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THERMOGAMAS webinar

Building thermodynamic formalism for hyperbolic random dynamical systems

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Joint work with Lucas Amorim (Université de Toulon), Benoît Saussol (Aix Marseille Université) and Sandro Vaienti (Université de Toulon)
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- $\sigma : \Omega \to \Omega$ is a homeomorphism, Ω is a compact metric space and $(\sigma, \mathbb{P})$ is ergodic.
- We also consider the projection $\pi(\omega, x) = \omega$.

A simple example

Example: Consider

$$\Theta : \Omega \times \mathbb{T}^2 \to \Omega \times \mathbb{T}^2$$
$$(\underline{\omega}, x) \mapsto (\sigma(\underline{\omega}), Ax + \omega_0)$$

- $\Omega = (\overline{B_\delta(0)})^{\mathbb{Z}}$, $\overline{B_\delta(0)}$ = closed ball of radius $\delta > 0$ centred at 0;
- $\underline{\omega} = (\omega_i)_{i\in\mathbb{Z}}$, $(\sigma\underline{\omega})_i = \omega_{i+1}$;
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- **Problem:** $h_{\text{top}}(\sigma) = \infty$ and $h_{\text{top}}(\sigma) \leq h_{\text{top}}(\Theta)$ since σ is the full shift on an uncountable alphabet.
- **First attempt:** fix a σ-invariant $\mathbb{P}$ and consider

$$\sup\{\, h_\mu(\Theta) : \ \pi_*\mu = \mathbb{P} \,\} \ \geq \ h_{\mathbb{P}}(\sigma).$$

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- Solution: apply a *random Brin–Katok theorem* to obtain finite local (quenched) entropy.

Relative entropy (Brin–Katok) and $\mathbb{P}$-relative equilibrium

Random Brin–Katok:

- $(\Theta; \sigma, \mathbb{P})$ RDS, $\Theta : \Omega \times M \to \Omega \times M$, $\Theta(\omega, x) = (\sigma\omega, T_\omega)$.
- μ is Θ-invariant, $\pi_*\mu = \mathbb{P}$ and $\mu = \mu_\omega(\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega)$.

Recall: $\pi(\omega, x) = \omega$. Ch. 5 of B. Hasselblatt and A. Katok, eds. *Handbook of dynamical systems. Vol. 1B*. Elsevier B. V., Amsterdam, 2006, pp. xii+1222.

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Random Bowen Ball: $B_n^\omega(x, \varepsilon) := \{y : \mathrm{dist}(T_\omega^k(y), T_\omega^k(x)) < \varepsilon,\ 0 \leq k < n\},\ T_\omega^k = T_{\sigma^{k-1}\omega} \circ \cdots \circ T_\omega$.

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$\mathbb{P}$-relative entropy: $h_\mu(\Theta\mid\mathbb{P}):=-\iint\lim_{\varepsilon\to0}\lim_{n\to\infty}\frac{1}{n}\log\mu_\omega\big(B_n^\omega(x,\varepsilon)\big)\,\mathrm{d}\mu_\omega(x)\,\mathrm{d}\mathbb{P}(\omega).$

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Definition ($\mathbb{P}$-relative equilibrium state)

Let $\mathcal{M}_{\mathbb{P}}(\Theta):=\{\mu:\ \mu$ is Θ-invariant and $\pi_*\mu=\mathbb{P}\}$. For measurable $\phi:\Omega\times M\to\mathbb{R}\cup\{-\infty\}$, we define the $\mathbb{P}$-relative topological pressure of ϕ as

$$P_{\mathrm{top}}(\phi\mid\mathbb{P})=\sup_{\nu\in\mathcal{M}_{\mathbb{P}}(\Theta)}\Big\{h_\nu(\Theta\mid\mathbb{P})+\int\phi\,\mathrm{d}\nu\Big\}.$$

A maximizer $\mu\in\mathcal{M}_{\mathbb{P}}(\Theta)$ is a $\mathbb{P}$-relative equilibrium state of ϕ; If $\phi\equiv 0$ we call μ a measure of maximal $\mathbb{P}$-relative entropy.

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Previous results

Random subshifts of finite type:

- Bogenschütz and Gundlach (1995)
- Mauldin and Urbański (2001)

Random countable topological Markov shifts:

Denker, Kifer, and Stadlbauer (2008)

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Random uniformly expanding maps:

- Kifer (1992)
- Baladi (1997)

Random non-uniformly expanding (without critical points):

- Arbieto, Matheus, and Oliveira (2004)
- Stadlbauer, Suzuki, and Varandas (2021)

Random non-uniformly hyperbolic interval maps (closed/open; discontinuities):

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Small perturbations of Axiom A diffeos:

- P.-D. Liu (1998)
- Chapter 5 of the book "Handbook of Dynamics" Hasselblatt and Katok (2006), written by Kifer and P.-D. Liu.

