Nonexpansive subspaces and Nivat's conjecture

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Introduction

- Periodic decomposition (Szabados's conjecture)
- Nonexpansiveness
- One-sided nonexpansiveness
- Nivat's conjecture

Let R denote $\mathbb{Z}$ or some finite field and let $\mathcal{A} \subset R$ be a finite alphabet with at least two elements.
We represent any configuration $\eta = (\eta_{\boldsymbol{g}})_{\boldsymbol{g} \in \mathbb{Z}^d}$ as a formal power series over d variables $x_1, \ldots, x_d$ with coefficients in $\mathcal{A}$, that is, as an element of

$$R[[X^{\pm 1}]] = \left\{ \sum_{\boldsymbol{g} \in \mathbb{Z}^d} a_{\boldsymbol{g}} X^{\boldsymbol{g}} : a_{\boldsymbol{g}} \in R \right\},$$

where $\boldsymbol{g} = (g_1, \ldots, g_d)$ and $X^{\boldsymbol{g}}$ is a shorthand for $x_1^{g_1} \cdots x_d^{g_d}$.
Let $R[X^{\pm 1}] \subset R[[X^{\pm 1}]]$ denote the set of Laurent polynomials with coefficients in R.

Given a Laurent polynomial

$$\varphi(X) = a_1 X^{\boldsymbol{u}_1} + \cdots + a_n X^{\boldsymbol{u}_n},$$

with $a_i \in R$ and $\boldsymbol{u}_i \in \mathbb{Z}^d$, and a configuration $\eta \in R[[X^{\pm 1}]]$, we may consider a new configuration $\varphi\eta \in R[[X^{\pm 1}]]$, where

$$(\varphi\eta)_{\boldsymbol{g}} = a_1 \eta_{\boldsymbol{g}-\boldsymbol{u}_1} + \cdots + a_n \eta_{\boldsymbol{g}-\boldsymbol{u}_n} \quad \forall \boldsymbol{g} \in \mathbb{Z}^d.$$

- A Laurent polynomial $\varphi(X) \in R[X^{\pm 1}]$ *annihilates* a configuration $\eta \in R[[X^{\pm 1}]]$ if $\varphi\eta = \mathbf{0}$.
- In this algebraic setting, $\eta \in R[[X^{\pm 1}]]$ is periodic of period $\boldsymbol{h} \in \mathbb{Z}^d$ if and only if $(X^{\boldsymbol{h}} - 1)\eta = \mathbf{0}$.
- If η has d periods linearly independents over $\mathbb{R}^d$, we say that η is fully periodic.

We say that a configuration $\eta \in \mathcal{A}^{\mathbb{Z}^d}$ has *low pattern complexity* if

$$|\{(X^{\boldsymbol{u}}\eta)|_{\mathcal{S}} : \boldsymbol{u} \in \mathbb{Z}^d\}| \leq |\mathcal{S}|$$

holds for some non-empty, finite set $\mathcal{S} \subset \mathbb{Z}^d$. If in addition $\mathcal{S}$ is *convex*, we say that η has *low convex pattern complexity*.

Theorem (Kari and Szabados [14])

Let $\eta \in \mathcal{A}^{\mathbb{Z}^d}$, with $\mathcal{A} \subset \mathbb{Z}$, be a configuration.

1. *If η has low pattern complexity, then η has a non-trivial annihilator.*
2. *If η has a non-trivial annihilator, then there exist periodic configurations $\eta_1, \ldots, \eta_m \in \mathbb{Z}[[X^{\pm 1}]]$ such that $\eta = \eta_1 + \cdots + \eta_m$.*

- Some configurations can not be expressed as a finite sum of periodic configurations defined on finite alphabets.

Let $\eta \in \mathcal{A}^{\mathbb{Z}^d}$, with $\mathcal{A} \subset R$, and suppose $\eta_1, \ldots, \eta_m \in R[[X^{\pm 1}]]$ are periodic configurations such that $\eta = \eta_1 + \cdots + \eta_m$.

- We call $\eta = \eta_1 + \cdots + \eta_m$ an *R-periodic decomposition*.
- An *R-periodic decomposition* $\eta = \eta_1 + \cdots + \eta_m$ where m is the smallest possible is called an *R-minimal periodic decomposition* and the number m is called the *R-order of* η.

We are interested in R-periodic decompositions for $R = \mathbb{Z}_p$ because:

- $(\mathbb{Z}_p)^{\mathbb{Z}^d}$ is a metrizable compact space (with the product topology);
- $\mathbb{Z}_p$-periodic decompositions arise naturally from $\mathbb{Z}$-periodic decompositions.

