Substitution Dynamical Systems on Infinite Alphabet

Ali Messaoudi

UNESP/IBILCE - São José do Rio Preto - SP

3 June, 2025

Introduction

Objective: We will study ergodic and dynamical properties of symbolic dynamical system associated to substitutions on an infinite countable alphabet.

Definition

A **substitution** *is a map* $\sigma : A \to A^*$ *where* A *is a countable set (alphabet) and* A^* *is the set of finite words.*

Example: $A = \{0, 1\}$, $\sigma(0) = 01$, $\sigma(1) = 0$.

We extend σ to A^* and to $A^{\mathbb{Z}_+}$ (the set of infinite words on A) by concatenation.

We assume that :

$$\forall a \in A, \lim_{n\to\infty} |\sigma^n(a)| = \infty.$$

Let $M_\sigma = (M_{ij})_{i,j\in A}$ where $M_{ij} = |\sigma(i)|_j$.

Example: $M_\sigma = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Finite Alphabet: If A is finite, $\exists u \in A^{\mathbb{Z}+}$ and $k \in \mathbb{N}$, $\sigma^k(u) = u$.

Let

$$\Omega_u = \overline{\{S^n(u),\ n \in \mathbb{Z}_+\}},$$

where S is the shift map. (Ω_u, S) is called the shift dynamical system associated to σ and u.

Classical result:

If σ is a primitive substitution (i.e.M_σ primitive matrix) on $A = \{0, \dots, d\},\ d \geq 1$, then the shift dynamical system (Ω, S) does not depend of the periodic point. Moreover it is minimal, uniquely ergodic with topological entropy 0 (see Michel [M], Queffélec [Q], Pytheas Fogg [P]).

Compact Alphabet: Several works, see for instance the two interting papers [DOP] of Durand, Ormes and Petite and [MRW] of Manibo, Rust and Walton.

Countably infinite alphabet:

For example: $A = \mathbb{Z}_+, \ \sigma(n) = 0(n+1), \ \forall n,$

$$M_\sigma = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Difficulty: $A^{\mathbb{Z}_+}$ is not compact, M_σ is infinite matrix and in general we don't have a periodic point of σ.

To any σ, we associate (Ω_σ, S) (shift dynamical system) where

$$\Omega_\sigma = \{u \in A^{\mathbb{Z}_+} : \text{any finite factor of } u \text{ occurs in } \sigma^n(a) \\ \text{for some } n \in \mathbb{N} \text{ and } a \in A\}.$$

1. An important study was initiated by Ferenczi in [F]. He considered the squared drunken substitution defined by

$$\sigma(n) = (n-2)nn(n+2), \ \ n \in 2\mathbb{Z}.$$

He proved that (Ω_σ, S) is not minimal, has non finite invariant measure. but has an infinite invariant measure μ which is shift ergodic and has Krengel entropy equal to 0.

Ferenczi also proved that for a class of constant length substitution σ for which M_σ is irreducible, aperiodic and positive recurrent, (Ω_σ, S) admits an ergodic probability invariant measure.

2. In a submitted work Bezugly, Jorgensen, Sanadhya [BJS], constructed stationary and non-stationary generalized Bratteli-Vershik models for a class of substitutions on an infinite countable alphabet. As a consequence, they proved that for a left determined substitution σ with M_σ irreducible, aperiodic and recurrent which is also of *bounded size* (the letters of all $\sigma(n)$ belong to the set

$$\{n-d, n-d+1, \ldots, n+d\}$$

where $d \in \mathbb{Z}$ with $d > 0$ is independent of n), there exists a shift invariant measure μ on Ω_σ.

Our Results:

In this work [DFMV], we give sufficient conditions on σ that guarantee that (Ω, S) has no finite invariant measure, or is unique ergodic and minimal. Our results hold in a more general context. We don't need to assume that the substitutions are bounded size.

