# Self Consistent Transfer Operators for Heterogeneous Networks

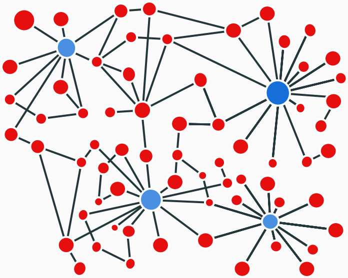
THERMOGAMAS

Herbert Milton Ccalle Maquera

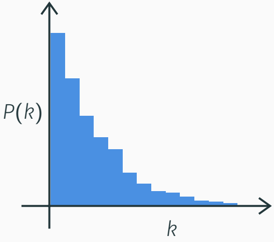
30-09-2025

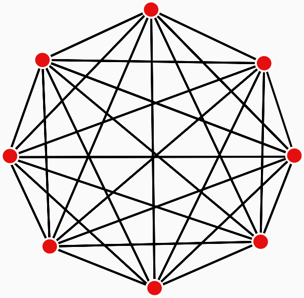
ICMC-USP

# Heterogeneous Coupled Maps

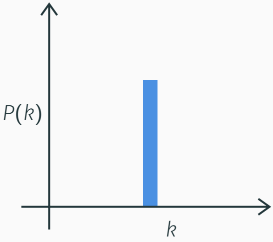


Heterogeneous Network



Homogeneous Network



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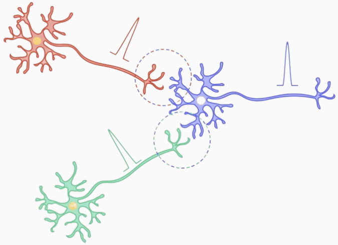
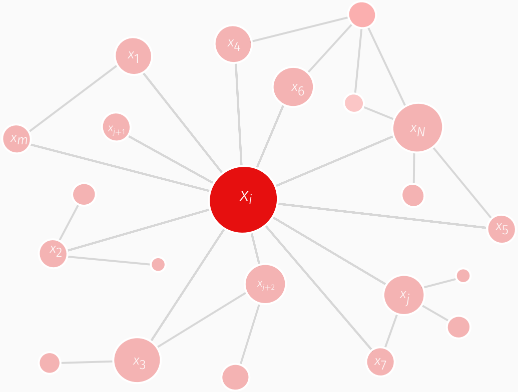


# Setting

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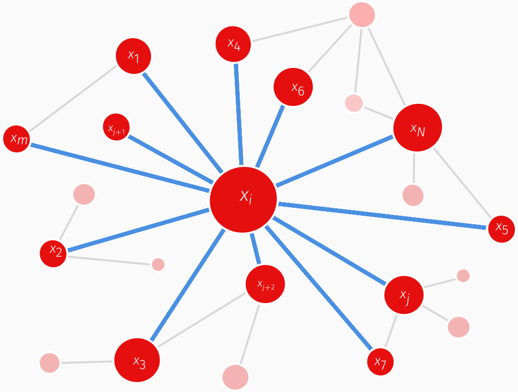



$$x_i(t+1) = f_i(x_i(t)) + \frac{\alpha}{N} \sum_{j=1}^{N} A_{ij} h(x_i(t), x_j(t)) \mod 1$$

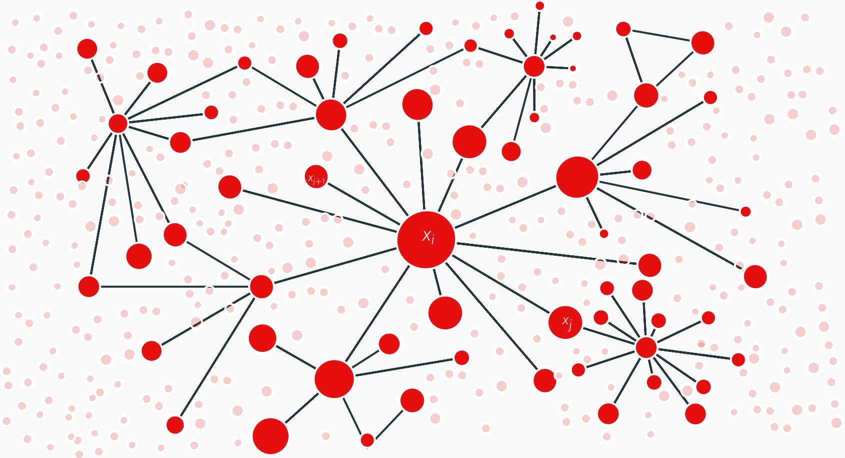
# Globally Coupled Maps

- **Kunihiko Kaneko** Period-Doubling of Kink-Antikink Patterns, Quasiperiodicity in Antiferro-Like Structures and Spatial Intermittency in Coupled Logistic Lattice. [*1984, Progress of Theoretical Physics*]
- ***P. Bálint, G. Keller, F. M. Selley and P. Toth*** *Synchronization versus stability of the invariant distribution for a class of GCM. [2018, Nonlinearity]*
- ***Stefano Galatolo*** *Self-Consistent Transfer Operators: Invariant Measures, Convergence to Equilibrium, Linear Response and Control of the Statistical Properties. [ 2022, Communications in Mathematical Physics]*
- ***F. M. Sélley and M. Tanzi*** *Synchronization for networks of Globally Coupled Maps in the thermodynamic limit. [2022, Journal of Statistical Physics]*

