On Phase Transitions to Aperiodic Order in Lattice Systems

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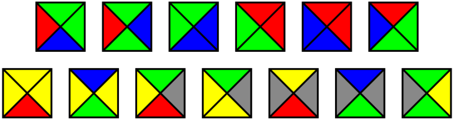
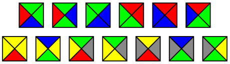


Introduction

- Fundamental problem in Statistical Physics : understand phase transitions between "disorder" and "order".
- **Toy models :** lattice models on $\mathbb{Z}^d$.
- Fundamental example : the 2D Ising model exhibits a ferromagnetic order below a critical temperature $T_c > 0$.
- In this talk : motivated by quasicrystals, we look for phase transitions between disorder and aperiodic long-range order as the inverse temperature $\beta = 1/T$ varies.
- Quasicrystals exhibit long-range order while simultaneously lacking periodicity.

  In this talk : quasicrystal $\approx$ aperiodic subshift.

LATTICE MODELS ON $\mathbb{Z}^d$

- A finite set S, for instance :
  - $S = \{-, +\}$ ("spins");
  - $S = \{0, 1\}$ ("empty/occupied" site);
  - $S = \Big\{$  $\Big\}$.
- Configurations : $\Omega \stackrel{\text{def}}{=} S^{\mathbb{Z}^d} = \{\omega = (\omega_i)_{i \in \mathbb{Z}^d}, \omega_i \in S\}$.
- Shift : $(T^j \omega)_i = \omega_{i+j}$.
- ▸ (Ω, T) d-dimensional full shift on "alphabet" S.
- Interaction (statisical physics) : $\Phi = (\Phi_\Lambda)_{\Lambda \Subset \mathbb{Z}^d}$, where $\Phi_\Lambda : \Omega \to \mathbb{R}$ is continuous.
- Potential (dynamical systems) : $\varphi : \Omega \to \mathbb{R}$ continuous.

Φ AND φ, GIBBS STATES AND EQUILIBIRUM STATES

- Interactions used to define Gibbs states, potentials used to define equilibrium states.
- From Φ to φ :

$$\varphi = \sum_{\Lambda \Subset \mathbb{Z}^d,\, 0 \in \mathbb{Z}^d} \frac{\Phi_\Lambda}{|\Lambda|} .$$

- From φ to Φ : easy for finite-range interactions/locally constant potentials, but messy in general.
- Today : I will not speak at all about Gibbs states.
- If $\sum_{\Lambda \Subset \mathbb{Z}^d,\, 0 \in \Lambda} \|\Phi_\Lambda\|_\infty < +\infty$ then { *shift-invariant* Gibbs states } = { equilibrium states}.
- Statistical physics versus dynamical systems :

$$\sum_{\Lambda \Subset \mathbb{Z}^d,\, 0 \in \Lambda} \|\Phi_\Lambda\|_\infty < +\infty \quad \text{"} \longleftrightarrow \text{"} \quad \sum_{n \geq 1} n^{d-1} \mathrm{var}_n \varphi < \infty$$

Equilibrium States

Given $\varphi : \Omega \to \mathbb{R}$ continuous and an inverse temperature β, a shift-invariant probability measure that maximize

$$\nu \mapsto h(\nu) + \beta \int \varphi \, \mathrm{d}\nu$$

is called an equilibrium state of φ.

Two special cases :

- $\beta = 0$. Then $\exists!$ eq. state which is the measure of maximal entropy on Ω (maximal disorder).
- $\beta \to +\infty$ (temperature going to 0). Intuitively, only φ matters, and configurations should be somewhat ordered.

Remark : Today, I will not say a lot of things about zero-temperature limits of equilibrium states.

If $\mu_{\beta\varphi}$ is an equilibrium state of φ, then

$$p_\varphi(\beta) := h(\mu_{\beta\varphi}) + \beta \int \varphi \, \mathrm{d}\mu_{\beta\varphi}$$

is the topological pressure of $\beta\varphi$, and

$$\beta \mapsto p_\varphi(\beta), \ \beta \in \mathbb{R} \,,$$

defines a continuous convex function (the pressure function).

The pressure function has a slant asymptote when $\beta \to +\infty$, with slope $\sup_{\nu \text{ shift-invariant}} \int \varphi \, d\nu$.

A μ attaining this maximum is called a maximizing measure for φ.

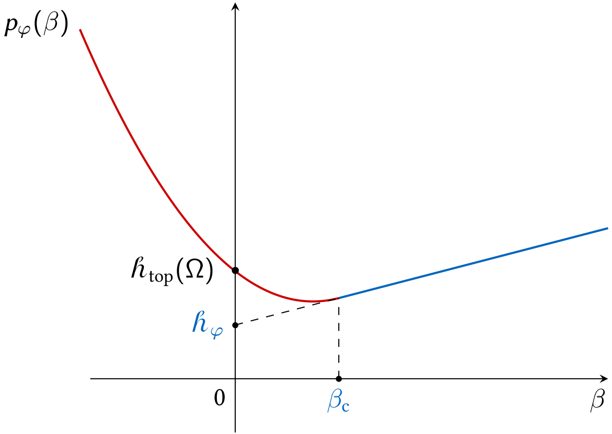
Denote by $\mathrm{ES}(\beta\varphi)$ the set of equilibrium states of $\beta\varphi$.

Folklore :

$$\{\mu \colon \exists \beta_n \to \infty,\ \mu_n \in \mathrm{ES}(\beta_n\varphi) \text{ with } \mu_n \rightsquigarrow \mu\}$$

$$\subseteq \{\text{maximizing measures for } \varphi\}$$

where $\rightsquigarrow$ denotes convergence in the vague topology.



where $h_\varphi = \sup\left\{ h(\eta) \,:\, \eta \text{ is maximizing for } \varphi \right\}$ and the slope of the blue part is $\sup_{\nu \text{ shift-invariant}} \int \varphi \,d\nu$.

