Blur shifts

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THERMOGAMAS
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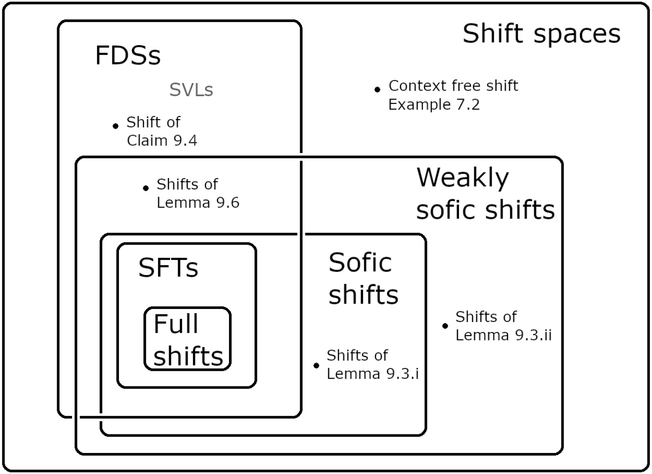
Symbolic dynamics

- Usual shift spaces with countable alphabets:
  - Sofic shift spaces
  - Weakly sofic shift spaces
  - Variable length shift spaces
  - Relationship between shift spaces and labeled graphs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Symbolic dynamics

- Blur shift spaces:
  - An alternative topology for shift spaces with infinite alphabets on the lattice $\mathbb{N}$;
  - Allow to make any shift space compact (or locally compact);
  - Applications in C^* problems and ergodic optimization.

Introduction

$\mathcal{A}$ an alphabet (any cardinality).

$$\mathcal{A}^{\mathbb{N}} := \{(x_i)_{i\in\mathbb{N}} : x_i \in \mathcal{A}\ \forall i \in \mathbb{N}\}.$$

Finite Words

For $\mathbf{x} = (x_i)_{i\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ and $\ell, k \in \mathbb{N}$, the finite word $(x_\ell \dots x_k) \in \mathcal{A}^{k-\ell+1}$ is denoted as $\mathbf{x}_{[\ell,k]}$.

Topology of $\mathcal{A}^{\mathbb{N}}$

Consider in $\mathcal{A}$ the discrete topology, and in $\mathcal{A}^{\mathbb{N}}$ the associated prodiscrete topology. A basis for this topology consists of cylinders:

$$[a_0 a_1 \ldots a_{n-1}] := \{(x_i)_{i\in\mathbb{N}} : x_j = a_j \ \forall j = 0, \ldots, n-1\}.$$

Compactness

$\mathcal{A}^{\mathbb{N}}$ is compact if and only if $\mathcal{A}$ is finite. If $\mathcal{A}$ is infinite, then $\mathcal{A}^{\mathbb{N}}$ is not locally compact. The topology is metrizable, and cylinders are clopen sets.

Shift Map

The **shift map** $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ is defined by:

$$\sigma\big((x_i)_{i\in\mathbb{N}}\big) = (x_{i+1})_{i\in\mathbb{N}}.$$

Shift Spaces

Given a set of forbidden words $\mathrm{F} \subset \bigcup_{n\geq 1} \mathcal{A}^n$, the **shift space** X_{F} is defined as:

$$X_{\mathrm{F}} := \{\mathbf{x} \in \mathcal{A}^{\mathbb{N}} : \mathbf{x}_{[\ell,k]} \notin \mathrm{F},\ \forall \ell, k \in \mathbb{N}\}.$$

Characterization of Shift Spaces

A set $\Lambda \subset \mathcal{A}^{\mathbb{N}}$ is a shift space if and only if it is closed in the topology of $\mathcal{A}^{\mathbb{N}}$ and σ-invariant, i.e., $\sigma(\Lambda) \subset \Lambda$.

Language of a Shift Space

For a shift space Λ, the set of **words of length** $n \geq 1$ that appear in Λ is denoted by:

$$B_n(\Lambda) := \{\mathbf{x}_{[0,n-1]} \in \mathcal{A}^n : \mathbf{x} \in \Lambda\}.$$

The **language** of Λ is:

$$B(\Lambda) := \bigcup_{n \geq 0} B_n(\Lambda),$$

where $B_0(\Lambda) := \{\epsilon\}$ with ϵ being the empty word.