Preliminaries

Definition

Let $M = (M_{ij})_{i,j \in \mathbb{Z}_+}$ *be an infinite nonnegative matrix. We say that* M *is* **irreducible** *if:* $\forall i, j \in \mathbb{Z}_+, \exists k \geq 1,\ M_{ij}^k = (M^k)_{ij} > 0.$

Let $i \in \mathbb{Z}_+$,

$$p_i = gcd\{n \in \mathbb{N},\ M_{ii}^n > 0\}$$

is called the **period** *of the state* i. *If* M *is irreducible, then there exists* $p \geq 1$ *such that* $p_i = p$ *for every* $i \in \mathbb{Z}_+$ *and we say that* M *has period* $p \geq 1$. *We say that an irreducible matrix* M *is* **aperiodic** *if* $p = 1$ *and periodic otherwise.*

Result: If $M = (M_{ij})_{i,j \in \mathbb{Z}_+}$ is an irreducible and aperiodic nonnegative matrix, then there exists $\lambda \in [0, \infty]$ (**Perron value** of M), such that

$$\lim_{n \to \infty} (M_{ij}^n)^{1/n} = \lambda, \ \forall i, j \in \mathbb{Z}_+.$$

Definition

We say that M is **transient** *if and only if* $\sum_{n=0}^{+\infty} \frac{M_{ij}^n}{\lambda^n} < +\infty$, *otherwise M is said to be* **recurrent**.

If M is recurrent and $\lim_{n \to \infty} \frac{M_{ij}^n}{\lambda^n} > 0$, *we say that M is* **positive recurrent**, *otherwise M is said to be* **null recurrent.**

Remark

1. The definition is an extension of the notions of recurrence and transience in the case of Markov chains and stochastic matrices (see the works of Vere Jones and the books of Kitchen and Seneta for instance).

2. It is known by Perron Frobenuis Theorem that if M is recurrent there are left and right eigenvectors l and r associated to λ and when the scalar product $l \cdot r$ is finite,if and only if M is positive recurrent.

3. In general, it can be difficult to check if a matrix is transient or positive-or null recurrent.

Example: The matrix associated to $\sigma(n) = 0(n+1)$ is irreducible, aperiodic and positive recurrent with Perron value $\lambda = 2$.

Definition

Let $M = (M_{ij})_{i,j\geq 0}$ be a nonnegative matrix. We say that M is **scrambling** *if there exists $a > 0$ such that*

$$\sum_{j=0}^{+\infty} \min(M_{ij}, M_{kj}) \geq a \text{ for all } i \neq k \in \mathbb{Z}_+.$$

Example:

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The notion of scrambling matrices comes from the theory of stochastic matrices.

Stochastic Matrix

Definition

Let $P = (P_{ij})_{i,j\geq 0}$ *be a nonnegative stochastic matrix. We say that* P *is*

- **ergodic** *if* $\lim_{n\to\infty} P_{ij}^n = \pi_j > 0$ *for all* $i, j \in \mathbb{N}$, *where* $(\pi_j)_{j\geq 0}$ *is a probability vector;*
- **strongly ergodic** *if* P *is ergodic and if*

$$\lim_{n\to\infty} \sup_{i\geq 0} \sum_{j=0}^{\infty} |P_{ij}^n - \pi_j| = 0.$$

Important classical result:

The following properties are equivalent:

1. P is strongly ergodic.
2. P is **uniformly geometrically ergodic**, i.e., there exist $\beta \in (0,1)$ and a constant $C > 0$ such that

$$|P_{ij}^n - \pi_j| \leq C\beta^n, \ \forall\, i, j, n \in \mathbb{Z}_+.$$

3. $\exists n \geq 1$, such that P^n is scrambling.

Main Results

In this work, we will assume that: $\exists C \geq 1$, $|\sigma(n| < C$, $\forall n$ (σ is bounded length substitution.

Theorem (1)

Let $\sigma : \mathbb{Z}_+ \to \mathbb{Z}_+^*$ be a substitution which has a periodic point u and $M = M_\sigma$ is irreducible and aperiodic. If M satisfies

$$\lim_{n \to +\infty} \sup_{i \in \mathbb{Z}_+} \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} = 0 \text{ for all } j \in \mathbb{Z}_+, \tag{1}$$

then the dynamical system (Ω_σ, S) has no finite invariant measure.