Let $\eta \in \mathcal{A}^{\mathbb{Z}^d}$, with $\mathcal{A} \subset \mathbb{Z}_+$, and suppose $\eta = \eta_1 + \cdots + \eta_m$ is a $\mathbb{Z}$-minimal periodic decomposition. For $p \in \mathbb{N}$ prime, we may consider the $\mathbb{Z}_p$-periodic decomposition

$$\overline{\eta} = \overline{\eta}_1 + \cdots + \overline{\eta}_m,$$

where the bar denotes the congruence modulo p.
In general, the $\mathbb{Z}_p$-order of $\overline{\eta}$ is lower or equal than the $\mathbb{Z}$-order of η.
However, we have the following result:

Theorem (Colle 2022)

Let $\eta \in \mathcal{A}^{\mathbb{Z}^2}$, with $\mathcal{A} \subset \mathbb{Z}_+$, be a configuration and suppose $\eta = \eta_1 + \cdots + \eta_m$ is a $\mathbb{Z}$-minimal periodic decomposition. Then, there exists $k \in \mathbb{N}$, with $\mathcal{A} \subset \{0, 1, \ldots, k-1\}$, such that $\overline{\eta} = \overline{\eta}_1 + \cdots + \overline{\eta}_m$ is a $\mathbb{Z}_p$-minimal periodic decomposition for all prime number $p \geq k$.

Definition

Let $\eta \in \mathcal{A}^{\mathbb{Z}^2}$ *be a configuration and let* $\ell \subset \mathbb{R}^2$ *be a line through the origin. Given* $t > 0$*, the* t*-neighbourhood of* ℓ *is defined as*

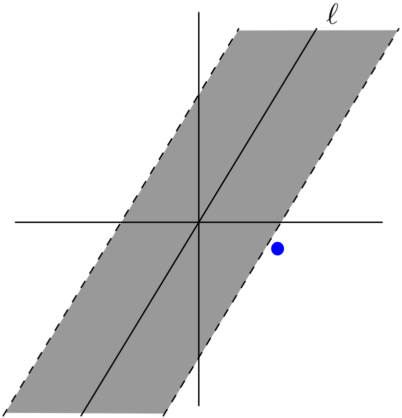
$$\ell^t = \{\boldsymbol{g} \in \mathbb{Z}^2 : \mathrm{dist}(\boldsymbol{g}, \ell) \leq t\},$$

where dist *denotes the Euclidean distance between a point and a set. We say that* ℓ *is an expansive line on* $\overline{Orb\,(\eta)}$ *if there exists* $t > 0$ *such that*

$$\forall\ x, y \in \overline{Orb\,(\eta)}, \quad x|\ell^t = y|\ell^t \implies x = y.$$

Otherwise, ℓ *is called a nonexpansive line on* $\overline{Orb\,(\eta)}$*. The set formed by the nonexpansive lines on* $\overline{Orb\,(\eta)}$ *is denoted* $\mathrm{NEL}(\eta)$*.*

- If $\overline{Orb\,(\eta)}$ is infinite, then $\mathrm{NEL}(\eta)$ has at least one element (Boyle-Lind Theorem [1]).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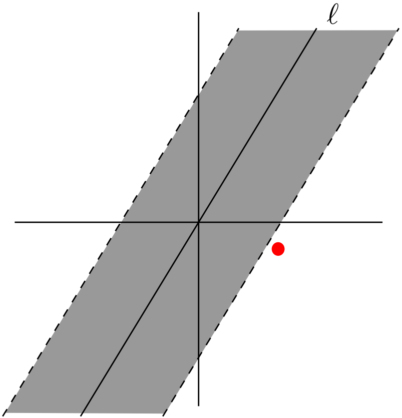
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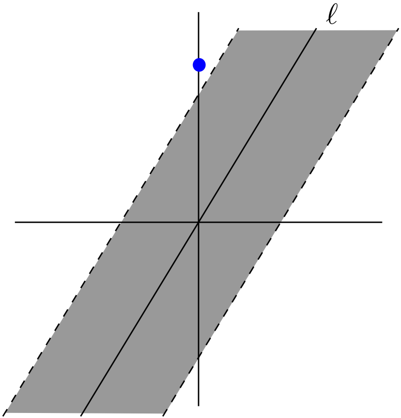
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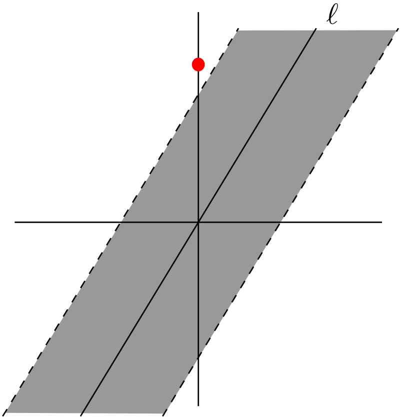
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Why are we interested in nonexpansiveness?

