Quasi limit theorems for open systems

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- I will present a few preliminary results of an ongoing paper with J Atnip, C Gonzalez-Tokman, G Froyland and Y Nakano.
- The following two contributions share our kind of investigation.
- M. M. Castro, J. S. W. Lamb, G. Olicon Mendez, and M. Rasmussen. *Existence and uniqueness of quasi-stationary and quasi-ergodic measures for absorbing Markov processes: a Banach lattice approach*, 2021. arXiv:2111.13791v6. To appear in Stochastic Processes and their Applications. FOR MARKOV CHAINS
- P. Collet, S. Martínez and J. San Martin, *Quasi-Stationary Distributions, Markov Chains, Diffusions and Dynamical Systems,* Springer, 2013. FOR GIBBS STATE
- Our results apply to deterministic dynamical systems and dynamical systems perturbed in a quenched way.
- Basic reference for quenched (fibred) systems: L Arnold: Random Dynamical Systems.

Random Dynamical Systems: quenched

- Suppose we have a complete metric space X and a probability space $(\Omega, \mathcal{F}, m)$ and an ergodic invertible map σ. Define I_ω (the FIBER) a closed subset of X and
- For each $\omega \in \Omega$ we consider the map

$$T_\omega : I_\omega \to I_{\sigma\omega}$$

- and the random composition

$$T_\omega^n = T_{\sigma^{n-1}\omega} \circ \cdots \circ T_{\sigma\omega} \circ T_\omega.$$

- Put $I_c := \cup_{\omega\in\Omega}\{\omega\} \times I_\omega$; writing $B \in I_c$ means $\forall\omega, B_\omega \in I_\omega, \{\omega\} \times B_\omega = B \cap \{\omega\} \times I_\omega$.
- Ex. Take σ an irrational rotation of the circle $\sigma\omega = \overline{\omega + \alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\Omega = \cup_{j=1}^M \Omega_j$, a finite partition of the circle.
- On each Ω_j the map $\omega \in \Omega_j \to T_\omega$ is constant.

$$T_\omega^3(x) = T_{\overline{\omega+2\alpha}} T_{\overline{\omega+\alpha}} T_{\overline{\omega}}(x).$$

- In this context we will consider a probability measure satisfying the equivariant condition:

$$\int_{I_\omega} f \circ T_\omega \, d\mu_\omega = \int_{I_{\sigma\omega}} f \, d\mu_{\sigma\omega}, \qquad f \in L^1(I, \mu_{\sigma\omega}). \tag{1}$$

Another example: the i.i.d. case

- Take a probability space $(\Omega, \mathcal{F}, m)$ and associate to each $\omega \in \Omega$ a map T_ω.
- Then construct $(\Omega^{\mathbb{Z}}, \mathcal{F}^{\mathbb{Z}}, m^{\mathbb{Z}})$ and take an element $\underline{\omega} \in \Omega^{\mathbb{Z}}$ as $\underline{\omega} = (\dots, \omega_{-1}, \omega_0, \omega_1, \dots)$, where $\omega_j \in \Omega$.
- Define the shift
$$(\sigma\underline{\omega})_i = \underline{\omega}_{i+1}$$
and put
$$\underline{\omega} \to T_{\underline{\omega}} = T_{\omega_0},$$
where ω_0 is the 0-th coordinate of $\underline{\omega}$.
- Then
$$T^n_{\underline{\omega}} = T_{\sigma^{n-1}\underline{\omega}} \circ \dots \circ T_{\sigma\underline{\omega}} \circ T_{\underline{\omega}} = T_{\omega_{n-1}} \circ \dots \circ T_{\omega_0}.$$
- Notice that in this scenario we could still get a *deterministic* skew system $T : X \to X$ given by
$$T(\omega, x) = (\sigma\omega, T_\omega(x))$$