Follower and Predecessor Sets

For $\mathbf{w} \in B(\mathcal{A}^{\mathbb{N}})$, the **follower set** in Λ is:

$$\mathcal{F}_\Lambda(\mathbf{w}) := \{a \in \mathcal{A} : \mathbf{w}a \in B(\Lambda)\}.$$

The **predecessor set** is:

$$\mathcal{P}_\Lambda(\mathbf{w}) := \{a \in \mathcal{A} : a\mathbf{w} \in B(\Lambda)\}.$$

Properties of Follower and Predecessor Sets

For a set $A \subset B(\mathcal{A}^{\mathbb{N}})$:

$$\mathcal{F}_\Lambda(A) = \bigcup_{\mathbf{w} \in A} \mathcal{F}_\Lambda(\mathbf{w}), \quad \mathcal{P}_\Lambda(A) = \bigcup_{\mathbf{w} \in A} \mathcal{P}_\Lambda(\mathbf{w}).$$

Constructing Blur shifts

Let $\mathcal{A}$ be an alphabet.

Step 1: Let $\mathcal{V} \subset 2^{\mathcal{A}}$ be any family of subsets of $\mathcal{A}$ such that

$$H \in \mathcal{V} \qquad \Rightarrow \qquad |H| = \infty$$

and

$$G, H \in \mathcal{V} \text{ and } G \neq H \qquad \Rightarrow \qquad |G \cap H| < \infty.$$

The sets in $\mathcal{V}$ will be said to be the **blurred sets** of $\mathcal{A}$.

Constructing Blur shifts

Label each $H \in \mathcal{V}$ with a symbol $\tilde{H}$, and denote by $\tilde{\mathcal{V}}$ the set of all symbols used to label blurred sets.

Step 2: Let $\bar{\mathcal{A}} := \mathcal{A} \cup \tilde{\mathcal{V}}$;

We remark that, although there is a bijection between $\mathcal{V}$ and $\tilde{\mathcal{V}}$, an element in $\mathcal{V}$ is a subset of $\mathcal{A}$ while an element in $\tilde{\mathcal{V}}$ is a symbol of $\bar{\mathcal{A}}$

Constructing Blur shifts

Step 3: Define the full shift $\bar{\mathcal{A}}^{\mathbb{N}}$ and consider the equivalence relation $\sim$ in $\bar{\mathcal{A}}^{\mathbb{N}}$ given by

$$(x_i)_{i\in\mathbb{N}} \sim (y_i)_{i\in\mathbb{N}} \in \bar{\mathcal{A}}^{\mathbb{N}}$$

$$\Updownarrow$$

$$\min\{j : x_j \in \tilde{\mathcal{V}}\} = \min\{j : y_j \in \tilde{\mathcal{V}}\} =: k, \ \textit{and } x_i = y_i, \ \forall\, i \leq k.$$

Definition

The space $\Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}} := \bar{\mathcal{A}}^{\mathbb{N}}_{/\sim}$ *is the* **full blur shift space** *of* $\mathcal{A}^{\mathbb{N}}$ *with resolution* $\mathcal{V}$.

Recall that $\tilde{H} \notin H$ for any $H \in \mathcal{V}$.

Given $H \in \mathcal{V}$ we will denote

$$\bar{H} := H \cup \{\tilde{H}\}$$

which is a subset of $\bar{\mathcal{A}}$ but not of $\mathcal{A}$, and $\tilde{H} \in \bar{H}$.Define

$$\bar{\mathcal{V}} := \{\bar{H} : \ H \in \mathcal{V}\}.$$

Note that $\bar{\mathcal{V}}$ is a family of subsets of $\bar{\mathcal{A}}$ which also satisfies the properties imposed in *Step 1* on the family $\mathcal{V}$.

- If $\mathrm{x} \in \mathcal{A}^{\mathbb{N}} \subset \bar{\mathcal{A}}^{\mathbb{N}}$, then $[\mathrm{x}]$, the equivalence class of x in $\Sigma_{\mathcal{A}^{\mathbb{N}}}^{\mathcal{V}}$ contains only x.

