INSTITUT FOURIER
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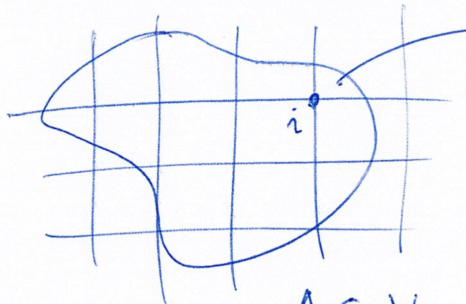
Extremal Ising Gibbs states on hyperbolic lattices

"Thermogamas" 2025/09/30

Ising model

$G = (V, E)$ infinite graph



$\Lambda \Subset V$

$\sigma_i \in \{-1, +1\}$ $\quad \sigma = (\sigma_i)_i$

$$H^\omega_\Lambda(\sigma) = -\sum_{\substack{i \sim j \\ i,j \in \Lambda}} \sigma_i \sigma_j - \sum_{\substack{i \in \Lambda \\ j \notin \Lambda \\ i \sim j}} \sigma_i \omega_j$$

$$\mu^\omega_\Lambda(\sigma) \propto \exp(-\beta H^\omega_\Lambda(\sigma))$$

Gibbs states

$$\mathcal{G}_\beta = \Big\{ \mu \in \mathcal{M}_1(\{-1,+1\}^V) : \quad \mu(\cdot \mid \sigma_{\Lambda^c})(\omega) = \mu^\omega_\Lambda \quad \forall \Lambda \Subset V, \ \mu\text{-ae } \omega \in \{\pm 1\}^V \Big\}$$

- "phase transition" $\mu^+ \neq \mu^-$ for β large on $\mathbb{Z}^d$, $d \geq 2$.

- $\mathcal{G}_\beta$ is a Choquet simplex $\rightsquigarrow$ $ex\,\mathcal{G}_\beta$ extremal elements

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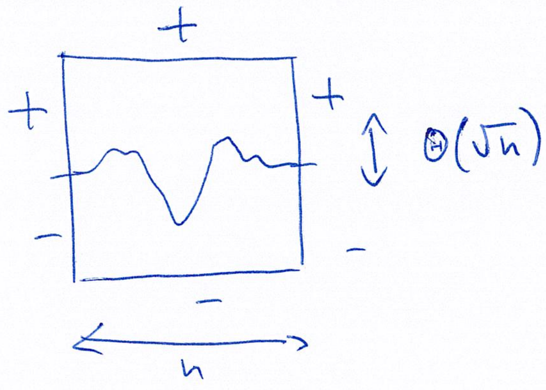
State of the art on $\mathbb{Z}^2$, $\mathbb{Z}^3$:

[Aizenman, Higuchi 80s] $\beta > \beta_c$

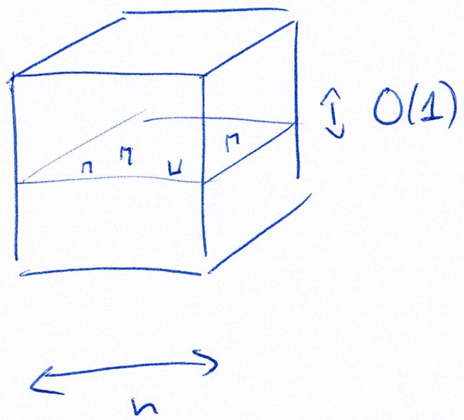


$$\mu^{\pm} = \frac{1}{2}(\mu^+ + \mu^-)$$

$$\rightsquigarrow \mathcal{G}_\beta = \{\alpha\mu^+ + (1-\alpha)\mu^- \,,\ \alpha \in [0,1]\}$$

$\text{ex}\,\mathcal{G}_\beta$ is $\boxed{\text{finite}}$

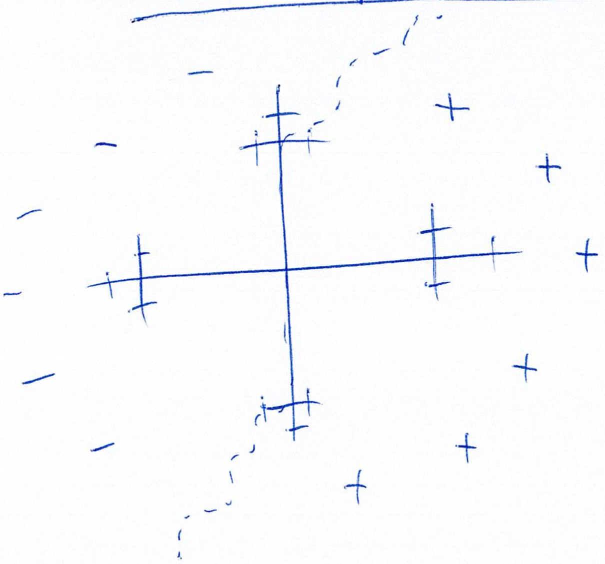
β large [Dobrushin 72]