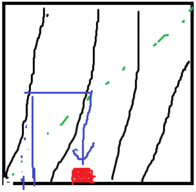
- In the following we could forget the presence of ω : the results will be still NEW.
- Invariance $\mu(T^{-1}A) = \mu(A)$ is replaced by equivariance $\mu_\omega(T_\omega^{-1}(A)) = \mu_{\sigma\omega}(A)$.
- The rate of decay could be affected by norms of functions along fibers $f_{\sigma^k\omega}(x), x \in I_{\sigma^k\omega}$. Use ergodicity of σ an especially temperedess. The function $a(\omega)$ is tempered if for ωa.e. we have $\lim_{|n|\to\infty} \frac{1}{n} \log |a(\sigma^n\omega)| = 0$, which is equivalent to say that $\forall \varepsilon, \exists A_\omega > 0, \forall n : a(\sigma^n\omega) \leq A_\omega e^{\varepsilon|n|}$.
- More important powers are replaced by compositions or cocycles; quasicompactness of operators are replaced by quasicompactness of cocycles.

- Given a dynamical system (X, μ, T) with μ which is T-invariant, Birkhoff ergodic theorem states that for any $f \in L^1(\mu)$:

$$\lim_{n\to\infty} \frac{1}{n} S_n(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \hat{f}(x), \ \ \mu - a.s.$$

with $\int f d\mu = \int \hat{f} d\mu$; $\hat{f}(Tx) = \hat{f}(x)$. If μ is ergodic then $\hat{f}(x) = \int f d\mu, \ \ a.s.$

- How could we formulate it for **open systems**?
- Namely for systems which admit a set (the hole), which absorbs the trajectories?



QUESTIONS and Holes

- When the orbit enters the hole it disappears. Do we have points whose orbit never disappears? The surviving set.
- How does mass escape into the hole? Escape rate.
- How could we describe the metastable statistics of convergence to the equilibrium state on the surviving set?
- The hole. Let H be a measurable set in I_c. For each $\omega \in \Omega$ the fiber sets $H_\omega \subseteq I_\omega$ are uniquely determined by the condition that $\{\omega\} \times H_\omega = H \cap (\{\omega\} \times I_\omega)$. For each $\omega \in \Omega$ and $n \geq 0$ we define

$$X_{\omega,n} := \left\{ x \in I_\omega : T_\omega^j(x) \notin H_{\sigma^j \omega} \text{ for all } 0 \leq j \leq n \right\} = \bigcap_{j=0}^{n} T_\omega^{-j} \left(X_{\sigma^j \omega, 0} \right) \quad (2)$$

to be the set of points in $X_{\omega,0}$ whose trajectories avoid the hole for n iterates. Note that $X_{\omega,0} = I_\omega \backslash H_\omega$ and $X_{\omega,n} = \{\tau(\omega,\cdot) > n\}$, where τ is the first hitting time to H (i.e. $\tau(\omega,x) = \inf\{j \geq 1 : T_\omega^j(x) \in H_{\sigma^j \omega}\}$). We call $X_{\omega,n}$ the *nth* *level surviving set*. We then naturally define

$$X_{\omega,\infty} := \bigcap_{n=0}^{\infty} X_{\omega,n} = \bigcap_{n=0}^{\infty} T_\omega^{-n}(X_{\sigma^n \omega,0}) \quad (3)$$

the *ω-surviving set*.

Example: the baker map I

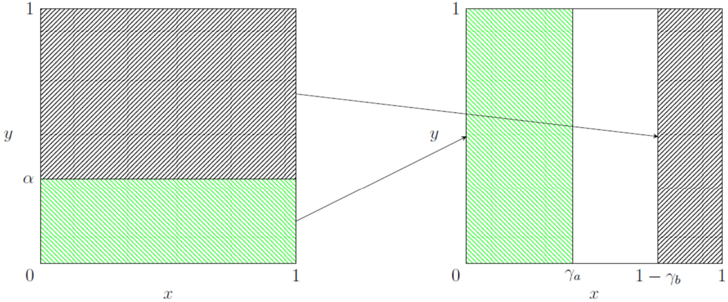
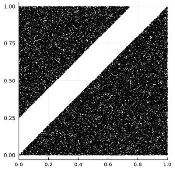


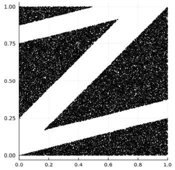
FIGURE 1. Action of the baker's map on the unit square. The lower part of the square is mapped to the left part and the upper part is mapped to the right part.