- It is a multifaceted dynamical condition which, in particular, plays an important role in the exploitation of dynamical systems;
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Let $\eta = \eta_1 + \cdots + \eta_m$ be a $\mathbb{Z}$-minimal periodic decomposition and suppose $\boldsymbol{h}_i \in \mathbb{Z}^2$ is a period for η_i, with $1 \leq i \leq m$. Since

$$\varphi(X) = (X^{\boldsymbol{h}_1} - 1) \cdots (X^{\boldsymbol{h}_m} - 1) \in \mathrm{Ann}_{\mathbb{Z}}(\eta),$$

then every nonexpansive line on $\overline{Orb\,(\eta)}$ contains a period of some η_i, with $1 \leq i \leq m$.

In his Ph.D. thesis [21], Michal Szabados conjectured that the converse also holds:

Conjecture (Szabados)

Let $\eta \in \mathcal{A}^{\mathbb{Z}^2}$, with $\mathcal{A} \subset \mathbb{Z}$, be a not fully periodic configuration and suppose $\eta = \eta_1 + \cdots + \eta_m$ is a $\mathbb{Z}$-minimal periodic decomposition. If $\ell \subset \mathbb{R}^2$ is a line through the origin and there exists $1 \leq i \leq m$ such that ℓ contains a period for η_i, then $\ell \in \mathrm{NEL}(\eta)$.

What is known?

Theorem (Colle 2022)

Let $\eta \in \mathcal{A}^{\mathbb{Z}^2}$, with $\mathcal{A} \subset \mathbb{Z}$, be a not fully periodic configuration with low convex pattern complexity and suppose $\eta = \eta_1 + \cdots + \eta_m$ is a $\mathbb{Z}$-minimal periodic decomposition. If $\ell \subset \mathbb{R}^2$ is a line through the origin and there exists $1 \leq i \leq m$ such that ℓ contains a period for η_i, then $\ell \in \mathrm{NEL}(\eta)$.

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What next?

- Szabados's conjecture in the general case (it remains open even for configurations with $\mathbb{Z}$-order 3).

Since the notion of expansiveness can be naturally extended to the multidimensional case, we have the following question:

Question

Let $\eta \in A^{\mathbb{Z}^d}$, with $\mathcal{A} \subset \mathbb{Z}$, be a not fully periodic configuration and suppose $\eta = \eta_1 + \cdots + \eta_m$ is a $\mathbb{Z}$-minimal periodic decomposition. If F is a nonexpansive $(d-1)$-dimensional subspace on $\overline{Orb(\eta)}$, then does F the span of $d-1$ periods of $\eta_1, \ldots, \eta_m$?

- it remains open even when η has low convex pattern complexity.

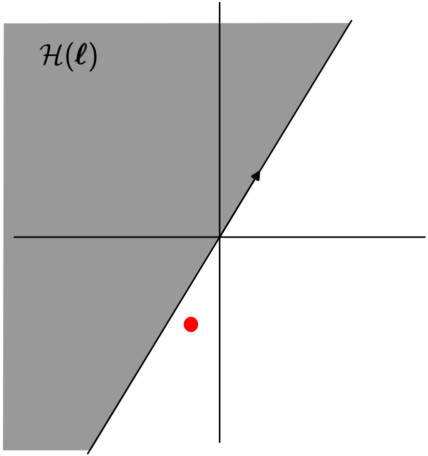
One-sided expansiveness

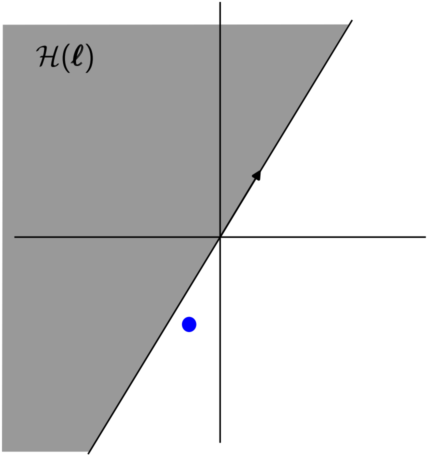
Given a line $\ell \subset \mathbb{R}^2$, we use $\boldsymbol{\ell}$ to denote the line ℓ endowed with a given orientation.