Hypothesis (H) and (H')

$$\Theta : \Omega \times M \to \Omega \times M$$
$$(\omega, x) \mapsto (\sigma\omega,\, T_\omega(x))$$

Ω: compact metric space
M: surface

$T_\omega : M \to M$
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(H1) (Hyperbolicity)

(H2) (Fibrewise mixing)

(H2') (Rapid fibrewise mixing)

(H) = (H1) + (H2) and **(H') = (H1) + (H2')**

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(H1) (Hyperbolicity) There exist family of deterministic cones $\mathcal{C} = \big(\mathcal{C}^-(x), \mathcal{C}^+(x)\big)_{x\in M} \subset TM$ s.t. for each $x \in M$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\mathrm{D}T_\omega^{-1}(x)\,\mathcal{C}^+(x) \subset \mathring{\mathcal{C}}^+\big(T_\omega^{-1}x\big),\; \mathrm{D}T_\omega(x)\,\mathcal{C}^-(x) \subset \mathring{\mathcal{C}}^-\big(T_\omega x\big).$$

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(H2) (Fibrewise mixing) Let $\delta > 0$, and $\{B_\delta(x_i)\}_{i=1}^k$ an open cover of M. Define

$$\mathcal{T}_{\delta,\mathrm{mix}}(\omega) = \mathcal{T}_{\mathrm{mix}}(\omega) := \min\left\{n \in \mathbb{N};\ T_\omega^n(B_\delta(x_i)) \cap B_\delta(x_j) \neq \emptyset,\ \forall 1 \leq i,j \leq k\right\}.$$

For every $\delta > 0$ there exists a constant $A = A(\delta) \geq 1$, satisfying

$$\mathbb{P}[\omega \in \Omega;\ \mathcal{T}_{\mathrm{mix}}(\omega) \leq A] > 0.$$

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(H2') (Rapid fibrewise mixing) For each $\delta > 0$, there exist $K = K(\delta), \kappa = \kappa(\delta) > 0$ s.t.

$$\mathbb{P}[\omega \in \Omega; \mathcal{T}_{\mathrm{mix}}(\omega) > n] \le K e^{-\kappa n} \text{ for every } n \in \mathbb{N}.$$

(H) = (H1) + (H2) and **(H') = (H1) + (H2')**

Examples

The two examples below satisfy Hypothesis **(H')**

(1) Let $A_1, \ldots, A_k \in \mathrm{SL}(2, \mathbb{N})$ and $\mathrm{Tr}(A_i) > 2$ for $i \in \{1, \ldots, k\}$.

$$\begin{aligned}\Theta : \{1, \ldots, k\}^{\mathbb{Z}} \times \mathbb{T}^2 &\to \{1, \ldots, k\}^{\mathbb{Z}} \times \mathbb{T}^2 \\ (\underline{\omega}, x) &\mapsto (\sigma\underline{\omega}, A_{\omega_0}(x))\end{aligned}$$

where σ is the left shift and $\mathbb{P}$ is a Gibbs state of σ.

(2) Let $T_1, \ldots, T_k$ are Anosov diffeomorphisms in $\mathbb{T}^2$ preserving the same deterministic family of cones in $T\mathbb{T}^2$

$$\begin{aligned}\Theta : \{1, \ldots, k\}^{\mathbb{Z}} \times \mathbb{T}^2 &\to \{1, \ldots, k\}^{\mathbb{Z}} \times \mathbb{T}^2 \\ (\underline{\omega}, x) &\mapsto (\sigma\underline{\omega}, T_{\omega_0}(x))\end{aligned}$$

where σ is the left shift and $\mathbb{P}$ is Bernoulli.

Results

Theorem 1 (Amorim-C.-Sassoul-Vaienti, 2025+)

Let $(\Theta;\sigma,\mathbb{P})$ *be a random dynamical system satisfying* **(H)** *and* $\phi:\Omega\times M\to\mathbb{R}$ *be a random potential such that*

$$\operatorname*{ess\,sup}_{\omega\in\Omega}\|\phi(\omega,\cdot)\|_{\mathcal{C}^\alpha}<\infty$$

for some $\alpha>0$. *Then, there exists a unique* $\mathbb{P}$*-relative equilibrium state* $\nu\in\mathcal{M}_{\mathbb{P}}(\Theta)$ *of* ϕ.