# Coupled map Networks

- **Koiller and Lai-Sang Young** Coupled map networks. [*2010, Nonlinearity*]
- ***Tiago Pereira, Sebastian van Strien and Matteo Tanzi*** *Heterogeneously coupled maps: hub dynamics and emergence across connectivity layers. [2020, J. Eur. Math. Soc.]*

# Thermodynamic Limit




To study the thermodynamic limit of heterogeneous coupled maps

# Mean Field Approximation

- Consider

$$x_i(t+1) = f(x_i(t)) + \frac{\alpha}{N}\sum_{j=1}^{N} A_{ij} h(x_i(t), x_j(t)) \mod 1, \text{ for } i = 1, 2, \ldots, N$$

Let $F : \mathbb{T}^N \to \mathbb{T}^N$,

$$\boxed{x_i(t+1) = F_i(x_1(t), \ldots, x_N(t)).}$$

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- Define $F_{\boldsymbol{\mu},i} : \mathbb{T} \to \mathbb{T}$

$$F_{\boldsymbol{\mu},i}(x_i) = f_i(x_i) + \frac{\alpha}{N}\sum_{j=1}^{N} \int A_{ij} h(x_i, y_j) d\boldsymbol{\mu}(\mathbf{y})$$

$F_{\boldsymbol{\mu}} : \mathbb{T}^N \to \mathbb{T}^N$,

$$\boxed{F_{\boldsymbol{\mu}}(\mathbf{x}) = (F_{\boldsymbol{\mu},1}(x_1), F_{\boldsymbol{\mu},2}(x_2), \ldots, F_{\boldsymbol{\mu},N}(x_N))}$$

# Dynamics

Evolution of the state of the system

$$\boxed{\mathcal{F}_N \boldsymbol{\mu} := (F_{\boldsymbol{\mu}})_* \boldsymbol{\mu} = ((F_{\boldsymbol{\mu},1})_* \mu_1, (F_{\boldsymbol{\mu},2})_* \mu_2, \ldots, (F_{\boldsymbol{\mu},N})_* \mu_N)}$$

# Homogeneous vs Heterogeneous

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# Challenges

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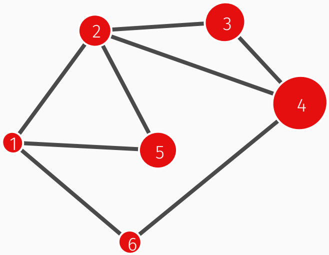
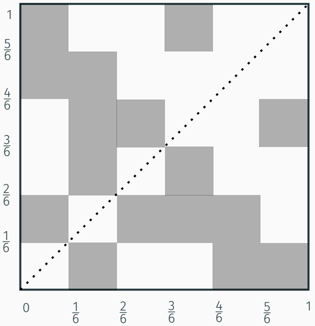
- Does a fixed point exist in the thermodynamic limit?
- Is this fixed point stable?

# Graphons



1
2
3
4
5
6

1
$\frac{5}{6}$
$\frac{4}{6}$
$\frac{3}{6}$
$\frac{2}{6}$
$\frac{1}{6}$
0
$\frac{1}{6}$
$\frac{2}{6}$
$\frac{3}{6}$
$\frac{4}{6}$
$\frac{5}{6}$
1

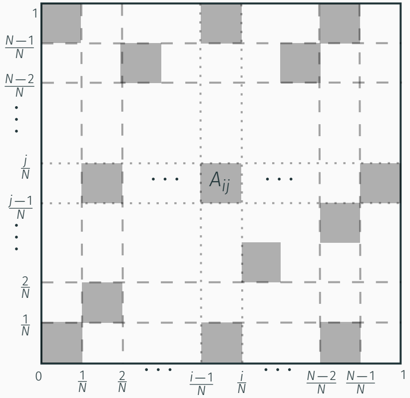
# $A_{ij}$ in terms of $W_N$

We divide the interval $[0,1]$ in $N$ sub intervals, for $i = 2, \dots, N$

$$I_1 := [0, 1/N], \quad I_i := \left(\frac{i-1}{N}, \frac{i}{N}\right]$$

define $W^{(N)}$ by

$$W^{(N)}\big|_{I_i \times I_j}(x,y) := A_{ij}$$



# Coupled maps in terms of $W^{(N)}$

So, for $z \in I_i$

$$\frac{1}{N}A_{ij} = \int_{\frac{j-1}{N}}^{\frac{j}{N}} dz' W^{(N)}(z, z')$$

For $\boldsymbol{\mu} \in \mathcal{M}(\mathbb{T})^{\otimes N}$ the mean field map is

$$F_{\boldsymbol{\mu},i}(x_i) = f_i(x_i) + \frac{\alpha}{N}\sum_{j=1}^{N}\int A_{ij}h(x_i, y_j)d\boldsymbol{\mu}(\boldsymbol{y})$$