Let $\eta \in \mathcal{A}^{\mathbb{Z}^2}$ be a configuration. An oriented line $\boldsymbol{\ell} \subset \mathbb{R}^2$ through the origin is called a *one-sided expansive direction on* $\overline{Orb(\eta)}$ if

$$\forall\, x, y \in \overline{Orb(\eta)}, \quad x|\mathcal{H}(\boldsymbol{\ell}) = y|\mathcal{H}(\boldsymbol{\ell}) \implies x = y.$$

Otherwise, $\boldsymbol{\ell}$ is called a *one-sided nonexpansive direction on* $\overline{Orb(\eta)}$. We use $\mathrm{ONED}(\eta)$ to denote the set formed by the one-sided nonexpansive directions on $\overline{Orb(\eta)}$.


$\mathcal{H}(\boldsymbol{\ell})$


$\mathcal{H}(\boldsymbol{\ell})$

- By the compactness of $\overline{Orb\,(\eta)}$, a line $\ell \in \mathrm{NEL}(\eta)$ if and only if, for some orientation, $\boldsymbol{\ell} \in \mathrm{ONED}(\eta)$.

Question

Let $\eta \in \mathcal{A}^{\mathbb{Z}^2}$, with $\mathcal{A} \subset \mathbb{Z}$, be a not fully periodic configuration with a non trivial annihilator. Then $\ell \in \mathrm{NEL}(\eta)$ if and only if $-\boldsymbol{\ell}, \boldsymbol{\ell} \in \mathrm{ONED}(\eta)$?

What is known?

- It fails for algebraic configurations (Ledrappie 3-dot system configuration).

Following the same lines in Colle 2022, we have:

Theorem

Let $\eta \in \mathcal{A}^{\mathbb{Z}^2}$ be a not fully periodic configuration with low convex pattern complexity. Then $\ell \in \mathrm{NEL}(\eta)$ if and only if $-\boldsymbol{\ell}, \boldsymbol{\ell} \in \mathrm{ONED}(\eta)$.

- If η has low convex pattern complexity and there exists $\ell \in \mathrm{NEL}(\eta)$ such that $-\boldsymbol{\ell}, \boldsymbol{\ell} \in \mathrm{ONED}(\eta)$, then $\overline{Orb\,(\eta)}$ has a periodic configuration.

This question is related to advances on a problem known as Nivat's conjecture:

Conjecture (Nivat)

If there exist $n, k \in \mathbb{N}$ *such that* $|\{(X^{\boldsymbol{u}}\eta)|_{R_{n,k}} : \boldsymbol{u} \in \mathbb{Z}^2\}| \leq nk$, *then* $\eta \in \mathcal{A}^{\mathbb{Z}^2}$ *is periodic.*

- There has been significant progress towards a proof of Nivat's conjecture.
- It fails in the multidimensional case (Cassaigne 2006)!

What next?

Since the notion of one-sided expansiveness can be naturally extended to the multidimensional case, we have the following question:

Question

What conditions must $\eta \subset A^{\mathbb{Z}^d}$ *satisfy to ensure that a* $(d-1)$*-dimension subspace* F *is nonexpansive on* $\overline{Orb(\eta)}$ *if and only if both half-spaces* F_+ *and* F_- *are one-sided nonexpansive on* $\overline{Orb(\eta)}$*? Is low convex pattern complexity enough?*

Question (Jarkko's question)

Let $D \subset \mathbb{Z}^d$ be a non-empty finite set. If $\mathcal{P} \subset \mathcal{A}^D$ is a finite set of patterns such that $|\mathcal{P}| \leq |D|$, does there exist a $\mathcal{P}$-consistent periodic configuration?

- For $d = 2$, the case of non-convex shapes remains open.
- For $d > 2$, nothing is known (even for convex shapes).

In a work in progress (joint with Eduardo Garibaldi), we have the following result:

Theorem

Let $\eta \in \mathcal{A}^{\mathbb{Z}^2}$, with $\mathcal{A} \subset \mathbb{Z}_+$, be a non-periodic configuration with low convex pattern complexity. Then there exist a $\mathbb{Z}$-minimal periodic decomposition $\vartheta = \vartheta_1 + \cdots + \vartheta_m$, with $\vartheta \in \overline{Orb(\eta)}$ non-periodic, a prime $p \in \mathbb{N}$, with $\mathcal{A} \subset \{0, 1, \ldots, p-1\}$, half planes $U_1, \ldots, U_m \subset \mathbb{Z}^2$ and half planes $V_1, \ldots, V_m \subset \mathbb{Z}^2$, with $U_i \cap V_i = \emptyset$, such that, for each $1 \leq i \leq m$, $\overline{\vartheta}_i|_{U_i}$ and $\overline{\vartheta}_i|_{V_i}$ are fully periodic, where the bar denotes the congruence modulo p.

- In particular, it means that ϑ is fully periodic, except on a finite union of strips.

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