Theorem 2 (Amorim-C.-Saussol-Vaienti, 2025+)

Let $(\Theta;\sigma,\mathbb{P})$ *be a random dynamical system satisfying* **(H')**, *and* $\phi:\Omega\times M\to\mathbb{R}$ *as in Theorem 1. Then* $\nu(\mathrm{d}\omega,\mathrm{d}x)=\nu_\omega(\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega)$ *exhibits quenched exponential decay of correlations for Hölder observables: For every* $p\in[1,\infty)$, $\exists C_p\in L^p(\Omega,\mathbb{P}),\alpha>0$ *such that for any* $f,g\in\mathcal{C}^\alpha$, $\forall n\in\mathbb{N}$

$$\left|\int_M f\circ T_\omega^n\cdot g\,\mathrm{d}\nu_\omega-\int_M f\,\mathrm{d}\nu_{\sigma^n\omega}\int_M g\,\mathrm{d}\nu_\omega\right|\le C_p(\omega)e^{-\alpha_0 n}\|f\|_{\mathcal{C}^\alpha}\|g\|_{\mathcal{C}^\alpha},$$

for $\mathbb{P}$*-a.e.* $\omega\in\Omega$

Results

Theorem 3 (Quenched CLT)

Let $(\Theta;\sigma,\mathbb{P})$ *satisfy* **(H')**, *let* $\phi:\Omega\times M\to\mathbb{R}$ *be as in Theorem 1, and let* $\nu=\nu_\omega\,\mathbb{P}(\mathrm{d}\omega)$ *be the* $\mathbb{P}$*-relative equilibrium state. Then* ν *satisfies the quenched CLT for any observable* $f:\Omega\times M\to\mathbb{R}$ *such that*

$$\operatorname*{ess\,sup}_{\omega\in\Omega}\|f(\omega,\cdot)\|_{\mathcal{C}^\alpha}<\infty.$$

Set

$$\tilde{f}(\omega,x):=f(\omega,x)-\int_M f(\omega,\cdot)\,\mathrm{d}\nu_\omega,\ \ S_n^\omega\tilde{f}(x):=\sum_{k=0}^{n-1}\tilde{f}\big(\sigma^k\omega,T_\omega^k(x)\big).$$

There exists $\Sigma^2\geq 0$*, independent of* ω*, such that for* $\mathbb{P}$*-a.e.* ω*,*

$$\frac{1}{\sqrt{n}}S_n^\omega\tilde{f}\xrightarrow[]{n\to\infty}\mathcal{N}(0,\Sigma^2),$$

Moreover, $\tilde{f}$ *is not a quenched coboundary, i.e.,*

it does not exist $\psi\in L^2(\mathbb{P}\otimes\nu)$ *such that* $\tilde{f}=g\circ\Theta-g$ *for* $\mathbb{P}$*-a.e.* ω, ν_ω*-a.e.* x.

then $\Sigma>0$.

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(i) Random Banach spaces: One finds an appropriate family of Banach spaces $(\mathcal{B}_\omega)_{\omega\in\Omega}$ such that $\mathcal{C}^\alpha$ is dense in $\mathcal{B}_\omega$ and

$$\mathcal{L}_\omega : \mathcal{B}_\omega \to \mathcal{B}_{\sigma\omega},\ \mathcal{L}_\omega(f) := (e^{\phi_\omega} f) \circ T_\omega^{-1}\ (\forall f \in \mathcal{C}^\alpha),$$

is bounded, where $\phi_\omega(\cdot) := \phi(\omega,\cdot)$. We also ask $\|\cdot\|_{\mathcal{B}_\omega} \le \|\cdot\|_{\mathcal{C}^\alpha}$

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such that $\mathcal{L}_\omega(\mathcal{C}_\omega) \subset \mathcal{C}_{\sigma\omega}$ and $\mathrm{span}\{\mathcal{C}_\omega\} = \mathcal{B}_\omega$.

(iii) Cone contraction: For $\mathbb{P}$-a.e. $\omega \in \Omega$ there exist random variables

$$\{\tau_i : \Omega \to \mathbb{N}\}_{i\in\mathbb{N}} \text{ with } \tau_1 \leq \tau_2 \leq \tau_3 \leq \dots$$

and constants $\Delta > 0$ and $\chi \in (0, 1)$ (independent of ω) such that, for all $k \geq 1$,

$$\sup_{f,g\in\mathcal{C}_{\sigma^{-\tau_k}\omega}} \mathrm{Hil}_{\mathcal{C}_\omega}\Big(\mathcal{L}^{(\tau_k)}_{\sigma^{-\tau_k}\omega} f,\ \mathcal{L}^{(\tau_k)}_{\sigma^{-\tau_k}\omega} g\Big) \leq \Delta\, \chi^{k-1};\ \Big(\text{recall that } \mathcal{L}^{(\tau_k)}_{\sigma^{-\tau_k}\omega}\mathcal{C}_{\sigma^{-\tau_k}\omega} \subset \mathcal{C}_\omega\Big),$$

where $\mathrm{Hil}_{\mathcal{C}_\xi}$ is the projective Hilbert metric on $\mathcal{C}_\xi$ and $\mathcal{L}^{(n)}_\xi := \mathcal{L}_{\sigma^{n-1}\xi} \circ \cdots \circ \mathcal{L}_\xi$.

For a cone $\mathcal{C}$ with order $\leq_\mathcal{C}$, set $\alpha(f, g) := \sup\{\lambda > 0 : \lambda g \leq_\mathcal{C} f\}$, $\beta(f, g) := \inf\{\mu > 0 : f \leq_\mathcal{C} \mu g\}$, and $\mathrm{Hil}_\mathcal{C}(f, g) := \log \frac{\beta(f,g)}{\alpha(f,g)}$.