$$\mu^{\pm} \in \text{ex}\,\mathcal{G}_\beta$$

$\mathcal{G}_\beta$ has $\boxed{\text{countably}}$ many extremal states

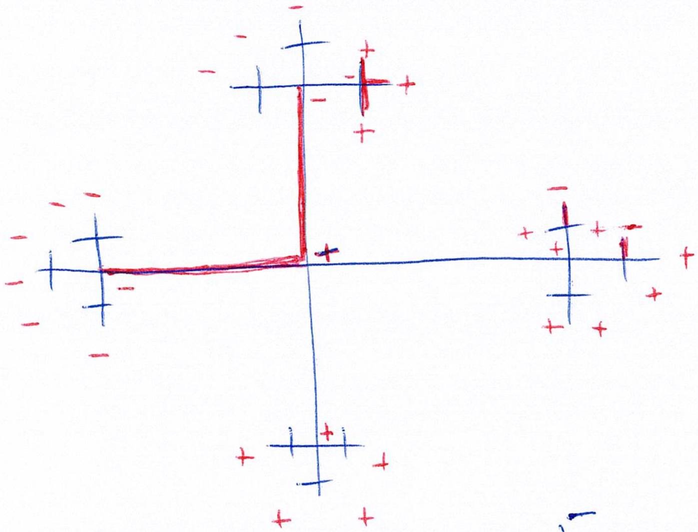
State of the art on regular trees



$\text{ex}\,\mathcal{G}_\beta$ is $\boxed{\text{uncountable}}$

[Higuchi 77]

More extremal inhomogeneous on trees $\mathcal{T}^d$



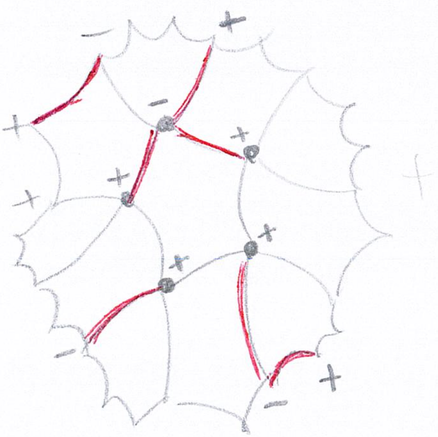
[Gandolfo, Ruiz, Shlosman 12]
[C, Külske, Le Ny 23]

$\omega \in \{-1,+1\}^V$
$D(\omega)$ set of disagreeing edges

Thm If $\boxed{\deg\max D(\omega) < \frac{d-1}{2}}$
then μ^ω is extremal
at low T (β large).

Extension to hyperbolic graphs

$\mathcal{L}_{p,q}$ regular tiling of $\mathbb{H}^2$ by p-gons
where each vertex has degree q.



Thm [C, D'Achille, Le Ny 25]
If $\deg\max D(\omega) < \frac{IC_{p,q}}{2}$
then $\mu^\omega \in \mathrm{ex}\mathcal{G}_\beta$ at low T.

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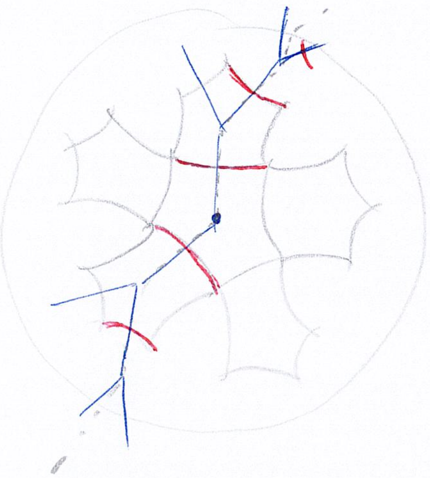
Corollary [CDALN, Häggström Jonasson Lyons 02]

On $\mathcal{L}_{p,q}$. ↳ explicit formula for $IC_{p,q}$.

At low T (large β),

ex $\mathcal{G}_\beta$ is uncountable as soon as

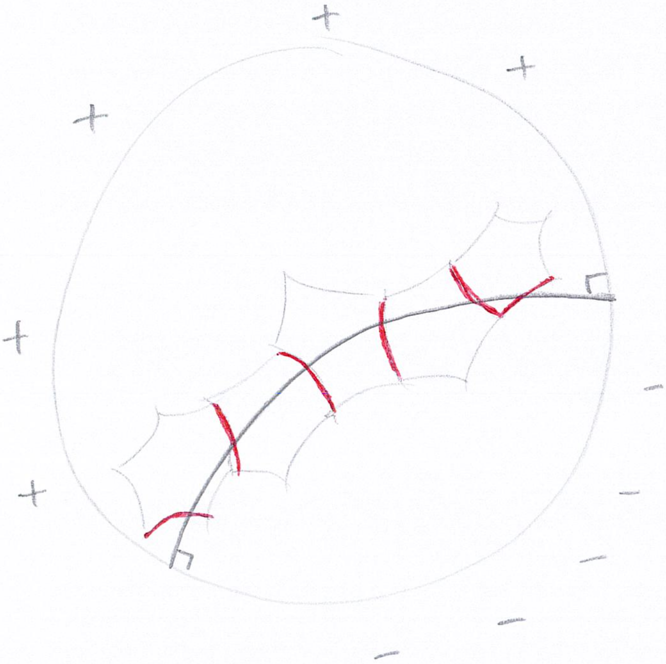
$$p>4,\quad q\geq 3,\quad \frac{1}{p}+\frac{1}{q}<\frac{1}{2},\quad IC_{p,q}>2$$



conditions to embed a tree of degree ≥ 2 in the dual $\mathcal{L}_{q,p}$ and fulfilling the sparsity condition if $D(\omega)$ is a branch (bi-infinite)

Theorem [Series Sinai 90]

continuous object!



Take γ geodesic of $\mathbb{H}^2$.

Define $\mu^{\pm,\gamma}$ Ising state

$\exists$ uncountably many <u>mutually singular</u> such states

<u>Q</u>: are they <u>extremal</u>?

Reformulation as a billiard question

"How many" billiard trajectories on $\mathcal{L}_{q,p}$ never bounce more than $\frac{q}{2}-1$ times on two adjacent faces of the q-gon?