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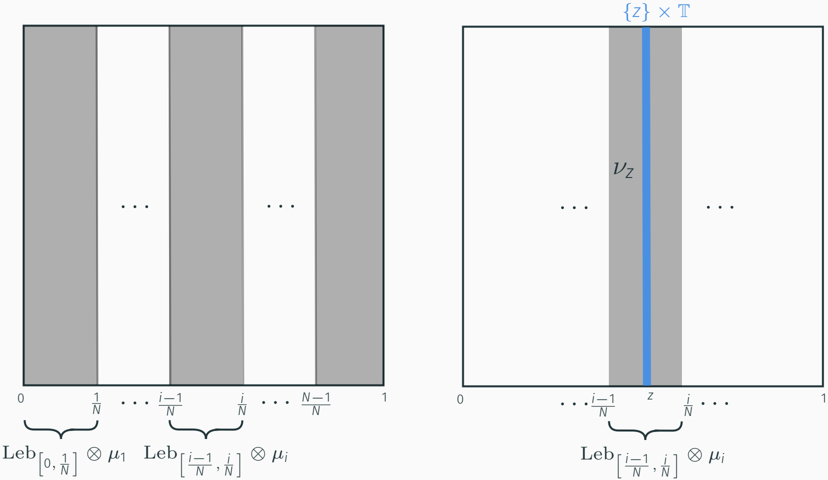
# Coupled maps in terms of $W^{(N)}$



# Coupled maps in terms of $W^{(N)}$

Then

$$\nu_z^{(N)} = \mu_i \text{ when } \; z \in I_i$$

So, $F_{\nu,z} : \mathbb{T} \to \mathbb{T}$ for $z$ in $I_i$ :

$$F_{\nu,z}^{(N)}(x) := f(x) + \alpha \int_0^1 \int_{\mathbb{T}} dz' d\nu_{z'}^{(N)}(y)\; W^{(N)}(z,z')\; h(x,y)$$

## SCTO in the thermodynamic limit

- $F_\nu : [0,1] \times \mathbb{T} \to [0,1] \times \mathbb{T}, \qquad F_\nu(z,x) = (z, F_{\nu,z}(x))$

  Where

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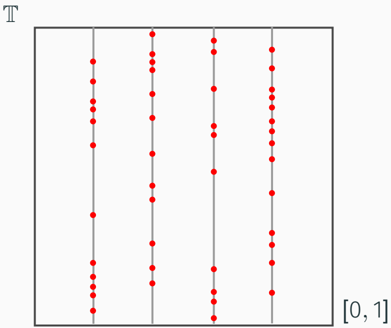
$$\boxed{\mathcal{F}_W \nu = (F_\nu)_* \nu}$$

# Local Contracting Dynamics

# Contraction on the fiber



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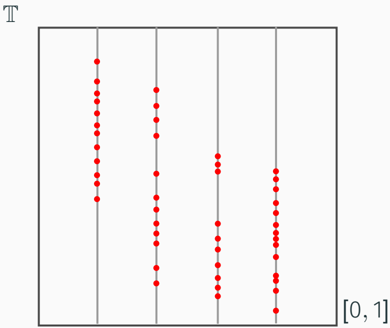

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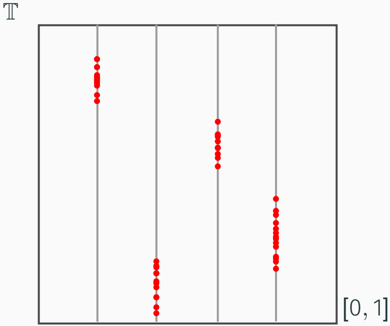

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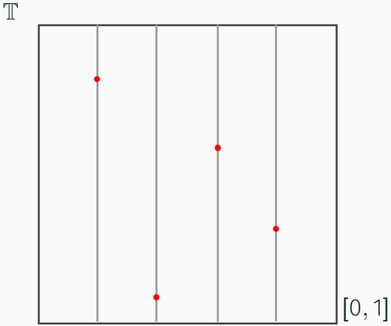

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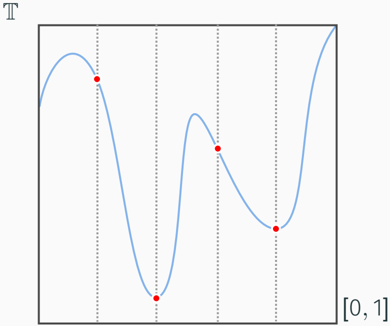

# Existence and uniqueness of fixed point for STO

Let $f : \mathbb{T} \to \mathbb{T}$ be a $\Lambda$-contraction in $I$. Let $h : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ be Lipschitz in the both variables. Let $W \in L^\infty([0,1]; L^1([0,1]))$

$$F_{\nu,z}(x) = f(x) + \alpha \int_0^1 \int_{\mathbb{T}} W(z,z')\, h(x,y)\, d\nu_{z'}(y)\, dz',$$

There exists $\hat{\alpha} > 0$ such that for all $\alpha < \hat{\alpha}$ the maps $F_{\nu,z}$ are uniform contractions. Then for almost every $z \in [0,1]$

$$\mathcal{F}_W \nu^* = \nu^*$$

where $\nu^*_z = \delta_{g(z)}$ for some $g : [0,1] \to \mathbb{T}$ measurable function.