Right and left eigenvectors, and spectral gap

Assume cones $\mathcal{C}_\omega \subset \mathcal{B}_\omega$ and dual cones $\mathcal{C}_\omega^* \subset \mathcal{B}_\omega^*$ and that (i)–(iii) from the previous slide hold. Then via standard arguments, for $\mathbb{P}$-a.e. ω we have that:

Adaptation of the Birkhoff cones arguments to the Liverani, Saussol, and Vaienti (1998) to the random context

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Right random eigenvector: recall that $\mathcal{L}_\omega : \mathcal{C}_\omega \to \mathcal{C}_{\sigma\omega}$, there exists $\mu_\omega \in \mathcal{C}_\omega$ and $\lambda_\omega > 0$ satisfying

$$\mu_\omega := \lim_{n\to\infty} \frac{\mathcal{L}^n_{\sigma^{-n}\omega}\mathbb{1}}{\|\mathcal{L}^{(n)}_{\sigma^{-n}\omega}\mathbb{1}\|_{\mathcal{B}_\omega}} \in \mathcal{C}_\omega, \ \mathcal{L}_\omega\mu_\omega = \lambda_\omega\mu_{\sigma\omega}.$$

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Spectral gap:

$$\mathcal{L}_\omega^{(n)} g = \lambda_\omega^{(n)}\,\ell_\omega(g)\,\mu_{\sigma^n\omega} + \lambda_\omega^{(n)} Q_\omega^{(n)}(g), \quad \|Q_\omega^{(n)}\|_{\mathcal{B}_\omega\to\mathcal{B}_{\sigma^n\omega}} \le C\,\chi^{N_\tau(\omega,n)},$$

where

$$\lambda_\omega^{(n)} := \prod_{j=0}^{n-1}\lambda_{\sigma^j\omega} \text{ and } N_\tau(\omega,n) = \#(\{\tau_i(\sigma^n\omega)\}_{i\in\mathbb{N}} \cap [1,n])$$

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Natural candidates

For each $f \in \mathcal{C}^\alpha(M)$, define $\nu_\omega(f) = \ell_\omega(f \cdot \mu_\omega)$. Observe that ν_ω is equivariant:

$$\nu_\omega(f \circ T_\omega) = \ell_\omega(f \circ T_\omega \cdot \mu_\omega) = \frac{1}{\lambda_\omega}(\mathcal{L}_\omega^*)\,\ell_{\sigma\omega}(f \circ T_\omega \cdot \mu_\omega) = \frac{1}{\lambda_\omega}\,\ell_\omega(f \cdot \mathcal{L}_\omega \mu_\omega) = \nu_{\sigma\omega}(f).$$

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Also, choosing $f, g \in \mathcal{C}^\alpha(M)$ we have

$$\begin{aligned}\nu_\omega(f \circ T_\omega^n \cdot g) &= \ell_\omega(f \circ T_\omega^n \cdot g \cdot \mu_\omega) = \frac{1}{\lambda_\omega^{(n)}}\,(\mathcal{L}_\omega^*)^{(n)}\ell_{\sigma^n\omega}(f \circ T_\omega^n \cdot g \cdot \mu_\omega) = \frac{1}{\lambda_\omega^{(n)}}\,\ell_{\sigma^n\omega}\left(f \cdot \mathcal{L}_\omega^{(n)}(g \cdot \mu_{\sigma^n\omega})\right) \\ &= \ell_{\sigma^n\omega}\left(f \cdot \ell_\omega(g \cdot \mu_\omega)\,\mu_\omega + f \cdot Q_\omega^{(n)}(g\mu_\omega)\right) = \nu_{\sigma^n\omega}(f)\,\nu_\omega(g) + \mathcal{O}\left(\left\|Q_\omega^{(n)}\right\|_{B_\omega \to B_{\sigma^n\omega}}\right).\end{aligned}$$

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Recall that $\|Q_\omega^{(n)}\|_{\mathcal{B}_\omega \to \mathcal{B}_{\sigma^n\omega}} \leq C\,\chi^{N_\tau(\omega,n)}$. In particular

$$\left|\int_M f \circ T_\omega^n \cdot g \, d\nu_\omega - \int_M f \, d\nu_{\sigma^n\omega} \int_M g \, d\nu_\omega\right| \leq C\chi^{N_\tau(\omega,x)} \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\alpha}.$$

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If one manages to do such a construction:

- $\nu = \nu_\omega(\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega)$ is a candidate for $\mathbb{P}$-relative equilibrium state for the potential

$$\phi + \log\left|\det\left(\mathrm{D}T_\omega|_{E^s(\omega,x)}\right)\right|.$$

- $\int \log \lambda_\omega \mathbb{P}(\mathrm{d}\omega)$ is a candidate for its $\mathbb{P}$-relative topological pressure.

