Eduardo Garibaldi   João Gomes   Marcelo Sobottka

THERMOGAMAS Webinar

Maximizing measures for countable alphabet shifts via blur shift spaces

(arXiv:2507.18736)

Ergodic Optimization

Framework.

- Let Σ be any one-sided shift space defined over a (countable) alphabet $\mathscr{A}$;
- Any potential $A : \Sigma \to \mathbb{R}$ which is bounded from above;
- We study the ergodic maximizing constant

$$\beta(A) = \sup\left\{ \int_\Sigma A\,d\mu \,:\, \mu \text{ is a } \sigma\text{-invariant probability measure} \right\}.$$

and the maximizing measures whose $\int A\,d\mu = \beta(A)$.

Main Objective

Provide sufficient conditions for the existence of maximizing probabilities measures.

Our approach is based on compactification method for countable alphabet shifts.

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Coercive hypothesis: $$\lim_{i\to\infty} \sup A|_{[i]} = -\infty,$$

the existence of maximizing measure was proved by:

- [2006 - Jenkinson, Mauldin and Urbański] for primitive countable alphabet subshifts of finite type;
- [2010 - Bissacot and Garibaldi] for primitive countable alphabet Markov shifts;
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Existence Theorem

Let Σ be a shift over a countable alphabet that satisfies both

- *finite cyclic predecessor assumption:* $\mathscr{P}(a) \cap \mathscr{F}_m(a)$ *is finite for every* $a \in \mathscr{A}$ *and for all* $m \geq 1$;
- *denseness of periodic measures:* *the set of ergodic probabilities supported on periodic orbits of* Σ *is (weak*) dense among the* σ*-invariant measures.*

Then, every upper semi-continuous potential A fulfilling

$$\limsup_{i \to \infty} \sup A|_{[i]} < \beta(A)$$

has a maximizing probability.

Remark. We define the predecessor and follower sets as

$$\mathscr{P}(w) = \{a \in \mathscr{L}_1 : aw \in \mathscr{L}\} \quad \text{and}$$
$$\mathscr{F}_m(w) = \{b \in \mathscr{L}_1 : wvb \in \mathscr{L} \text{ for some } v \in \mathscr{L}_{m-1}\}.$$

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Blur Shift Space

- Let Σ be a shift space over a countable alphabet.
- A set $\mathscr{V} = \{B_1, \ldots, B_s\} \subset 2^{\mathscr{A}}$ be a finite resolution of blurred sets if:
  - B_r is infinite for each $1 \leq r \leq s$;
  - $B_i \cap B_j$ is finite for all $1 \leq i \neq j \leq s$; and $\mathscr{L}_1 \setminus \bigcup_{r=1}^{s} B_r$ is finite.

We define the Blur Shift space $\hat{\Sigma}$ with resolution $\mathscr{V}$ associated with Σ as

Blur Shifts

$$\hat{\Sigma} \subset (\mathscr{A} \sqcup \mathscr{V})^{\mathbb{N}}$$

Original Shift Σ

$$(x_0, x_1, x_2, \ldots) \in \Sigma$$

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$(x_0, x_1, x_2, \dots) \in \Sigma$	$(w, B_r, B_r, \dots) \in \partial_{B_r}\Sigma$

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The convergence on the Blur shift $\hat{\Sigma}$ can be described as follows.

Lemma

1. The sequence $\{x^n\} \subset \hat{\Sigma}$ converges to $x \in \Sigma$ if, and only if, for every positive integer M, there exists an integer $N > 0$ such that $n > N$ implies $x_i^n = x_i$ for all $1 \leq i \leq M$;
2. The sequence $\{x^n\} \subset \hat{\Sigma}$ converges to $(w, B_r, B_r, \ldots) \in \partial\Sigma$ if, and only if, for every finite subset $S \subset B_r$, there exists an integer $N > 0$ such that $n > N$ implies $x_i^n = w_i$ for all $0 \leq i < \ell(w)$ and $x_{\ell(w)}^n = B_r$ or $x_{\ell(w)}^n \in B_r \setminus S$.

Example. $(w, k, x_n, x_{n+1}, \ldots) \longrightarrow (w, B_r, B_r, \ldots)$ as $k \in B_r$ tends to ∞.

From this topological structure, we obtain

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Blur Shift Map

We now introduce the (blur) shift map $\hat{\sigma} : \hat{\Sigma} \longrightarrow \hat{\Sigma}$ as the usual left shift map.

Example. The fixed points

$$(B_r, B_r, \ldots) = \hat{\sigma}\,(B_r, B_r, \ldots)$$

will absorbs all points of $\partial\Sigma$, i.e., $\hat{\sigma}^{\ell(w)}\,(w, B_r, B_r, \ldots) = (B_r, B_r, \ldots)$.