(ES($\beta\varphi$) is the set of equilibrium states of $\beta\varphi$.) We focus on $\beta \geq 0$.

Definition :

A freezing phase transition occurs for φ at β_c if

- $\mathrm{ES}(\beta\varphi) = \mathrm{ES}(\beta'\varphi)$ for all $\beta, \beta' > \beta_c$,
- $\mathrm{ES}(\beta\varphi) \neq \mathrm{ES}(\beta'\varphi)$ for all $\beta < \beta_c < \beta'$.

Theorem

A freezing phase transition occurs for φ at β_c if and only if $p_\varphi(\beta)$ is as in the previous picture.

Let us reverse the perspective : we are given a subshift Ω_0 of Ω and we look for a φ that exhibits a freezing phase transition.

Definition :
Let Ω_0 be a proper subshift of Ω. A continuous potential $\varphi\colon\,\Omega\to\mathbb{R}$ is said to *exhibit freezing on* Ω_0 if it has a freezing phase transition at some β_{c} *and* Ω_0 is the smallest subshift which contains the supports of all measures in $\{\mu\colon \exists\beta_n\to\infty,\ \mu_n\in \mathrm{ES}(\beta_n\varphi)$ with $\mu_n\rightsquigarrow\mu\}$.

Theorem (J.-R. C., Tamara Kucherenko, Anthony Quas, arXiv 2025)
For any proper subshift Ω_0 of Ω, one can construct a continuous potential φ that exhibits freezing on Ω_0 for some $\beta_{\mathrm{c}}>0$.
Moreover, for all $\beta>\beta_{\mathrm{c}}$, $\mathrm{ES}(\beta\varphi)$ is the set of measures of maximal entropy on Ω_0.

Previous results

- All previous results were only in dimension 1 ($d = 1$).
- The first result is Hofbauer's example with a φ freezing on 0^∞ at $\beta_c = 1$ ($\Omega = \{0, 1\}^{\mathbb{N}}$).
- For instance, in an ongoing work, Bédaride, Cassaigne, Hubert and Leplaideur construct a φ which freezes on the Thue-Morse subshift. Their method relies on the properties of that subshift and uses Ruelle's Perron-Frobenius operator.
- There is a *non-constructive* result by Buzzi et al. where Ω_0 can be any subshift with zero topological entropy.

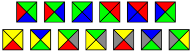
A class of potentials that cannot freeze

Theorem (J.-R. C., Tamara Kucherenko, Anthony Quas, arXiv 2025)
Let $\varphi : \Omega \to \mathbb{R}$ be a continuous potential.
If $\sum_n n^{d-1} \operatorname{var}_n \varphi < \infty$ then φ cannot exhibit a freezing phase transition.

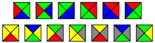
(Where $\operatorname{var}_n(\varphi) = \sup\{|\varphi(\omega) - \varphi(\omega')| : x, y \in X, \operatorname{dist}(\omega, \omega') \leq 2^{-n}\}$.)

An example of “quasi-crystal” $(d = 2)$: Kari-Culik tiling

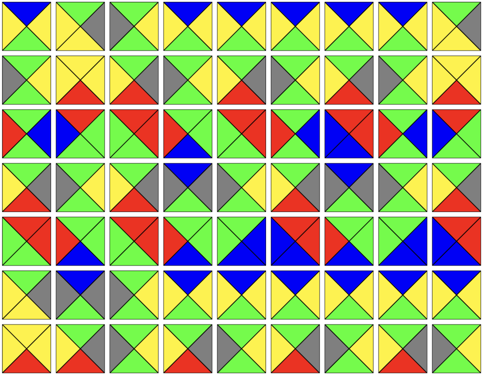
Take

$$S = \{$$  $$\}.$$

Place a copy of any of one of these 1×1 squares centered at $(i, j) \in \mathbb{Z}^2$, without rotating them.

Thus $\Omega = \{$  $\}^{\mathbb{Z}^2}$ is a two-dimensional full shift on 13 “symbols”.

What happens if we select *only* the configurations such that **the colors of the adjacent edges match**?

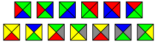


A portion of a Kari-Culik tiling.

(The spaces are only there to improve visualization.)

Facts About Kari-Culik Tilings

- Using the Kari-Culik tiles and following the color-matching rules, *one can tile the plane and there are no periodic configurations.*

- The set of all Kari-Culik tilings is a subshift of finite type of
$\Big\{$  $\Big\}^{\mathbb{Z}^2}$ (closed and invariant under the shift action).

This defines an aperiodic subshift of finite type (which has positive topological entropy).

Our theorem applies : there is a continuous φ freezing on the Kari-Culik subshift of finite type at some $\beta_c > 0$.

Remark : There are other examples of aperiodic tilings, constructed using other methods (Ammann, Jeandel-Rao, Labbé, etc.), which are also Wang tilings (dominoes), and have zero topological entropy.

Some open problems

- Is there a version of Conze-Guivarc'h theorem for $d \geq 2$?
- Support stability of maximzing measures : striking difference between $d = 1$ and $d \geq 2$ for "penalty" potentials and subshifts of finite type. (See Gonschorowski, Anthony Quas and Siefken.)
  Open question : find a stable quasicrytal in dimension $d \geq 2$.
- Genericity questions in dimension $d \geq 2$ for Lipschitz/Hölder potentials.
  For instance, is there an appropriate version of Conteras result in higher dimension?

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