# Proof

Consider

$$\mathcal{G} := \{g : [0,1] \to I \subset \mathbb{T}\,, \text{measurable and bounded}\},\ d_\infty\text{-metric}$$

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$$\mathcal{T}(g)(z) := F_{\nu,z}\big(g(z)\big).$$

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Then the map $\mathcal{T}$ is a contraction on $(\mathcal{G}, d_\infty)$.

$$\mathcal{T}(g^*) = g^*$$

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and

$$(\mathcal{F}_W \nu^*)_z = \nu_z^*.$$

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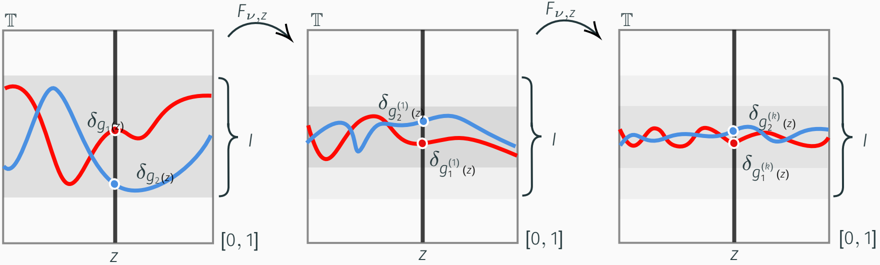


# Expanding maps

# Definitions

**Definition 1 (Self Consistent Transfer Operator)**

*Consider* $f \in \mathcal{C}^3(\mathbb{T}, \mathbb{T})$, $h \in \mathcal{C}^3(\mathbb{T} \times \mathbb{T}, \mathbb{R})$, *and* $W \in L^\infty([0,1], L^1([0,1], \mathbb{R}))$, *for any* $\nu \in \mathcal{M}_{1,\mathrm{Leb}}([0,1] \times \mathbb{T})$ *and almost any* $z \in [0,1]$ *we define*

$$F_{\nu,z}(x) = f(x) + \alpha \int_0^1 dz' \int_{\mathbb{T}} d\nu_{z'}(y)\, W(z,z')\, h(x,y) \tag{1}$$

*and*

$$\mathcal{F}\nu = (F_\nu)_* \nu$$

# Spaces we consider

## Wasserstein distance

Consider the notion of distance between measures given by

$$W^1(\mu, \nu) = \sup_{\substack{\mathrm{Lip}(g) \leq 1 \\ \|g\|_\infty \leq 1}} \left[ \int g \cdot d\mu - \int g \cdot d\nu \right] \tag{2}$$

We define

$$\|\mu\|_{W^1} := W^1(0, \mu).$$

## Disintegrations with Lebesgue Marginal

Consider $\mathcal{M}_1([0,1] \times \mathbb{T})$ is the set of probability measures

$$\mathcal{M}_{1,\mathrm{Leb}} := \{\nu \in \mathcal{M}_1([0,1] \times \mathbb{T}) : \ \nu(A \times \mathbb{T}) = \mathrm{Leb}(A) \ \ \forall A \subset [0,1] \text{ meas.}\}$$

# Spaces we consider

## BV Seminorms of Densities on $\mathbb{T}$

Consider

$$\mathcal{B}_{BV^i} := \{\psi \in L^1(\mathbb{T}) : \|\psi\|_{BV^i} < \infty\}$$

with

$$\|\psi\|_{BV^i} := |\psi|_{BV^i} + \|\psi\|_{L^1}$$

and

$$|\psi|_{BV^i} := \sup_{\substack{g \in \mathcal{C}^i(\mathbb{T},\mathbb{R}) \\ \|g\|_\infty \leq 1}} \int_{\mathbb{T}} g^{(i)}(s)\psi(s)ds$$

## Fiberwise Regularity and Variation Control

For $i = 1, 2$.

$$\tilde{\mathcal{B}}_{BV^i,M} := \{\varphi \in L^1([0,1] \times \mathbb{T}, \mathbb{R}) : \varphi_z \in \mathcal{B}_{BV^i,M} \text{ for a.e. } z \in [0,1]\}$$

# Spaces we consider

**Weak space**

$$\mathcal{B}_w := \{\varphi : [0,1] \times \mathbb{T} \to \mathbb{R} : \|\varphi\|_{\text{“1”}} < \infty\}$$

where

$$\|\varphi\|_{\text{“1”}} := \int_{[0,1]} \|\varphi_z\|_{W^1} dz$$

# Space we consider

$BV^1$-oscillation of $\varphi$:

$$\mathbf{osc}_{BV^1}(\varphi,\omega,r) := \operatorname*{ess\,sup}_{z,\bar{z}\in B(\omega,r)} |\varphi_{\bar{z}} - \varphi_z|_{BV^1} \tag{3}$$

and

$$\mathbf{var}_{p,BV^1}(\varphi) = \sup_{r>0} \frac{1}{r^p} \int_{[0,1]} \mathbf{osc}_{BV^1}(\varphi,\omega,r)\, d\omega. \tag{4}$$

**Stronger space**

$$\mathcal{B}_s := \{\varphi : [0,1]\times\mathbb{T} \to \mathbb{R} : \|\varphi\|_s < \infty\}$$

where

$$\|\varphi\|_s := \mathbf{var}_{p,BV^1}(\varphi) + \|\varphi\|_{\text{“1”}}$$

## Spaces we consider

**Admissible Set of Regular Densities**

To construct a suitable domain for the application of Schauder's fixed-point theorem

$$\mathcal{A}_{\mathsf{M}} := \mathcal{B}_{S,M} \cap \tilde{\mathcal{B}}_{BV^1,M_1} \cap \tilde{\mathcal{B}}_{BV^2,M_2} \cap \mathcal{M}_{1,\mathsf{Leb}} \tag{5}$$

where $\mathsf{M} = (M_1, M_2, M)$.