Remark

The last result holds for a class of substitutions σ such that M_σ is transient and σ has constant length, or M_σ is recurrent and has a left Perron eigenvector $l = (l_i)_{i\geq 0} \notin l^1$.

Example: Let $\sigma := \sigma_{a,b,c}$ defined by

$$\sigma(0) = 0^{a+b}1^c \text{ and } \sigma(n) = (n-1)^a n^b (n+1)^c \text{ for all } n \geq 1$$

where a, b, c are nonnegative integers such that $a > 0$, $c > 0$ and $i^k = ii \ldots i$ (k times). The matrix M_σ is irreducible and aperiodic.

$$M_\sigma = \begin{bmatrix} a+b & c & 0 & 0 & 0 & 0 & 0 & \cdots \\ a & b & c & 0 & 0 & 0 & 0 & \cdots \\ 0 & a & b & c & 0 & 0 & 0 & \cdots \\ 0 & 0 & a & b & c & 0 & 0 & \cdots \\ 0 & 0 & 0 & a & b & c & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We can prove that if If $c \geq a$, then M_σ is null recurrent and satisfies condition (1) of theorem 1 and hence (Ω_σ, S) has no finite invariant measure.

Theorem (2)

Let σ be a substitution on $A = \mathbb{Z}_+$ with a periodic point u and such that $M = M_\sigma$ is irreducible, aperiodic and positive recurrent, then (Ω_u, S) has a shift invariant measure μ which is finite if and only if M_σ has a left Perron eigenvector $l \in l^1$. Moreover, if σ has constant length and M_σ has a power that is scrambling, then (Ω_u, S) is minimal and has a unique shift invariant probability measure μ.

Remark

1. σ is not necessarily bounded size.

2. M_σ has a power that is scrambling, if and only if for some $n \geq 1$

$$\forall i, k \in A,\ \exists j \in A \text{ which occurs in } \sigma^n(i) \text{ and } \sigma^n(k). \tag{2}$$

Example: 1. M_σ associated to $\sigma(n) = 0(n+1)\forall n$ is scrambling.

2. M_σ associated to $\sigma(n) = (n-1)^a n^b (n+1)^c,\ a, b, c \geq 1$ are not scrambling.

Theorem (3)

Let σ be a non-constant substitution on $A = \mathbb{Z}_+$ with a periodic point u and such that $M = M_\sigma$ is irreducible, aperiodic and positive recurrent. Assume that M_σ has a right Perron eigenvector $r = (r_i)_{i \geq 0} \in l^\infty$ and there exists a positive integer such that M_σ^n is scrambling, then the dynamical system (Ω_σ, S) is minimal and has a unique invariant probability measure.

Example:

Let τ be given by

$$\tau(n) = 0^{a_n}(n+1), \text{ for all } n \geq 0.$$

where $0 \leq a_i \leq C$ for all $i \geq 0$ for some fixed $C > 0$ and $a_0 > 0$ and $\lim sup\, a_n \geq 1$.

$$M_\tau = \begin{bmatrix} a_0 & 1 & 0 & 0 & 0 & 0 & \dots \\ a_1 & 0 & 1 & 0 & 0 & 0 & \dots \\ a_2 & 0 & 0 & 1 & 0 & 0 & \dots \\ a_3 & 0 & 0 & 0 & 1 & 0 & \dots \\ a_4 & 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The Perron eigenvalue of M_τ is the unique real number $\lambda > 1$ satisfying

$$1 = \sum_{i=0}^{\infty} a_i \lambda^{-i-1}.$$

$$l = (1, 1/\lambda, \ldots, 1/\lambda^n, \ldots) \text{ and } r = (1, \alpha_1, \ldots, \alpha_n, \ldots),$$

where

$$\alpha_n = \lambda^n - \sum_{i=0}^{n-1} a_i \lambda^{n-i-1} \text{ for all } n \geq 1.$$

Since $l \cdot r < +\infty$, M_σ is positive recurrent.