Example: the baker map II: simulations by G Froyland

Figure: Original hole in white

Figure: First iteration: X_1 is in black





Example: the baker map III

Figure: Second iteration

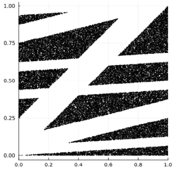
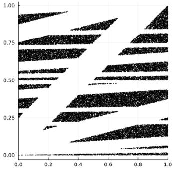


Figure: Third iteration



Example: the baker map IV

Figure: Fourth iteration

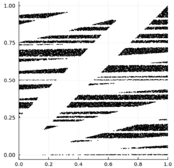
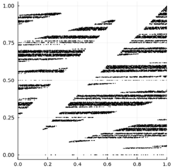


Figure: Fifth iteration



- A probability measure ζ on $(\Omega \times X, F \otimes B)$ is said to be a RANDOM PROBABILITY MEASURE relative to m if it has marginal m (i.e. $\zeta \circ \pi_1^{-1} = m$, where $\pi_1(\omega, x) = \omega$). This property holds if and only if the disintegrations $\{\zeta_\omega\}_{\omega \in \Omega}$ of ζ with respect to the partition $\{\{\omega\} \times\}_{\omega \in \Omega}$ satisfy the following properties:
  1. For every $B \in B$, the map $\Omega \ni \omega \longmapsto \zeta_\omega(B) \in [0, 1]$ is measurable,
  2. For m-a.e. $\omega \in \Omega$, the map $B \ni B \longmapsto \zeta_\omega(B) \in [0, 1]$ is a Borel probability measure.
- We denote with μ the measure on the skew system of the form $d\mu = d\mu_\omega dm$.
- CONDITIONALLY INVARIANT
  Det $\forall n : \eta(B) = \frac{\eta(T^{-n}B \cap X_n)}{\eta(X_n)}$
  Ran $\forall n : \rho_{\sigma^n\omega}(B) = \frac{\rho_\omega(T_\omega^{-n}B \cap X_{\omega,n})}{\rho_\omega(X_{\omega,n})}$
- YAGLOM LIMIT
  Det $\frac{\zeta(T^{-n}B \cap X_n)}{\zeta(X_n)} \to \eta(B)$
  Ran $\frac{\zeta_\omega(T_\omega^{-n}B \cap X_{\omega,n})}{\zeta_\omega(X_{\omega,n})} - \rho_{\sigma^n\omega}(B) \to 0$
- CONDITIONING LIMIT
  Det $\frac{\zeta(B \cap X_n)}{\zeta(X_n)} \to$ some measure
  Ran $\frac{\zeta_\omega(B \cap X_{\omega,n})}{\zeta_\omega(X_{\omega,n})} \to$ some random measure

- From the abstract point of view, we will assume the existence of a measure (CONFORMAL) and of operators (transfer or Perron-Fröbenius) verifying:
- Det: $\int L_c(\phi)\psi d\nu_c = \lambda \int \phi \ \psi \circ T d\nu_c, \ \lambda = \int L_c(1) \ d\nu_c$
- Ran:
  $\int L_{\omega,c}(\phi_\omega)\psi_{\sigma\omega} d\nu_{\sigma\omega,c} = \lambda_\omega \int \phi_\omega \ \psi_{\sigma\omega} \circ T_\omega d\nu_{\omega,c}, \ \lambda_\omega = \int L_{\omega,c}(1_\omega) \ d\nu_{\sigma\omega,c}$
- **Warning: the choice of the functional spaces is essential.**
- (Ex: for billiard systems and hyperbolic diffeos with singularities,

  $$L_c : \mathcal{B} \to \mathcal{B}, L_c(h)(\phi) = h(\phi \circ T),$$

  where $h \in \mathcal{B}$, and ϕ is a test function.)
- We will consider iterates of these operators