# Main Results and Proofs

- For every $N \in \mathbb{N}$ consider $\{x_i^{(N)}\}_{N=1,i=1}^{\infty,N}$ where $x_i^{(N)}$ is distributed according to $\mu_i^{(N)}$ defined as

$$\mu_i^{(N)}(A) := N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \nu_z(A) dz.$$

- Define the random variables

$$(y_1^{(N)}, ..., y_N^{(N)}) := F(x_1^{(N)}, ..., x_N^{(N)})$$

and call $\eta_i^{(N)}$ the distribution of $y_i^{(N)}$

# Convergence of the finite-dimensional system to the STO

**Theorem 1**

*Assume that for a given $z_* \in [0,1]$:*

i) $W^{(N)}(z_*,\cdot) \to W(z_*,\cdot)$ *in* $L^1([0,1])$ *for* $N \to \infty$;

ii) $z \mapsto W(z,\cdot)$ *is Lipschitz near* $z_*$;

iii) $\nu \in \mathcal{M}_{1,\mathrm{Leb}}([0,1]\times\mathbb{T})$ *has a disintegration* $\{\nu_z\}_{z\in[0,1]}$ *that is Lipschitz near* $z_*$.

*Then,*

$$\lim_{N\to\infty} \eta^{(N)}_{\lceil z_* N\rceil} = (\mathcal{F}\nu)_{z_*} \quad \textit{weakly.}$$

# Proof: Convergence of the finite-dimensional system to the STO

$$\int_{\mathbb{T}} g(y)\left(d(\mathcal{F}\nu)_{z_*}(y) - d\eta_i^{(N)}(y)\right) =$$

$$= \underbrace{\int_{\mathbb{T}} g(F_{\nu,z_*}(y))d\nu_{z_*}(y)}_{\mathcal{I}_1} - \underbrace{\int_{\mathbb{T}^N} g\left(f(x_i) + \frac{\alpha}{N}\sum_{j=1}^{N} A_{ij}h(x_i,x_j)\right)\prod_{j=1}^{N} d\mu_j^{(N)}(x_j)}_{\mathcal{I}_2}$$

To give estimates for the integral $\mathcal{I}_2$

$$g\left(f(x_i) + \frac{\alpha}{N}\sum_{j=1}^{N} A_{ij}h(x_i,x_j)\right) \approx g\left(f(x_i) + \int \frac{\alpha}{N}\sum_{j=1}^{N} A_{ij}h(x_i,x_j')\prod_{j=1}^{N} d\mu_j^{(N)}(x_j')\right)$$

then

$$\mathcal{I}_2 \approx \mathcal{I}_1 + O(N^{-1/3})$$

# Existence of the fixed point for the STO on a graphon

**Theorem 2 (Existence of the fixed point)**
*Let $f \in \mathcal{C}^3(\mathbb{T}, \mathbb{T})$, $h \in \mathcal{C}^3(\mathbb{T} \times \mathbb{T}, \mathbb{R})$, $W \in L^\infty([0,1], L^1([0,1], \mathbb{R}))$ be a graphon with $\mathtt{var}_{p,L^1}(W) < \infty$ for some $p \in (0,1]$, and $\mathcal{F}$ the associated STO. Consider the admissible set of regular densities $\mathcal{A}_\mathsf{M}$. Then there exist $\alpha_0 > 0$ and $M_0$, $M_1$, $M_2 > 0$ such that for all $|\alpha| < \alpha_0$, $\mathcal{F}$ has a fixed point $\varphi^*$ in the closure of $\mathcal{A}_\mathsf{M}$ in $\mathcal{B}_W$.*

**Corollary 1**
*For every fixed point $\varphi_*$ of $\mathcal{F}$ with absolutely continuous disintegration, $\varphi_z^* \in \mathcal{C}^2(\mathbb{T}, \mathbb{R})$ for a.e. $z \in [0,1]$.*

# Proof: Theorem 2

**Lemma 1 (Uniform Expansion and Distortion Bounds for Fiber Maps)**

*Consider* $\nu \in \mathcal{M}_{1,\mathrm{Leb}}$ *and* $F_{\nu,z}$ *the restriction of the dynamics to the fiber* $\{z\} \times \mathbb{T}$*. If*

$$|\alpha| \cdot \|W\|_{L^\infty([0,1],L^1([0,1],\mathbb{R}))} < \hat{\alpha} := \frac{\inf_x |f'(x)| - 1}{\|h\|_{\mathcal{C}^1}}$$

*then for any* $\nu \in \mathcal{M}_{1,\mathrm{Leb}}$ *and a.e.* $z \in [0,1]$*, the map* $F_{\nu,z}$ *is uniformly expanding, and*

$$\|F_{\nu,z}\|_{\mathcal{C}^3} < K, \quad \textit{and} \quad \sup_{x\in\mathbb{T}} \left| \frac{F''_{\nu,z}(x)}{(F'_{\nu,z})^2(x)} \right| < K'.$$

# Proof: Theorem 2

**Proposition 1 (Lipschitzness of Fiber maps for $\mathcal{C}^k$ norms)**

*For $k \geq 0$, let $h \in \mathcal{C}^k(\mathbb{T} \times \mathbb{T}, \mathbb{R})$, $f \in \mathcal{C}^k(\mathbb{T}, \mathbb{T})$, $\nu \in \mathcal{M}_{1,\mathrm{Leb}}$ and $W \in L^\infty([0,1], L^1([0,1], \mathbb{R}))$ then, for $z, \bar{z} \in A_W$ we have that*

$$\|F_{\nu,z} - F_{\nu,\bar{z}}\|_{\mathcal{C}^k} \leq \alpha \, \|h\|_{\mathcal{C}^k} \|W(z,\cdot) - W(\bar{z},\cdot)\|_{L^1}$$