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Main challenge: There is no thermodynamic formalism for *diffeomorphisms* based on Birkhoff cones. Existing results (Baladi–Tsujii, 2007; Gouëzel–Liverani, 2008) rely on spectral methods (anisotropic Banach spaces) instead.

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(C1) Positivity along leaves:

$$\int_{\gamma_\omega} \varphi(x)\, \rho_\omega(x)\, \mathrm{d}m_{\gamma_\omega}(x) > 0.$$

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(C2) Log–Hölder in the density:

$$\frac{\int_{\gamma_\omega} \varphi\,\rho_\omega\,\mathrm{d}m_{\gamma_\omega}}{\int_{\gamma_\omega} \varphi\,\varsigma_\omega\,\mathrm{d}m_{\gamma_\omega}} \le e^{b\mathrm{Hil}_{\mathcal{D}_\omega(\gamma_\omega)}(\rho_\omega,\varsigma_\omega)}$$

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(C3) Hölder across leaves (holonomy): for holonomy-related $(\widetilde{\gamma}_\omega, \gamma_\omega)$, if $\widetilde{\rho}_\omega$ is the transport/normalisation of ρ_ω,

$$\frac{\int_{\widetilde{\gamma}_\omega} \varphi\, \widetilde{\rho}_\omega\, \mathrm{d}m_{\widetilde{\gamma}_\omega}}{\int_{\gamma_\omega} \varphi\, \rho_\omega\, \mathrm{d}m_{\gamma_\omega}} \le e^{c\left(d_u(\widetilde{\gamma}_\omega, \gamma_\omega)\right)^\eta}$$

Cone definition

We define $\mathcal{C}_\omega$ as the closure of $C_\omega \subset \{\varphi : M \to \mathbb{R};\ \varphi \text{ bounded and measurable}\}$ with respect to the norm

$$\|f\|_\omega := \|f\|^{\sup_s}_{\omega,a,\kappa} + \frac{1}{b}\|f\|^{\Theta_s}_{\omega,a,\kappa} + \frac{1}{c}\|f\|^{d_u}_{\omega,\eta},$$

where

$$\|f\|^{\sup_s}_{\omega,a,\kappa} := \sup_{\gamma_\omega \in \Gamma^s_\delta(\omega)} \sup_{\substack{\rho_\omega \in \mathcal{D}_\omega(\gamma_\omega) \\ \int_{\gamma_\omega} \rho_\omega \, dm_{\gamma_\omega} = 1}} \left| \int_{\gamma_\omega} f \rho \, dm_{\gamma_\omega} \right|,$$

$$\|f\|^{\Theta_s}_{\omega,a,\kappa} := \sup_{\gamma \in \Gamma^s_\delta(\omega)} \sup_{\substack{\rho^1_\omega, \rho^2_\omega \in D_\omega(\gamma_\omega) \\ \int_{\gamma_\omega} \rho^1_\omega \, dm_{\gamma_\omega} = 1 \\ \int_{\gamma_\omega} \rho^2_\omega \, dm_{\gamma_\omega} = 1}} \frac{\left| \int_{\gamma_\omega} f \rho^1_\omega \, dm_{\gamma_\omega} - \int_{\gamma_\omega} f \rho^2_\omega \, dm_{\gamma_\omega} \right|}{\mathrm{Hil}_{\mathcal{D}_{\omega(\gamma_\omega)}}(\rho^1_\omega, \rho^2_\omega)},$$

$$\|f\|^{d_u}_{\omega,\nu,a,\kappa_1} := \sup_{\substack{(\gamma_\omega, \tilde{\gamma}_\omega) \in \Gamma^s_\delta(\omega) \times \Gamma^s_\delta(\omega) \\ \text{nearby pair}}} \sup_{\rho \in \mathcal{D}_\omega(\gamma_\omega)} \frac{\left| \int_{\gamma_\omega} f \rho_\omega \, dm_{\gamma_\omega} - \int_{\tilde{\gamma}_\omega} f \tilde{\rho}_\omega \, dm_{\tilde{\gamma}_\omega} \right|}{d_u(\gamma_\omega, \tilde{\gamma}_\omega)^\eta}.$$

And consider $\mathcal{B}_\omega := \mathrm{span}(\mathcal{C}_\omega)$.

The general strategy required us to

(i) Construct random Banach spaces $\mathcal{B}_\omega$, (ii) Construct random cones $\mathcal{C}_\omega$

(iii) Construct stopping times that generate cone contractions, i.e. existence of random variables $\tau_1 \leq \tau_2 \leq \ldots : \Omega \to \mathbb{N}$, satisfying $\mathrm{diam}_{\mathrm{Hil}_{\mathcal{C}_\omega}} \left(\mathcal{L}^{(\tau_m)}_{\theta^{-\tau_m}\omega} \mathcal{C}_{\theta^{-\tau_m}\omega} \right) \leq \Delta \chi^{m-1}$.

Using **(H1)** (hyperbolicity), we have established (i)+(ii). We now use **(H2)** (fibrewise mixing) to obtain (iii).