  In such a case we shall identify $[\mathrm{x}]$ with the point x itself.

- If $\mathrm{x} \in \bar{\mathcal{A}}^{\mathbb{N}} \setminus \mathcal{A}^{\mathbb{N}}$, then $[\mathrm{x}]$ contains infinitely many points and to represent it we will pick $(y_i)_{i\in\mathbb{N}} \in [\mathrm{x}]$ such that $y_i = x_i$ for all $i < n := \min\{i : x_i \in \tilde{\mathcal{V}}\}$ and $y_i = x_n = \tilde{H}$ for $i \geq n$.

Thus, we are going to identify

$$\begin{aligned}\Sigma_{\mathcal{A}^{\mathbb{N}}}^{\mathcal{V}} &\equiv \{(x_i)_{i\in\mathbb{N}} \in \bar{\mathcal{A}}^{\mathbb{N}} :\ x_i = \tilde{H} \in \tilde{\mathcal{V}} \Rightarrow x_{i+1} = \tilde{H}\} \\ &= \mathcal{A}^{\mathbb{N}} \cup \{(x_0 \ldots x_{n-1}\tilde{H}\tilde{H}\tilde{H}\ldots) :\ x_0\ldots x_{n-1} \in B(\mathcal{A}^{\mathbb{N}}), \tilde{H} \in \tilde{\mathcal{V}}\}.\end{aligned}$$

Hence, we can define on it the shift map $\sigma : \Sigma_{\mathcal{A}^{\mathbb{N}}}^{\mathcal{V}} \to \Sigma_{\mathcal{A}^{\mathbb{N}}}^{\mathcal{V}}$ in the usual way.

Definition

We say that $\Lambda' \subset \Sigma_{\mathcal{A}^{\mathbb{N}}}^{\mathcal{V}}$ *is a* **blur shift space** *with resolution* $\mathcal{V}$ *if and only if there exists a shift space* $\Lambda \subset \mathcal{A}^{\mathbb{N}}$ *such that*

1. $\Lambda = \{(x_n)_{n\in\mathbb{N}} \in \Lambda' : \ x_n \in \mathcal{A} \ \forall n \in \mathbb{N}\}$;
2. $(a_0 \ldots a_{n-1}\tilde{H}\tilde{H}\ldots) \in \Lambda'$ *for some* $\tilde{H} \in \tilde{\mathcal{V}} \quad \Longleftrightarrow \quad a_0 \ldots a_{n-1} \in B(\Lambda)$ *and* $|\mathcal{F}_\Lambda(a_0 \ldots a_{n-1}) \cap H| = \infty$.

(Λ' verifies the infinite-extension property)

Under the above notations, we have that Λ' *is the blur shift space of* Λ *with resolution* $\mathcal{V}$*, and denote* $\Lambda' = \Sigma_\Lambda^{\mathcal{V}}$.

If $\mathcal{V} = \emptyset$ corresponds to the maximum resolution for a blur shift, and $\Sigma_{\Lambda}^{\mathcal{V}} = \Lambda$.

On the other hand, $\mathcal{V} = \{\mathcal{A}\}$ corresponds to the minimum resolution (Ott-Tomforde-Willis shift spaces).

Given a blur shift space $\Sigma_\Lambda^\mathcal{V}$, denote:

- $\mathcal{V}_\Lambda := \{H \in \mathcal{V} : \ |B_1(\Lambda) \cap H| = \infty\}$
- $\tilde{\mathcal{V}}_\Lambda := \{\tilde{H} : \ H \in \mathcal{V}_\Lambda\}$
- $\bar{\mathcal{V}}_\Lambda := \{\bar{H} : \ H \in \mathcal{V}_\Lambda\}$.
- $\mathcal{L}_\infty^\mathcal{V}(\Lambda) := \Lambda$
- $\mathcal{L}_n^\mathcal{V}(\Lambda) := \{(x_i)_{i\in\mathbb{N}} \in \Sigma_\Lambda^\mathcal{V} : x_n \in \tilde{\mathcal{V}}_\Lambda$ and $x_{n-1} \notin \tilde{\mathcal{V}}_\Lambda\}$, for $n \in \mathbb{N}$
- $\partial^\mathcal{V}\Lambda := \bigcup_{n\in\mathbb{N}} \mathcal{L}_n^\mathcal{V}(\Lambda)$