If there exists $k \geq 1$, such that $a_{kn} \geq 1$ for all $n \in \mathbb{Z}_+$, then $\inf\{\alpha_n,\ n \in \mathbb{Z}_+\} > 0$.. Moreover M_τ^k is scrambling. thus (Ω_σ, S) has a unique probability invariant measure.

Idea for the Proof of Theorem 1

Assume that $u = u_0 u_1 \ldots = \sigma(u)$. We have that for all $j \in \mathbb{Z}_+$

$$\lim_{n \to +\infty} \sup_{a \in A} \frac{|\sigma^n(a)|_j}{|\sigma^n(a)|} = 0. \tag{3}$$

Now, assume that (Ω_σ, S) has a finite invariant measure. Let μ be a finite ergodic invariant measure. By Birkhoff's ergodic theorem, we deduce that for μ almost all $x \in \Omega_u$

$$\lim_{N \to \infty} \frac{1}{N} card\{0 \leq k \leq N-1 : S^k(x) \in [j]\} = \mu[j] \quad \forall j \in \mathbb{Z}_+. \tag{4}$$

Idea for the Proof of Theorem 1

Now, let $x \in \Omega_\sigma$ satisfying (4) and $N \in \mathbb{N}$. Let $V = u_m \dots u_{m+N-1}$, $m \in \mathbb{N}$ be a prefix of x. The word V can be written as

$$V = v_0\sigma(v_1)\dots\sigma^{n-1}(v_{n-1})\sigma^n(v_n)\sigma^{n-1}(w_{n-1})\dots\sigma(w_1)w_0, \quad (5)$$

where $n \geq 1$ is an integer and v_i, $i \in \{0,\dots,n\}$, w_j, $j \in \{0,\dots,n-1\}$, are elements of L_u possibly empty words of lengths smaller or equal to $K = \max\{|\sigma(b)|,\ b \in A\}$ and v_n is not empty.

Idea for the Proof of Theorem 1

We have that

$$\frac{1}{N} card\{0 \leq k \leq N-1, S^k(x) \in [j]\} = \frac{|V|_j}{|V|} = \frac{|\sigma^n(v_n)|_j + \sum_{k=0}^{n-1}(|\sigma^k(v_k)|_j + |\sigma^k(w_k)|_j)}{|\sigma^n(v_n)| + \sum_{k=0}^{n-1}(|\sigma^k(v_k)| + |\sigma^k(w_k)|)}.$$

By (3), we deduce that

$$\lim_{k\to\infty} \sup\left\{\frac{|\sigma^k(v)|_j}{|\sigma^k(v)|},\ v \in L_u,\ |v| \leq K\right\} = 0. \tag{6}$$

Using (6), and the Stolz-Cesaro Theorem, we deduce that

$$\lim_{n\to\infty} \frac{|\sigma^n(v_n)|_j + \sum_{k=0}^{n-1}(|\sigma^k(v_k)|_j + |\sigma^k(w_k)|_j)}{|\sigma^n(v_n)| + \sum_{k=0}^{n-1}(|\sigma^k(v_k)| + |\sigma^k(w_k)|)} = 0.$$

Therefore

$$\lim_{N\to\infty} \frac{1}{N}\, card\{0 \leq k \leq N-1, S^k(x) \in [j]\} = \mu[j] = 0.$$

Absurd!

Idea for the Proof of Theorem 2

Let $\sigma : A \to A^*$ and $t \geq 2,\ A_t = \{a_1 \ldots a_t,\ \text{ factor of } u\}$.

Let $\sigma_t : A_t \to A_t^*$ defined by: If $w = w_0 \ldots w_{t-1} \in A_t$ and $\sigma(w) = y_0 \ldots y_{|\sigma(w_0)|-1} y_{|\sigma(w_0)|} \cdots y_{|\sigma(w)|-1}$, then

$$\sigma_t(w) = (y_0 \ldots y_{t-1})(y_1 \ldots y_t) \ldots (y_{|\sigma(w_0)|-1} \cdots y_{|\sigma(w_0)|+t-2}). \quad (7)$$

Note that

$$|\sigma_t(w_0 \ldots w_{t-1})| = |\sigma(w_0)|, \quad (8)$$

σ_t was defined in [M] (in the case of substitutions on finite alphabets).