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Lemma 4

Recall (from the mixing condition H2) that

$$\mathcal{T}_{\mathrm{mix}}(\omega) := \min \left\{ n \in \mathbb{N};\, T^n_\omega(B_{\delta/2}(x_i)) \cap B_{\delta/2}(x_j) \neq \emptyset,\ \forall 1 \leq i, j \leq k \right\}.$$

Then there exists $D_1, D_2 > 1$ independent of ω such that for any $n \geq \mathcal{T}_{\mathrm{mix}}(\omega)$

$$\sup_{f,g \in \mathcal{C}} \mathrm{Hil}_{\mathcal{C}_\omega} \left(\mathcal{L}^{(n)}_\omega f, \mathcal{L}^{(n)}_\omega g \right) \leq D_1 + D_2^{\mathcal{T}_{\mathrm{mix}}(\omega)}.$$

In particular if $\mathcal{T}_{\mathrm{mix}}(\omega) \leq A$, then $\sup_{f,g \in \mathcal{C}_\omega} \mathrm{Hil}_{\mathcal{C}_\omega} \left(\mathcal{L}^{(n)}_\omega f, \mathcal{L}^{(n)}_\omega g \right) \leq D_1 + D_2^A := \Delta$.

See Liverani (1995) or X. Liu (2024) (see also Atnip–Froyland–Gonzalez-Tokman–Vaienti (2021) and Buzzi (1999)).

(iii) Construction of stopping times that generate cone contractions

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$$\tau_0 = 0 < \tau_1 < \tau_2 < \cdots \text{ with } N(\sigma^{-\tau_m}\omega) \leq A \text{ and } \tau_m - \tau_{m-1} \geq A \text{ for every } m \in \mathbb{N}.$$

Let Δ as before and $\chi := \tanh(\Delta/4) \in (0,1)$. Then for any $m \in \mathbb{N}$,

$$\sup_{f,g \in \mathcal{C}_{\sigma^{-\tau_m}\omega}} \mathrm{Hil}_{\mathcal{C}_{\sigma^{-\tau_m}\omega}}\left(\mathcal{L}^{(\tau_m)}_{\sigma^{-\tau_m}\omega} f,\, \mathcal{L}^{(\tau_m)}_{\sigma^{-\tau_m}\omega} g\right) \leq \chi^{m-1}\,\Delta.$$

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Consequence of the previous lemma and

$$\mathcal{L}^{(\tau_m)}_{\sigma^{-\tau_m}\omega}\,\mathcal{C}_{\sigma^{-\tau_m}\omega} = \mathcal{L}^{(\tau_1)}_{\sigma^{-\tau_1}\omega} \circ \mathcal{L}^{(\tau_2-\tau_1)}_{\sigma^{-\tau_2}\omega} \circ \cdots \circ \mathcal{L}^{(\tau_m-\tau_{m-1})}_{\sigma^{-\tau_m}\omega}\,\mathcal{C}_{\sigma^{-\tau_m}\omega} \subset \mathcal{C}_\omega\,,$$

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From the discussion in the previous slides, we can construct

- $\nu_\omega(f) := \ell_\omega(f\,\mu_\omega)$, with $\mathcal{L}_\omega \mu_\omega = \lambda_\omega\,\mu_{\sigma\omega}$ and $(\mathcal{L}_\omega)^* \ell_{\sigma\omega} = \lambda_\omega\,\ell_\omega$;
- $\left| \int_M f \circ T^n_\omega\, g \,\mathrm{d}\nu_\omega - \int f \,\mathrm{d}\nu_{\sigma^n\omega} \int_M g \,\mathrm{d}\nu_\omega \right| \leq \chi^{N_\tau(\omega,n)} \|f\|_{C^\alpha} \|g\|_{C^\alpha}$, with

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In particular, if (**H2′**) is satisfied ($\mathbb{P}[\mathcal{T}_{\mathrm{mix}}(\omega) > n] \leq C_0 e^{-\kappa n}$), then the decay is exponentially fast (depending on ω).

Showing that ν is the unique $\mathbb{P}$-relative equilibrium state

Assume that the RDS $(\Theta; \sigma, \mathbb{P})$ satisfies Hypothesis (**H**). Let $\nu = \nu_\omega(\cdot)\mathbb{P}(\mathrm{d}\omega)$, with $\nu_\omega(f) = \ell_\omega(f \cdot \mu_\omega)$.

- We show that

$$\int \log \lambda_\omega \,\mathrm{d}\mathbb{P}(\omega) = P_{\mathrm{top}}\left(\phi + \log \left| \det\left(DT_\omega|_{E^s(\omega,x)}\right)\right| \;\middle|\; \mathbb{P}\right),$$

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- We prove quenched CLT by the standard Nagaev-Guivarc'h perturbation method for complex cones, Chapter 7 of Hafouta, and Kifer (2018).