  $$L^n_{\omega,c}(f) = L_{\sigma^{n-1}\omega,c} \cdots L_{\omega,c}(f)$$

  which become powers in the deterministic setting

  $$L^n_c(f).$$

- We will consider a weight function $g : X \to (0, \infty)$.
- We assume there exists a Banach space $(\mathcal{B}, \|\cdot\|)$ of complex-valued functions on X such that
- there is $C_0 > 0$ for which

$$\|f \cdot v\| \leq C_0 \|f\| \|v\|, \qquad \|f\|_\infty \leq C_0 \|f\| \tag{4}$$

  for any $f, v \in \mathcal{B}$,
- $\mathcal{B}$ is dense in $L^1(\nu_{\omega,c})$ for any $\omega \in \Omega$,
- $\hat{X}_n, g \in \mathcal{B}$ a.s. for any $n \in \mathbb{N}$ (recall that $\hat{A}$ is the indicator function of A),
- the fiberwise transfer operator $\mathcal{L}_{\omega,c} : \mathcal{B} \to \mathcal{B}$ (associated with g) given by

$$\mathcal{L}_{\omega,c}(f)(x) := \sum_{T_\omega(y)=x} f(y) g_\omega(y), \quad f \in \mathcal{B},\ x \in X_{\sigma\omega} \tag{5}$$

  is well-defined
  for each $\omega \in \Omega$,
- the map

$$(\omega, x) \mapsto (\mathcal{L}_c f)_\omega(x) \tag{6}$$

  is measurable for any $\mathcal{B}$-valued random variable f, where

$$(\mathcal{L}_c f)_\omega := \mathcal{L}_{\sigma^{-1}\omega,c} f_{\sigma^{-1}\omega} \qquad \text{for } \omega \in \Omega.$$

Operators on the open system I

- We now introduce the operators for the open systems

$$L_\omega f = L_{\omega,c}(f 1_{X_{\omega,0}})$$

  and its iterates

$$L^n_\omega f = L^n_{\omega,c}(f 1_{X_{\omega,n-1}})$$

- Suppose one could prove the existence of a random measure $\nu_{\omega,\infty}$ supported on $X_{\omega,\infty}$ and such that (Assumption **A**):
  1. $\lambda_\omega \int f_\omega d\nu_{\omega,\infty} = \int L_\omega f_\omega d\nu_{\sigma\omega,\infty}, \ \lambda_\omega = \int L_\omega 1 d\nu_{\sigma\omega,\infty}$
  2. $L_\omega \phi_\omega = \lambda_\omega \phi_{\sigma\omega}, \ \int \phi_\omega d\nu_{\omega,\infty} = 1.$
  3. $\|L^n_\omega f_\omega - \nu_{\omega,\infty}(f_\omega)\phi_{\sigma^n\omega}\|_\infty \le D\|f\|\kappa^n$
- Then a few interesting results follow, notably
- The measure $\mu_{\omega,\infty} = \phi_\omega \nu_{\omega,\infty}$ is equivariant

$$\int f_{\sigma\omega} \circ T_\omega d\mu_{\omega,\infty} = \int f_{\sigma\omega} d\mu_{\sigma\omega,\infty}$$

- **Assumption A** is basically the quasi-compactness in the deterministic case and the quasi-compactness for random cocycles using MET.