$$\Sigma_\Lambda^{\mathcal{V}} = \bigcup_{n\in\mathbb{N}\cup\{\infty\}} \mathcal{L}_n^{\mathcal{V}}(\Lambda) = \Lambda \cup \partial^{\mathcal{V}}\Lambda$$

$$B_1(\Sigma_\Lambda^{\mathcal{V}}) = B_1(\Lambda) \cup \tilde{\mathcal{V}}_\Lambda$$

Proposition

Let $\Sigma_\Lambda^{\mathcal{V}} \subset \Sigma_{\mathcal{A}^{\mathbb{N}}}^{\mathcal{V}}$ be a blur shift space. Then

1. $\sigma(\Sigma_\Lambda^{\mathcal{V}}) \subset \Sigma_{\sigma(\Lambda)} \subset \Sigma_\Lambda^{\mathcal{V}}$;
2. $\sigma(\mathcal{L}_n^{\mathcal{V}}) = \mathcal{L}_{n-1}^{\mathcal{V}}$, $\forall n \geq 1$ *and* $\sigma(\mathcal{L}_0^{\mathcal{V}}) = \mathcal{L}_0^{\mathcal{V}}$.

In general $\Sigma_{\sigma(\Lambda)} \not\subset \sigma(\Sigma_\Lambda)$ even when $\sigma(\Lambda) = \Lambda$.

Defining a topology

1. Consider on $\mathcal{A}$ the discrete topology;
2. Consider on $\bar{\mathcal{A}}$ we consider the same open sets of $\mathcal{A}$ plus the sets $U \subset \bar{\mathcal{A}}$ that have the property that if $\tilde{H} \in U$ then $H \setminus F \subset U$ for some finite $F \subset H$;
3. On the full shift $\bar{\mathcal{A}}^{\mathbb{N}}$ we consider the product topology $\tau_{\bar{\mathcal{A}}^{\mathbb{N}}}$;
4. On $\Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}}$ we define the **quotient topology** denoted as $\tau_{\Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}}}$;
5. On $\Sigma^{\mathcal{V}}_{\Lambda}$ we define induced topology $\tau_{\Sigma^{\mathcal{V}}_{\Lambda}}$.

Generalized cylinders

For any $w_0 \dots w_{n-1} \in B(\mathcal{A}^{\mathbb{N}})$, $\bar{H} \in \bar{\mathcal{V}}$, and $F \subset H$ a finite set define:

$$Z(w_0 \dots w_{n-1}) := \{\mathrm{x} \in \Sigma_{\mathcal{A}^{\mathbb{N}}} : x_i = w_i, 0 \leq i \leq n-1\}$$

and

$$Z(w_0 \dots w_{n-1}\bar{H}, F) := \{\mathrm{x} \in \Sigma_{\mathcal{A}^{\mathbb{N}}} : x_i = w_i, 0 \leq i \leq n-1, x_n \in \bar{H} \setminus F\}.$$

Proposition

The family of all generalized cylinders is a clopen basis for $\mathcal{T}_{\Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}}}$.

Topological properties

- $\Sigma_{\mathcal{A}^{\mathbb{N}}}^{\mathcal{V}}$ is a Hausdorff space;
- $\Sigma_{\mathcal{A}^{\mathbb{N}}}^{\mathcal{V}}$ is a regular space;
- $\Sigma_{\Lambda}^{\mathcal{V}}$ is always a Fréchet-Urysohn space
- $\Sigma_{\Lambda}^{\mathcal{V}}$ is separable $\iff B_1(\Lambda)$ is countable;
- $\Sigma_{\Lambda}^{\mathcal{V}}$ is second countable $\iff B_1(\Lambda)$ and $\mathcal{V}_\Lambda$ are countable;
- $\Sigma_{\Lambda}^{\mathcal{V}}$ is first countable $\iff \forall H \in \mathcal{V}_\Lambda$, $H \cap B_1(\Lambda)$ is countable.