Key takeaways

Results

- We have established existence and uniqueness of relative equilibrium states for hyperbolic random dynamical systems on surfaces, under the uniform Hölder bound

$$\operatorname*{ess\,sup}_{\omega} \|\phi(\omega,\cdot)\|_{\mathcal{C}^\alpha} < \infty.$$

- We have obtained quenched exponential decay of correlations and a quenched CLT via the basis large deviation estimates, i.e.

$$\mathbb{P}[N \leq n] \leq K\, e^{-\kappa n}.$$

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Thank you for your attention!

Topological pressure via the random eigenvalues

Let $\nu = \nu_\omega(\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega)$ and assume $(\mathbf{H}) = (\mathbf{H_1}) + (\mathbf{H_2})$ (Hyperbolicity + Fibrewise mixing).

- We show that

$$\int \log \lambda_\omega \, \mathrm{d}\mathbb{P}(\omega) = P_{\mathrm{top}}\left(\Theta,\ \phi(\omega, x) + \log \left| \det\left(DT_\omega|_{E^s(\omega,x)} \right) \right| \;\middle|\; \mathbb{P}\right),$$

  by adapting an argument of Parmenter–Pollicott (2021) which reads

Theorem 5 (Parmenter–Pollicott (2021))

$T : M \to M$ is a mixing Anosov map (or has an Axiom A attractor) and $\phi : M \to \mathbb{R}$ is continuous, then for any piece of stable manifold $\gamma \subset M$,

$$P\left(\phi + \log\left|\det(DT|_{E^s})\right|,\ T\right) \;=\; \lim_{n\to\infty} \frac{1}{n} \log \int_\gamma \exp\left(\sum_{i=0}^{n-1} \phi \circ T^{-i}(y) \right) \mathrm{d}m_\gamma(y),$$

where m_γ is the induced Riemannian measure on γ.

The measure ν is a $\mathbb{P}$-relative equilibrium state

Let $\nu = \nu_\omega(dx)\mathbb{P}(d\omega)$ and assume $(\mathbf{H}) = (\mathbf{H_1}) + (\mathbf{H_2})$ (Hyperbolicity + Fibrewise mixing).

- We show that

$$\int \log \lambda_\omega \, d\mathbb{P}(\omega) \;=\; P_\nu\left(\Theta,\; \phi(\omega,x) + \log\left|\det\left(DT_\omega|_{E^s(\omega,x)}\right)\right| \;\middle|\; \mathbb{P}\right),$$

By establishing a weak Gibbs property:

Proposition 1 (Weak Gibbs property)

Let $\varepsilon > 0$ be small enough. Then, there exist functions $K_\varepsilon \in L^1(\Omega,\mathbb{P})$ and measurable functions $c_\varepsilon, C_\varepsilon : \Omega \to (0,\infty)$, such that for $\mathbb{P}$-a.e. $\omega \in \Omega$ there exists a strictly increasing sequence $\{n_k(\omega)\}_{k\in\mathbb{N}}$ such that for every $k \in \mathbb{N}$ and and $x \in M$:

$$c_\varepsilon(\omega)e^{-K_\varepsilon(\sigma^{-n_k(\omega)}\omega)} \le \frac{\nu_\omega(B_\omega^{n_k(\omega)}(x,\varepsilon))}{[\lambda^{(n_k(\omega))}_{\sigma^{-n_k(\omega)}\omega}]^{-1}\exp\left(S_n\bar{\phi}_{\sigma^{-n_k(\omega)}\omega}\left((T^n_{\sigma^{-n_k(\omega)}\omega})^{-1}x\right)\right)} \le C_\varepsilon(\omega)e^{K_\varepsilon(\sigma^{-n_k(\omega)}\omega)},$$

where $B^n_\omega(x,\varepsilon) = \{y \in M : \forall 0 \le i \le n-1, d(T_\omega^{-i}x, T_\omega^{-i}y) \le \varepsilon\}$ is the backward dynamical ball.

The proof of the above proposition borrows some ideas from Stadlbauer–Suzuki–Varandas (2021).

Uniqueness of $\mathbb{P}$-relative equilibrium states

Let $\nu = \nu_\omega(\mathrm{d}x)\,\mathbb{P}(\mathrm{d}\omega)$ and assume $(\mathbf{H}) = (\mathbf{H_1}) + (\mathbf{H_2})$ (hyperbolicity + fibrewise mixing).