# Proof: Theorem 2

**Proposition 2 (Invariance)**

*Under the assumptions of Lemma 1 and for $|\alpha| < \hat{\alpha}$, there are $M_1$, $M_2 > 0$ such that*

$$\mathcal{F}(\tilde{\mathcal{B}}_{BV^1,M_1} \cap \tilde{\mathcal{B}}_{BV^2,M_2} \cap \mathcal{M}_{1,\mathbf{Leb}}) \subset \tilde{\mathcal{B}}_{BV^1,M_1} \cap \tilde{\mathcal{B}}_{BV^2,M_2} \cap \mathcal{M}_{1,\mathbf{Leb}}$$

**Proof.**

The following Lasota-Yorke inequalities hold

$$|F_*\varphi|_{BV^1} \leq \lambda_1|\varphi|_{BV^1} + R_1||\varphi||_{L^1}$$

$$|F_*\varphi|_{BV^2} \leq \lambda_2|\varphi|_{BV^2} + R_2|\varphi|_{BV^1} + R_3||\varphi||_{L^1}$$

with $\lambda_1, \lambda_2 \in [0, 1)$ and $R_1, R_2, R_3 > 0$

$\square$

**Proposition 3 (Lipschitzness of Fiber maps for BV norm)**
*For any* $M_1$, $M_2 > 0$ *there is a constant* $K_\# > 0$ *such that for any* $\varphi \in \mathcal{M}_{1,\mathbf{Leb}} \cap \tilde{\mathcal{B}}_{BV^1,M_1} \cap \tilde{\mathcal{B}}_{BV^2,M_2}$,

$$|(F_{\varphi,z})_*\varphi_{\bar{z}} - (F_{\varphi,\bar{z}})_*\varphi_{\bar{z}}|_{BV^1} \leq K_\# \|W(z,\cdot) - W(\bar{z},\cdot)\|_{L^1}$$

*for all* $\bar{z}, z \in A_W \subset [0,1]$ *and* $\bar{z}$ *in the full measure set for which* $\varphi_{\bar{z}} \in \mathcal{B}_{BV^1,M_1} \cap \mathcal{B}_{BV^2,M_2}$.

**Proof.**
Fix $\psi \in \mathcal{B}_{BV^2}$, and let $g \in \mathcal{C}^1$ with $\|g\|_\infty \leq 1$.

$$\begin{aligned}\int_{\mathbb{T}} g'[(F_z - F_{\bar{z}})_*\psi] &\leq \int_{\mathbb{T}} \left(\frac{g \circ F_z}{F'_z} - \frac{g \circ F_{\bar{z}}}{F'_{\bar{z}}}\right)\psi' + \int_{\mathbb{T}} \left(\frac{g \circ F_z}{(F'_z)^2}F''_z - \frac{g \circ F_{\bar{z}}}{(F'_{\bar{z}})^2}F''_{\bar{z}}\right)\psi \\ &\leq K_\# \|W(z,\cdot) - W(\bar{z},\cdot)\|_{L^1}\end{aligned}$$

$\square$

# Proof: Theorem 2

**Lemma 2 (Invariance of admissible set)**

*Assume that* $\mathtt{var}_{p,L^1}(W) < \infty$*. Under the hypotheses of Lemma 1, consider the set*

$$\mathcal{A}_{\mathsf{M}} := \mathcal{B}_{s,M} \cap \tilde{\mathcal{B}}_{BV^1,M_1} \cap \tilde{\mathcal{B}}_{BV^2,M_2} \cap \mathcal{M}_{1,\mathbf{Leb}}$$

*with* $M_1, M_2 > 0$ *as in Proposition 2. Then, there is* $\alpha_0 > 0$ *sufficiently small,* $M > 0$*, and* $\bar{n} \in \mathbb{N}$ *such that provided* $|\alpha| < \alpha_0$

$$\mathcal{F}^n(\mathcal{A}_{\mathsf{M}}) \subset \mathcal{A}_{\mathsf{M}} \qquad \forall n \geq \bar{n}.$$

# Proof: Theorem 2

Pick $\varphi \in \mathcal{A}_{\mathsf{M}}$

$$\begin{aligned} |(\mathcal{F}^n\varphi)_z - (\mathcal{F}^n\varphi)_{\bar{z}}|_{BV^1} &= |(F^n_{\varphi,z})_*\varphi_z - (F^n_{\varphi,\bar{z}})_*\varphi_{\bar{z}}|_{BV^1} \\ &\leq |(F^n_{\varphi,z})_*\varphi_z - (F^n_{\varphi,z})_*\varphi_{\bar{z}}|_{BV^1} + |(F^n_{\varphi,z})_*\varphi_{\bar{z}} - (F^n_{\varphi,\bar{z}})_*\varphi_{\bar{z}}|_{BV^1} \\ &\leq \tau|\varphi_z - \varphi_{\bar{z}}|_{BV^1} + O(\alpha)\|W(z,\cdot) - W(\bar{z},\cdot)\|_{L^1} \end{aligned}$$

so,

$$\mathtt{var}_{p,BV^1}(\mathcal{F}^n\varphi) \leq \tau\ \mathtt{var}_{p,BV^1}(\varphi) + O(\alpha)\ \mathtt{var}_{p,L^1}(W)$$

then

$$\|\mathcal{F}^n\varphi\|_s \leq \tau\,\|\varphi\|_s + \bar{B}$$

So for $M \geq M_0$ the set $\mathcal{A}_{\mathsf{M}}$ is invariant where $M_0 = \bar{B}(1-\tau)^{-1}$.