Operators on the open system II

- The measure
$$\eta_\omega(B) = \frac{\nu_{\omega,c}(1_B \phi_\omega 1_{X_{\omega,0}})}{\nu_{\omega,c}(\phi_\omega 1_{X_{\omega,0}})}$$
is conditionally invariant.
- η_ω is the Yaglom limit of any other measure absolutely continuous wrt $\nu_{\omega,c}$ and with positive Radon-Nykodim derivative, class $P_{(\nu_{\omega,c,+})}$
- Each element of $P_{(\nu_{\omega,c,+})}$ has a (conditioning) limit and in particular $\nu_{\omega,\infty}$ is the conditioning limit of $\nu_{\omega,c}$ and $\mu_{\omega,\infty}$ of η_ω
- The previous results were known in a few cases for deterministic systems (see Demers, Young, Liverani, Maume, Todd, etc, they are new for random systems. We now begin to state results which are new in both framework).
- In the case of billiards quoted above, Demers proved that the conditionally invariant measure ν is simply the eigenmeasure of $L = 1_{X_0} L_c(1_{X_0})$
$$L\nu = \lambda_e \nu,$$
where $\nu(X_n) = \lambda_e^n$, λ_e being the escape rate.
- In that case ν is singular wrt to the Lebesgue measure, but it is its Yaglom limit, so the escape rate could be practically computed.

- We remind that $\zeta_{\omega,n}(B) = \frac{\zeta_\omega(B \cap X_{\omega,n})}{\zeta_\omega(X_{\omega,n})}$ and η_ω is the conditionally invariant. Choose $\zeta \in P_{(\nu_{\omega,c},+)}$.
- We already said that $\zeta_{\omega,n}$ has a (conditioning) limit $\zeta_{\omega,\infty}$.

Theorem: Quasi Exponential Mixing

Under **Assumption A**, there exist are $\kappa \in (0,1)$ and a measurable function $C : \Omega \to (0,\infty)$ such that

$$\left| \zeta_{\omega,n}\left(\left(f_{\sigma^k\omega} \circ T_\omega^k\right) v_\omega\right) - \eta_{\sigma^k\omega,n-k}(f_{\sigma^k\omega})\zeta_{\omega,\infty}(v_\omega)\right| \leq \frac{C(\omega)\|\psi_\omega\|\|\phi_\omega\|}{\inf \phi_{\sigma^k\omega} \inf \psi_\omega} \|f_{\sigma^k\omega}\|_\infty \|v\| \kappa^k$$

for every $\zeta \in P_{(\nu_{\omega,c},+)}$ with density ψ, measurable $f : \mathcal{I}_c \to \mathbb{R}$, $v \in \mathcal{B}$, integers $n \geq 1$, $1 \leq k \leq n$ and m-a.e. $\omega \in \Omega$. Moreover, if $\omega \mapsto \log \|f_\omega\|_\infty$ is in $L^1(m)$ then the right-hand side goes to zero exponentially fast as $k, n \to \infty$.

- We should add $\inf \phi_\omega$ is tempered; moreover $\log \|f_\omega\|_\infty$ is also tempered.

New results II

- We could now state the ergodic theorem for our open systems

Theorem: Quasi Strong Law of Large Numbers

Under **Assumption A** Then, $\mu_{\omega,\infty}$ is the unique quasi-ergodic probability measure with respect to $P_{(\nu_{\omega,c},+)}$.
Furthermore, for each $\zeta_\omega \in P_{(\nu_{\omega,c},+)}$ and $\mathcal{B}$-valued random variable $f_\omega : I_\omega \to \mathbb{R}$ with $\|f\|_\infty \in L^1(m)$, there exist sets $\Gamma_{\omega,n} \subseteq X_{\omega,n}$ with $\zeta_{\omega,n}(\Gamma_{\omega,n}) = 1$ a.s. for each $n \geq 0$ such that for m-a.e. $\omega \in \Omega$ and any $\{x_n\}_{n\geq 0} \in \prod_{n=0}^{\infty} \Gamma_{\omega,n}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f_{\sigma^j\omega} \circ T_\omega^j(x_n) = \int f_\omega \, d\mu_\infty .$$

- We now establish a few limit theorems
- We first define the dynamical variance.
- Let $(S_n f)_\omega(x) := \sum_{j=0}^{n-1} f_{\sigma^j\omega} \circ T_\omega^j(x)$.
  Assume that
  $$\mu_{\omega,\infty}(f_\omega) = 0 \quad m\text{-almost surely.}$$
  Then,
  $$(s^2(f))_\omega := \mu_{\omega,\infty}\left(f_\omega^2\right) + 2\sum_{n=1}^{\infty} \mu_{\sigma^{-n}\omega,\infty}\left(f_\omega \circ T^n_{\sigma^{-n}\omega} \cdot f_{\sigma^{-n}\omega}\right)$$
  exists m-almost surely. Set
  $$\Sigma^2(f) := \int s^2(f)\, dm.$$
  Furthermore,
  $\Sigma^2(f) > 0$ if and only if f is not cohomologous to a constant;

