oliveira-Sobottka-24)*.

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Open question

Which measures set can be obtain as $\underset{\beta\to\infty}{Acc}\left(\mathcal{M}^{\mathrm{Therm}}(\beta\varphi)\right)$ and how does it interact with $\mathcal{M}_0^\ell$ and $\mathcal{M}^{\mathrm{Asym}}(F,\lambda)$?

Links with local potential

Let $\Omega = \mathcal{A}^{\mathbb{Z}^2}$ and $\varphi : \Omega \to \mathbb{R}$, define

$$\mathcal{G}(\beta) = \left\{ \mu \text{ which maximises } \nu \longmapsto h_\sigma(\nu) - \beta \int \varphi \mathrm{d}\nu \right\} \text{ and } \mathcal{G}(\infty) = \underset{\beta \to \infty}{\mathrm{Acc}}(\mathcal{G}(\beta))$$

Theorem *(Gayral-Taati-S-23)*

Let $\mathcal{K} \subset \mathcal{M}(\mathcal{A}^{\mathbb{N}})$ be a Π_2-computable connected compact. There exists a local potential $\varphi : \mathcal{B}^{\mathbb{Z}^2} \to \mathbb{R}$ such that $\mathcal{G}(\infty)$ is affine isomorphic to $\mathcal{K}$.
Moreover $\mathcal{G}(\infty)$ is uniformly approached.

"From PCA's to Equilibrium Systems and Back" (Goldstein-Lebowitz-Maes-89):
Let F be a CA on $\mathcal{A}^{\mathbb{Z}}$ of neighboor $\mathbb{U}$. There is a correspondance betweenn $\mathcal{M}_\epsilon$ and $\mathcal{G}(\varphi_\epsilon)$ where

$$\begin{array}{rccl} \varphi_\epsilon : & \mathcal{A}^{\mathbb{Z}^2} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & -\log \mathbf{P}\left(F_\epsilon([x]_{\mathbb{U}\times\{0\}}) = x_{(0,1)}\right) \end{array}$$

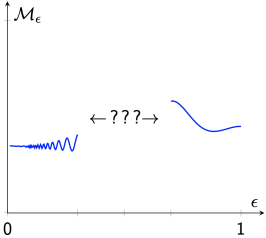
So $\mathcal{M}_0^\ell$ is in correspondance with $\underset{\epsilon \to 0}{\mathrm{Acc}}(\mathcal{G}(\varphi_\epsilon))$.

Phase transition

Open question

Characterize $\mathbf{Uni}(F) = \{\epsilon : |\mathcal{M}_\epsilon| = 1\}$

- Hight level noise: any CA are ergodic *(Marcovici-Taati-S-19)*
- Low level noise:
  - almost all CA are ergodic
  - there exists a CA not ergodic for small noise *(Gacs-01)*



Theorem *(Ilkka-Marsan-S-25)*

There exists a CA, $0 < \epsilon_1 < \epsilon_2 < \epsilon_3$ such that:

$$]0, \epsilon_1] \cup [\epsilon_3, 1] \subset \mathbf{Uni}(F) \text{ and } \epsilon_2 \notin \mathbf{Uni}(F)$$