- **Deterministic result (Carrasco–Rodríguez-Hertz, 2023).** If $T : M \to M$ is a centre isometry (i.e. $TM = E^s \oplus E^c \oplus E^u$ with E^c having special properties) and $\overline{\phi}$ is a suitable potential (in our context $E^c = \varnothing$ and $\overline{\phi}$ is Hölder), and if there exist measures μ^u, μ^s such that
  - (i) the (Rokhlin) disintegration of μ^u along unstable leaves $\{\mu^u_x\}$ is absolutely continuous with respect to the leafwise Lebesgue measure;
  - (ii) the disintegration of μ^s along stable leaves $\{\mu^s_x\}$ is absolutely continuous with respect to the leafwise Lebesgue measure;
  - (iii) *Margulis property:* for every $x \in M$,

$$T_*\mu^u_x = e^{-P_{\mathrm{top}}(\overline{\phi})+\overline{\phi}}\,\mu^u_{T(x)}, \quad T_*\mu^s_x = e^{P_{\mathrm{top}}(\overline{\phi})-\overline{\phi}}\,\mu^s_{T(x)};$$

  - (iv) T is topologically mixing,

  then T admits a unique equilibrium state for $\overline{\phi}$.
- **Random adaptation.** We adapt the above result by observing that ℓ_ω and μ_ω are Margulis measures for the potential

$$\overline{\phi}(\omega, x) := \phi(\omega, x) + \log\left|\det(DT_\omega|_{E^s(\omega,x)})\right|$$

  and that both μ_ω and ℓ_ω have full support.

Quenched CLT

Let $\nu = \nu_\omega(\mathrm{d}x)\,\mathbb{P}(\mathrm{d}\omega)$ and assume $(\mathbf{H}') = (\mathbf{H_1}) + (\mathbf{H}_2')$ (hyperbolicity + fibrewise rapid mixing). Take

$$f_\omega \in L^\infty(\Omega, C^\alpha(M)) \text{ with } \nu_\omega(f_\omega) = 0 \text{ and } S_n f(\omega, x) := \sum_{k=0}^{n-1} f_{\sigma^k \omega}(T_\omega^k x).$$

Normalise, without loss of generality, to $\lambda_\omega \equiv 1$ by $\bar{\phi}_\omega := \phi_\omega - \log \lambda_\omega$.

Nagaev–Guivarc'h on complex cones. Consider the twist operator

$$\mathcal{L}_{\omega,t} g := \mathcal{L}_\omega\big(e^{itf_\omega} g\big), \; |t| \le t_0.$$

Cone contraction (complex Hilbert metric) gives a uniform spectral gap for small $|t|$:

$$\mathcal{L}^{(n)}_{\omega,t} = \lambda_\omega(t)^{(n)} \Pi_{\sigma^n \omega}(t) + Q^{(n)}_{\omega,t}, \quad \|Q^{(n)}_{\omega,t}\| \le C e^{-\chi n},$$

with $t \mapsto \lambda_\omega(t)$ analytic at 0, $\lambda'_\omega(0) = 0$, $\lambda''_\omega(0) = \Sigma^2$. Hence $\log \nu_\omega\big(e^{itS_n f}\big) = \frac{1}{2}\sigma^2 t^2 n + o(nt^2)$, and Lévy's theorem completes the proof.

- Chapter 7 of Y. Hafouta and Y. Kifer. *Nonconventional limit theorems and random dynamics*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018, pp. xiii+284
- J. Atnip, G. Froyland, C. González-Tokman, and S. Vaienti. “Thermodynamic Formalism and Perturbation Formulae for Quenched Random Open Dynamical Systems”. In: *Dissertationes Mathematicae* (2024). to appear; see arXiv:2307.00774. arXiv: 2307.00774 [math.DS]
- Sequencial Billiards: M. F. Demers and C. Liverani. *Central Limit Theorem for Sequential Dynamical Systems*. 2025. arXiv: 2502.07765 [math.DS]

The $\log\left|\det\, dT|_{E^s}\right|$ correction

We focus on the deterministic case. Take $\phi = -\log|\det\, dT|$, we know for this choice of potential the eigenvectors of $\mathcal{L}_\phi f = (e^\phi f) \circ T^{-1}$ should give rise to the SRB measure.
From Pesin's formula

$$P_{\mathrm{top}}\left(-\log\left|\det\, dT|_{E^u}\right|\right) = h_{\mu_{\mathrm{SRB}}} - \int \log\left|\det\, dT|_{E^u}\right| d\mu_{\mathrm{SRB}} = 0.$$

we observe that a correction is needed.
Angle identity:

$$|\det\, dT_x| = \frac{\alpha(Tx)}{\alpha(x)}\left|\det\, dT_x|_{E^s}\right|\left|\det\, dT_x|_{E^u}\right|, \quad \alpha(x) = \sin\angle(E^s_x, E^u_x).$$

Coboundary:

$$-\log|\det\, dT_x| + \log\left|\det\, dT_x|_{E^s}\right| = -\log\left|\det\, dT_x|_{E^u}\right| + \log\alpha(Tx) - \log\alpha(x).$$

Leafwise change of variables (stable leaf γ):

$$\int_\gamma \mathcal{L}_\phi f \; dm_\gamma = \int_\gamma (e^\phi f)\circ T^{-1}\, dm_\gamma = \int_{T^{-1}\gamma} e^\phi f\left|\det\, dT|_{E^s}\right| dm_{T^{-1}\gamma} = \int_{T^{-1}\gamma} e^{\phi+\log\left|\det\, dT|_{E^s}\right|} f \; dm_{T^{-1}\gamma}.$$