# Proof: Theorem 2

**Lemma 3**

*The set* $\mathcal{A}_{\mathbf{M}} := \mathcal{B}_{s,M} \cap \tilde{\mathcal{B}}_{BV^1,M_1} \cap \tilde{\mathcal{B}}_{BV^2,M_2} \cap \mathcal{M}_{1,\mathbf{Leb}}$ *is convex.*

Proof.

- $\mathbf{var}_{p,BV^1}(\tau\varphi + (1-\tau)\psi) \leq \tau \; \mathbf{var}_{p,BV^1}(\varphi) + (1-\tau) \; \mathbf{var}_{p,BV^1}(\psi)$.
- $\|\tau\varphi + (1-\tau)\psi\|_{\text{“1”}} \leq \tau \; \|\varphi\|_{\text{“1”}} + (1-\tau) \; \|\psi\|_{\text{“1”}}$
- For a.e. $z \in [0,1]$, $\psi_z, \varphi_z \in \mathcal{B}_{BV^i,M_i}$ then

$$\|\tau\varphi_z + (1-\tau)\psi_z\|_{BV^i} \leq M_i.$$

$\square$

**Lemma 4**

*The set $\mathcal{A}_{\mathrm{M}}$ is relatively compact in $\mathcal{B}_w$.*

Proof.

- $\mathcal{B}_{\text{p-BV}} = \{\varphi : [0,1] \times \mathbb{T} \to \mathbb{R} : \ \|\varphi\|_{\text{p-BV}} < \infty\}$ with $\|\varphi\|_{\text{p-BV}} = \|\varphi\|_{\text{“1”}} + \boldsymbol{var}_{p,W^1}(\varphi)$, then $\mathcal{B}_{\text{p-BV,M}}$ is relatively compact in $\mathcal{B}_w$
- $\|\cdot\|_{\text{p-BV}} \leq \|\cdot\|_s$ and $\mathcal{B}_{s,M} \subset \mathcal{B}_{\text{p-BV},M}$.
- $\bar{\mathcal{B}}_{s,M}$ is compact in $\mathcal{B}_w$. Since $\mathcal{A}_{\mathrm{M}} \subset \mathcal{B}_{s,M}$. So $\bar{\mathcal{A}}_M$ is a closed subset of a compact space then $\mathcal{A}_{\mathrm{M}}$ is relatively compact in $\mathcal{B}_w$.

$\square$

# Proof: Theorem 2

**Lemma 5**

*The SCTO is Lipschitz continuous*

$$\mathcal{F} : (\mathcal{B}_{S,M} \cap \mathcal{M}_{1,\mathbf{Leb}}, \|\cdot\|_{\text{“1”}}) \to (\mathcal{B}_{S,M} \cap \mathcal{M}_{1,\mathbf{Leb}}, \|\cdot\|_{\text{“1”}})$$

Proof.

$$\|\mathcal{F}\varphi - \mathcal{F}\psi\|_{\text{“1”}} \leq (\kappa + 2\alpha \|h\|_{\mathcal{C}^1} \|W\|_\infty) \|\varphi - \psi\|_{\text{“1”}}.$$

$\square$

Proof of theorem 2.

- Lemma 2: $\mathcal{A}_{\mathsf{M}}$ is invariant under the action of the STO.
- Lemma 5: $\mathcal{F} : \mathcal{A}_{\mathsf{M}} \to \mathcal{A}_{\mathsf{M}}$ is Lipschitz continuous.
- Lemma 4: $\overline{\mathcal{A}}_M$ is relatively compact in $\mathcal{B}_w$.
- Lemma 3: $\overline{\mathcal{A}}_M$ is convex.
- The map $\mathcal{F}|_{\mathcal{A}_{\mathsf{M}}}$ uniformly continuous then there exists a unique continuous extension $\overline{\mathcal{F}} : \overline{\mathcal{A}}_M \to \mathcal{B}_w$ satisfying $\overline{\mathcal{F}}|_{\mathcal{A}_{\mathsf{M}}} = \mathcal{F}$. Moreover, since $\mathcal{F}^n(\mathcal{A}_{\mathsf{M}}) \subset \mathcal{A}_{\mathsf{M}}$, continuity ensures that $\overline{\mathcal{F}}^n(\overline{\mathcal{A}}_M) \subset \overline{\mathcal{A}}_M$. Therefore, Schauder's Fixed Point Theorem guarantees the existence of a fixed point $\varphi^* \in \overline{\mathcal{A}}_M$.