Idea for the Proof of Theorem 2

Lemma

(1) M_σ irreducible and aperiodic with Perron value $\lambda \implies M_{\sigma_t}$ too.

(2) M_σ is positive recurrent (resp. null recurrent, transient) $\implies$ M_{σ_t} too.

(3) σ is a constant length and M_σ is strongly ergodic $\implies$ M_{σ_t} too.

Existence of invariant measure

Let $u = u_0 u_1 \ldots = \sigma(u) \in \Omega_\sigma$. For $j \in A$ set

$$\mu[j] := \lim_{n\to\infty} \frac{|\sigma^n(u_0)|_j}{\lambda^n} = \lim_{n\to\infty} \frac{M^n_{u_0,j}}{\lambda^n}.$$

For $t \geq 2$ integer and $I_t = i_1 \ldots i_t \in A_t$ set

$$\mu[i_1 \ldots i_t] = \lim_{n\to\infty} \frac{|\sigma^n(u_0)|_{i_1\ldots i_t}}{\lambda^n} = \lim_{n\to\infty} \frac{(M^n_t)_{U_t,I_t}}{\lambda^n} \tag{9}$$

where $U_t = u_0 \ldots u_{t-1}$ and $I_t = i_1 \ldots i_t$.

Existence of invariant measure

For every integer $t \geq 1$ and $I = i_1 \dots i_t \in A_t$ we have to prove that

$$\mu[I] = \sum_{b \in A, Ib \in A_{t+1}} \mu[Ib]. \tag{10}$$

and

$$\mu[I] = \sum_{a \in A, aI} \mu[aI]. \tag{11}$$

For the proof of the last 2 equalities, we use Fatou Lemma for series and also our Lemma that connect a right Perron eigenvector of M_σ and M_{σ_t}.

Unique ergodicity

The idea is similar the case of primitive substitution of finite alphabet.

Idea for the Proof of Theorem 3

The proof is more technical. We use properties of σ_t. In particular if M_σ is scrambling, then M_{σ_t} is too. We also use Stolz-Cezaro Theorem.

Remark

The proof of minimality in the case where M_σ^n is scrambling for some integer $n \geq 1$ is not difficult.

We can prove that the dynamical system associated a $\sigma_{a,b,c}$ are not minimal even in the case where M_σ is positive recurrent $(a > c)$.

It will be interesting to find σ such that (Ω, S) is minimal and M_σ has no scrambling power.

Bibliography

- [BJS] S. Bezugly, P. E. T. Jorgensen, S. Sanadhya, *Measures and generalized Bratteli diagrams for dynamics of infinite alphabet-substitutions*, ArXiv:2203.14127, 2022.
- [DuO] F. Durand, N. Ormes, S. Petite, *Self-induced systems*. J. Anal. Math. 135(2):725?756, 2018.
- [DFMV] W. Domingos, S. ferenczi, A. Messaoudi, G. Valle, *Invariant measure for substitutions on countable alphabets*, submitted, 2022.

Bibliography

[F] S. Ferenczi, *Substitution Dynamical Systems on infinite alphabets*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 7, 2315–2343.

[K] B. P. Kitchens, *Symbolic Dynamics, one-sided, two-sided and countable state Markov shifts*, Universitext, Springer-Verlag, 1998.

[M] P. Michel, *Stricte ergodicité d?ensembles minimaux de substitutions*, C.R. Acad. Sc. Paris 278 (1974), 811?813.

Bibliography

- [P] N. Pytheas Fogg, *Substitutions in dynamics, arithmetics and combinatorics*, Lecture Notes in Math., Vol 1794, Springer-Verlag, 2002.
- [Q] M. Queffélec, *Substitution Dynamical systems- Spectral analysis*, Lecture Notes in Math., Vol 1294, Springer-Verlag, 1987.
- [S] E. Seneta, *Non negative matrices an Markov chains*, Springer, 1973.

OBRIGADO!