Metrizability

Theorem

Suppose $\Sigma_\Lambda^{\mathcal{V}}$ is a blur shift which is first countable and such that at least one of the following conditions holds:

1. *$\mathcal{V}_\Lambda$ is countable;*
2. *Each $H \in \mathcal{V}_\Lambda$ has just a finite number of elements that appear in some other set of $\mathcal{V}_\Lambda$ (but it is possible that some element appears in infinitely many sets of $\mathcal{V}_\Lambda$).*

Then $\Sigma_\Lambda^{\mathcal{V}}$ is metrizable.

Metrizability

Corollary

If a blur shift is second countable, then it is metrizable.

Metrizability

Corollary

Let $\Sigma_\Lambda^{\mathcal{V}}$ be a compact blur shift. The following statements are equivalent:

1. $B_1(\Lambda)$ *is countable;*
2. $\Sigma_\Lambda^{\mathcal{V}}$ *is first countable;*
3. $\Sigma_\Lambda^{\mathcal{V}}$ *is second countable;*
4. $\Sigma_\Lambda^{\mathcal{V}}$ *is separable;*
5. $\Sigma_\Lambda^{\mathcal{V}}$ *is metrizable.*

Compactness

Theorem

A blur shift $\Sigma_\Lambda^{\mathcal{V}}$ is compact if and only if $\mathcal{V}_\Lambda$ is a finite family of sets which covers all except a finite number of elements of $B_1(\Lambda)$.

Local compactness

Corollary

If for any nonempty letter $a \in B_1(\Lambda)$ and $\mathrm{u} \in B(\Lambda)$ there are a finite number of sets in $\mathcal{V}_\Lambda$ that cover all except a finite number of elements of $\mathcal{F}_\Lambda(a\mathrm{u})$, then Σ_Λ is locally compact. If the previous property also holds for the empty word ϵ, then Σ_Λ is compact.

Ergodic optimization

Theorem (Gomes-Garibaldi-Sobottka 2025)

Let Λ be a topological transitive column-finite countable Markov shift. For every subordinate bounded above upper semi-continuous function $A : \Lambda \to \mathbb{R} \cup \{-\infty\}$, there exists a maximizing probability σ-invariant measure λ on Λ, that is,

$$\int_\Lambda A\, d\lambda = \sup \left\{ \int_\Lambda A\, d\mu \ : \ \mu \text{ is } \sigma\text{-invariant probability} \right\}.$$

Graph algebras and countable Markovian edge shifts

Theorem (Ott-Tomforde-Willis 2014)

Let E and F be countable graphs with no sinks and no sources. If Λ_E and Λ_F are conjugated via a length-preserving conjugacy, then $C^*(E)$ and $C^*(F)$ are isomorphic.

(Ott-Tomforde-Willis shifts use resolution $\nu = \{A\}$)

Graph algebras and countable Markovian shifts

Theorem (Gonçalves-Royer 2015)

Let E and F be two ultragraphs with no sinks that satisfy Condition (RFUM). If Λ_E and Λ_F are conjugated via a length-preserving conjugacy, then $C^*(E)$ and $C^*(F)$ are isomorphic.

(Gonçalves-Royer ultragraph shifts use resolution ν adequately chosen for each given ultragraph.)

Open problems

1. To find a complete set of sufficient and necessary conditions for a blur shift to be metrizable.
2. To construct metrics for non-second countable blur shift spaces.
3. To study the chaotic behaviour of blur shifts for distinct resolutions.
4. Given a classical shift space, is there some ‘natural’ resolution compatible with the dynamical and algebraic structures?

Conjecture

Let $\Lambda \subset \mathcal{A}^{\mathbb{N}}$ and $\Gamma \subset \mathcal{B}^{\mathbb{N}}$ be two weakly sofic shifts whose associated labeled graphs are left-resolving and such that there are only finitely many vertexes that are source of each fixed label. Let Σ_Λ and Σ_Γ be the respective blur shifts for the 'natural' resolutions. Suppose that Σ_Λ and Σ_Γ hold the condition RFUM. If Σ_Λ and Σ_Γ are topologically conjugate via a lenght-preserving generalized sliding block code, then the C^*-algebras associated to the labeled graphs of Λ and Γ are isomorphic.

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THANK YOU