$\square$

# Exponential Stability of the Fixed Point

**Theorem 3 (Exponential Stability of the Fixed Point)**

*Let $\varphi^* \in \mathcal{A}_{\mathsf{M}}$ be a fixed point of the SCTO. The fixed point is unique and locally exponentially stable meaning that there is $\delta > 0$ and constants $K > 0$, $\rho > 0$ such that for every $\varphi \in \mathcal{M}_{1,\mathsf{Leb}}$ with $\|\varphi_z - \varphi_z^*\|_{\mathcal{C}^1} \leq \epsilon$ for a.e. $z \in [0,1]$, the following holds*

$$\sup_{x \in \mathbb{T}} |(\mathcal{F}^t \varphi)_z(x) - \varphi_z^*(x)| \leq K e^{-\rho t}.$$

Consider

$$\mathcal{V}_a := \left\{ \psi \in \mathcal{C}(\mathbb{T}, \mathbb{R}) : \frac{\psi(x)}{\psi(y)} \leq e^{a|x-y|} \right\}$$

define

$$\tilde{\mathcal{V}}_a := \left\{ \nu \in \mathcal{M}_{\mathsf{Leb}}([0,1] \times \mathbb{T}) : \frac{d\nu_z}{dLeb} \in \mathcal{V}_a \text{ for a.e. } z \right\}.$$

For $\nu$, $\nu' \in \tilde{\mathcal{V}}_a$, define the distance

$$\tilde{\theta}_a(\nu, \nu') := \operatorname*{ess\,sup}_{z \in [0,1]} \theta_a \left( \frac{d\nu_z}{dLeb}, \frac{d\nu'_z}{dLeb} \right).$$

# Proof: Theorem 3

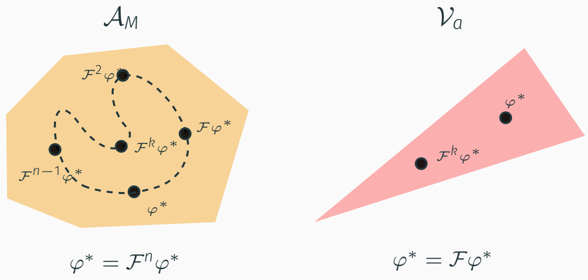
**Proposition 4**

*If $\alpha$ is sufficiently small, there is $a > 0$ such that $\mathcal{F}$ sends $\tilde{\mathcal{V}}_a$ into $\tilde{\mathcal{V}}_{\eta a}$ for some $\eta \in [0, 1)$. Furthermore, suppose that $\varphi, \psi \in \mathcal{M}_{1,\mathrm{Leb}} \cap \tilde{\mathcal{V}}_a$ and for almost every $z \in [0, 1]$, $\psi_z \in \mathcal{C}^2(\mathbb{T})$ with $\operatorname{ess\,sup}_{z\in[0,1]} \|\psi_z\|_{\mathcal{C}^2} < \infty$. Then there is $\gamma \in [0, 1)$ such that*

$$\tilde{\theta}_a(\mathcal{F}\varphi, \mathcal{F}\psi) \leq \gamma\, \tilde{\theta}_a(\varphi, \psi).$$

$$\begin{aligned}
\theta_a((\mathcal{F}\varphi)_z, (\mathcal{F}\psi)_z) &= \theta_a((F_{\varphi,z})_*\varphi_z, (F_{\psi,z})_*\psi_z) \\
&\leq \theta_a((F_{\varphi,z})_*\varphi_z, (F_{\varphi,z})_*\psi_z) + \theta_a((F_{\varphi,z})_*\psi_z, (F_{\psi,z})_*\psi_z) \\
&\leq \lambda\theta_a(\varphi_z, \psi_z) + C_\# \alpha \operatorname*{ess\,sup}_{z'\in[0,1]} \theta_a(\varphi_{z'}, \psi_{z'}).
\end{aligned}$$

# Proof: Exponential Stability of the Fixed Point



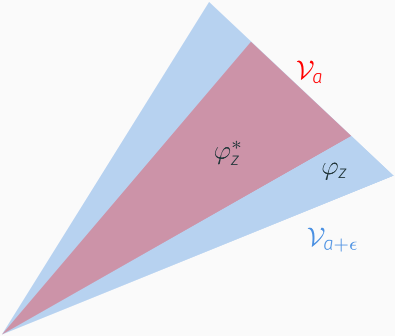

Note that $\operatorname{ess\,sup}_{z\in[0,1]} \|(\mathcal{F}^i\varphi^*)_z\|_{\mathcal{C}^2} < \infty$ for $0 \le i \le n-1$

$$\tilde{\theta}_a(\varphi^*, \mathcal{F}\varphi^*) = \tilde{\theta}_a\left(\mathcal{F}^n\varphi^*, \mathcal{F}^{n+1}\varphi^*\right) \le \lambda^n \tilde{\theta}_a(\varphi^*, \mathcal{F}\varphi^*)$$

which implies $\theta_a(\varphi^*, \mathcal{F}\varphi^*) = 0$, then

$$\varphi^* = \mathcal{F}\varphi^*$$

# Proof: Exponential Stability of the Fixed Point




$$\tilde{\theta}_{a+\varepsilon}(\mathcal{F}\varphi,\varphi^*) = \operatorname*{ess\,sup}_{z\in[0,1]} \theta_{a+\varepsilon}\left((\mathcal{F}\varphi)_z,(\mathcal{F}\varphi^*)_z\right) \leq \gamma \operatorname*{ess\,sup}_{z\in[0,1]} \theta_{a+\varepsilon}\left(\varphi_z,\varphi_z^*\right).$$

By iterating this process

$$\tilde{\theta}_{a+\varepsilon}(\mathcal{F}^n\varphi,\varphi^*) \leq \gamma^n\,\tilde{\theta}_{a+\varepsilon}\left(\varphi,\varphi^*\right)$$

for uniqueness

$$\tilde{\theta}_a(\hat{\nu},\nu^*) = \tilde{\theta}_a(\mathcal{F}\hat{\nu},\mathcal{F}\nu^*) \leq \gamma\,\tilde{\theta}_a(\hat{\nu},\nu^*)$$

Thank you!

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