Theorem: Quasi Central Limit Theorem

Under **Assumption A**. Let $f : X \to \mathbb{R}$ be a $\mathcal{B}$-valued random variable. Assume that $\mu_{\omega,\infty}(f) = 0$ m-a.s. and $\Sigma^2(f) > 0$. Then, for any $\zeta \in P_{(\nu_{\omega,c},+)}$ and $a \in \mathbb{R}$,

$$\lim_{n\to\infty} \zeta_{\omega,n}\left(\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1} f_{\sigma^j\omega} \circ T_\omega^j \leq a\right) = \frac{1}{\sqrt{2\pi\Sigma^2(f)}}\int_{-\infty}^{a} \exp\left(-\frac{z^2}{2\Sigma^2(f)}\right)\, dz \tag{7}$$

We emphasise that the right-hand-side of (7) is a (non-random) constant.

Theorem: Quasi Large Deviation Principle

Under **Assumption A**. Let $f : X \to \mathbb{R}$ be a $\mathcal{B}$-valued random variable. Assume that $\mu_{\omega,\infty}(f) = 0$ m-a.s. and $\Sigma^2(f) > 0$. Then, for any $\zeta \in P_{(\nu_{\omega,c},+)}$, there is a nonnegative, non-random, continuous, strictly convex function $c : (-\delta_0, \delta_0) \to \mathbb{R}$ with some $\delta_0 > 0$ such that c vanishes at 0 and that for any $\zeta \in P_{\nu_{\omega,c},+}(\nu_c)$ and $\delta \geq 0$,

$$\lim_{n\to\infty} \frac{1}{n} \log \zeta_{\omega,n} \left(\frac{1}{n} \sum_{j=0}^{n-1} f_{\sigma^j\omega} \circ T_\omega^j > \delta \right) = -c(\delta) \quad , \; m - a.s.$$

New results VI

Theorem: Quasi Berry-Essen Theorem

Under **Assumption A**. Let $f : X \to \mathbb{R}$ be a $\mathcal{B}$-valued random variable. Assume that $\mu_{\omega,\infty}(f) = 0$ m-a.s. and $\Sigma^2(f) > 0$. Then for any $\zeta \in P_{(\nu_{\omega,c},+)}$,, there is $C = C_{f,\zeta}$ such that

$$\sup_{a\in\mathbb{R}} \left| \zeta_{\omega,n}\left(\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} f_{\sigma^j\omega} \circ T^j_\omega(\omega,x) \leq a \right) - \frac{1}{\sqrt{2\pi\Sigma^2(f)}} \int_{-\infty}^{a} \exp\left(-\frac{z^2}{2\Sigma^2(f)} \right) \, dz \right| \leq \frac{C}{\sqrt{n}}$$

Twsted operator: the Nagaev-Guivarc'h approach I

- The limit theorem are proved by adapting the Nagaev-Guivarc'h method in terms of the twisted operator to the cocycle of transfer operators.
- This theory, which used an implicit function result of Hennion, was proposed in our two papers:
  1. D. DRAGICEVIC, G. FROYLAND, C. GONZALEZ-TOKMAN, S. VAIENTI, A spectral approach for quenched limit theorems for random expanding dynamical systems, Comm.Math. Phys., (2018).
  2. D . DRAGICEVIC, G. FROYLAND, C. GONZALEZ-TOKMAN, S. VAIENTI, A spectral approach for quenched limit theoems for random hyperbolic dynamical systems, Trans. Amer Math. Soc, (2020).
- We used the quasi-compactness of the cocycle (and therefore the Multiplicative Ergodic Theorem) plus the implicit function theorem .