On language stable subshifts

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Basic topological notions

Definition

Let (X, T) be a topological dynamical system, X a topological space. An automorphism $\phi\colon X \to X$ is an homeomorphism s.t.

$$\phi \circ T = T \circ \phi.$$

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Q: What can we say on $\mathrm{Aut}(X, T)$ as a group? Commutative? Amenable? What are the subgroups? the quotients?...

Q: What do dynamical properties of (X, T) say about properties of $\mathrm{Aut}(X, T)$ and vice versa ?

Q: How does $\mathrm{Aut}(X, T)$ acts on X? On T-invariant measures?

Subshifts

Let A be a finite alphabet.
$A^{\mathbb{Z}}$ endowed with the product topology.
The shift map

$$\begin{array}{rcl} \sigma : A^{\mathbb{Z}} & \to & A^{\mathbb{Z}} \\ (x_n)_{n\in\mathbb{Z}} & \mapsto & (x_{n+1})_{n\in\mathbb{Z}} \end{array}$$

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Similarly

$$X_{\mathcal{F}} = \{(x_n)_n \in A^{\mathbb{Z}}; x_i \cdots x_{i+m} \notin \mathcal{F} \ \forall m, i\}, \text{ where } \mathcal{F} \subset A^*.$$

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Example

- subshift $X_{\mathcal{F}}$ of finite type (SFT): $\mathcal{F}$ is finite.
  Ex $\mathcal{F} = \{11\}$, golden mean shift

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Conjugacies of subshifts are given by cellular automaton.

Curtis-Lyndon-Hedlund

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Pb: find a (generic) family of subshifts with characteristic measure including the mentioned cases.

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Idea: use notion of minimal forbidden word

Béal-Mignosi-Restivo-Sciortino (00)

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The language of X

$$\mathcal{L}(X) = \{x_i \cdots x_j ; x \in X, i < j\}.$$

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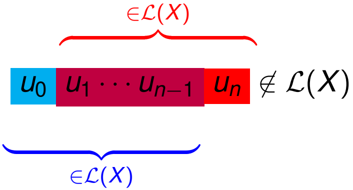

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If $u_0 \cdots u_n$ is a minimal forbidden word of X,

- The word $u_1 \cdots u_{n-1} \in \mathcal{L}(X)$ is the middle of the forbidden word $u_0 u_1 \cdots u_{n-1} u_n$.
- It is a bispecial word: i.e. $\exists a_1 \neq a_2, b_1 \neq b_2 \in A$ s.t.

$$a_1 u_1 \cdots u_{n-1} b_1 \text{ and } a_2 u_1 \cdots u_{n-1} b_2 \in \mathcal{L}(X)$$

Characterization of minimal forbidden words

The extension graph of $u \in \mathcal{L}(X)$ is the bipartite graph $\mathcal{E}(u)$ where

- left vertices are $\{a \in A; au \in \mathcal{L}(X)\}$;
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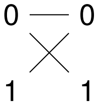
Example

$\mathbf{x} = 01\underline{00101}\overline{00100}\underline{10100}101001001010010\cdots$

$\mathcal{E}(010)$



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Proposition

A word $u \in \mathcal{L}(X)$ is the middle of a minimal forbidden word $\iff$ its bipartite extension graph $\mathcal{E}(u)$ is not complete.

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- For a general subshift X,

$$X = \{(x_n)_n \in A^{\mathbb{Z}}; x_i \cdots x_{i+m} \notin \mathcal{M}(X) \quad \forall m, i\}$$
$$\mathcal{L}(X) = A^* \setminus A^*\mathcal{M}(X)A^*$$

$\mathcal{M}(X)$ uniquely characterizes $\mathcal{L}(X)$.

Definition (Cyr-Kra)

A subshift X is language stable (LS) if the set

$$L\mathcal{M}(X) = \{n \in \mathbb{N}; \mathcal{M}(X) \cap A^n \neq \emptyset\}$$

has a zero lower uniform density, i.e.

$$\lim_{n \to +\infty} \min_{t \geq 0} \frac{1}{n} |L\mathcal{M}(X) \cap \{t+1, \ldots, t+n\}| = 0.$$

Equivalently, the distance between 2 consecutive elements in $L\mathcal{M}(X)$ is unbounded.

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- Sofic shifts (not SFT) are **not** language stable.

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$$L\mathcal{M}(X) \subset \{0, 1, 2^n, 2^n 3 \quad n \in \mathbb{N}\} + 2.$$

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X is well approximated by SFT when X is language stable.

Theorem

The family of language stable subshifts is

- *invariant under conjugacies*

  Béal-Mignosi-Restivo-Sciortino *(00)*

- *generic*

  Cyr-Kra *(21)*

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- $\exists$ non LS subshift with arbitrary entropy.
- $\exists$ non LS subshift with $n \log \log n$ complexity

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The proof based on :

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The proof based on :

- uniform bound on number of special words of a given length
- Fine and Wilf theorem (if X is aperiodic)

This provides the lengths of bispecial words form a zero density set.

On automorphisms of LS subshifts

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Theorem (Cyr-Kra-P)

Assume that X is LS and the gaps in $L\mathcal{M}(X)$ growth fast enough (explicit)
Then for any factor Y of X the $Aut(Y, \sigma)$-action admits an invariant measure:

$$\exists \text{ measure } \mu; \quad \mu(\phi^{-1}(\cdot)) = \mu(\cdot) \quad \forall \phi \in Aut(Y, \sigma).$$

Restrictions on LS subshifts

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If X is irreducible and LS, then $Aut(X, \sigma)$ is a LEF group

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Gordon-Vershik

The group G is Locally Embeddable into Finite groups (LEF) if for every finite set $K \subset G$, there exists a finite group H and a map $\varphi\colon G \to H$ such that the following hold:

1. $\varphi(k_1 k_2) = \varphi(k_1)\varphi(k_2)$ for all $k_1, k_2 \in K$
2. the restriction of φ to K is injective.

LEF	not LEF
$\mathbb{Z}^d, \mathbb{F}_d, \mathbb{Q}$	$\langle a, b; ba^n b^{-1} = a^m\rangle$ $n > m \geq 2$
resid. finite	Thompson group $V \& T$

Restrictions on LS subshifts

$$\mathrm{Aut}(X,\sigma) = \{\phi\colon X \to X; \phi \circ \sigma = \sigma \circ \phi\} \ni \sigma.$$

Theorem (Cyr-Kra-P)

If X is irreducible and LS, then $Aut(X,\sigma)$ is a LEF group

There exists subshifts where $\mathrm{Aut}(X,\sigma)$ is not LEF.