The shift map $\hat{\sigma}$ is continuous only on $\hat{\Sigma} \setminus \hat{\mathcal{L}}_0$.

Example. The discontinuity of $\hat{\sigma}$ can be observed from the convergences of the following sequences, as $k \in B_r$ tends to ∞,

$$\begin{aligned} (k, 0, 0, \ldots) &\to (B_r, B_r, \ldots) \qquad \text{and} \\ \hat{\sigma}\,(k, 0, 0, \ldots) &\to (0, 0, \ldots) \neq \hat{\sigma}\,(B_r, B_r, \ldots)\,. \end{aligned}$$

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Invariant Measures

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Convexity

Proposition

$$\mathcal{M}\big(\hat{\Sigma}, \hat{\sigma}\big) = \mathrm{Conv}\left(\mathcal{M}\left(\Sigma, \sigma\right) \sqcup \left\{\delta_{(B_r, B_r, \dots)} \;:\; 1 \le r \le s\right\}\right).$$

Sketch of the Proof.

1. If $\hat{\mu} \in \mathcal{M}\big(\hat{\Sigma}, \hat{\sigma}\big)$ and $\hat{\mu}(\Sigma) = 1$, then $\hat{\mu} \in \mathcal{M}\left(\Sigma, \sigma\right)$;
2. If $\hat{\nu} \in \mathcal{M}\big(\hat{\Sigma}, \hat{\sigma}\big)$, then $\hat{\nu}\big(\bigsqcup_{k \ge 1} \hat{\mathcal{L}}_k\big) = 0$. In particular,

$$\hat{\nu}\big(\hat{\mathcal{L}}_0\big) = \hat{\nu}\left(\partial\Sigma\right) \qquad \text{and} \qquad \hat{\nu}\left(\left\{\left(B_r, B_r, \dots\right)\right\}\right) = \hat{\nu}\left(\partial_{B_r}\Sigma\right);$$

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1. If $\hat{\mu} \in \mathcal{M}\big(\hat{\Sigma},\hat{\sigma}\big)$ and $\hat{\mu}(\Sigma) = 1$, then $\hat{\mu} \in \mathcal{M}\left(\Sigma,\sigma\right)$;
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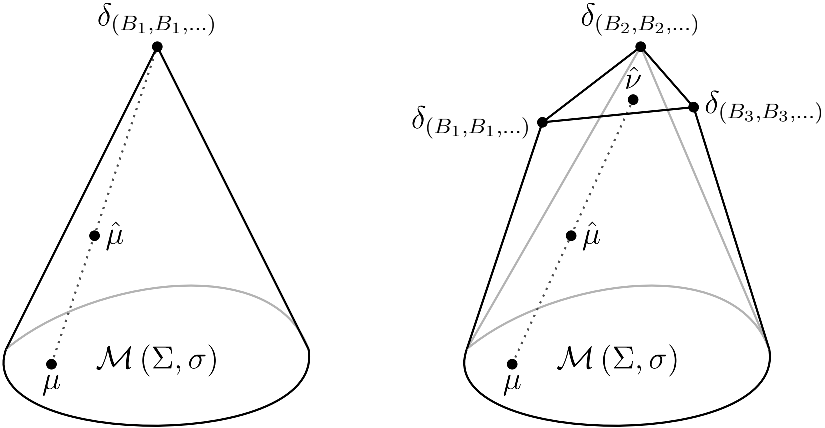

Remark.

The phenomenon of escape of mass described on [Iommi and Velozo 21] can be translated as a transference of mass from Σ to points of $\partial\Sigma$.

Example. Note that

$$\frac{1}{2}\delta_{(k,0,k,0,\dots)} + \frac{1}{2}\delta_{(0,k,0,k,\dots)} \xrightarrow{*} \frac{1}{2}\delta_{(B_r,B_r,\dots)} + \frac{1}{2}\delta_{(0,B_r,B_r,\dots)};$$

as $k \in B_r$ tends to ∞. This limit probability is not invariant.

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Lemma

Let Σ be a shift over a countable alphabet verifying the finite cyclic predecessor assumption and the denseness of periodic measures. Then,

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Sketch of the Proof.

1. Considers a sequence of periodic measures $\dot{\mu}_l$ that converge to $\bar{\mu}$;
2. We analyze three cases based on the unboundedness of symbols or of periods associated to the (periodic) points on the support of $\dot{\mu}_l$;
3. In all cases, we construct an appropriate neighborhood Z of $(a, B_r, B_r, \ldots)$ and by **finite cyclic predecessor assumption**, it is shown that:
   - Either $\dot{\mu}_l\,(Z) = 0$ for large l;
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Potentials

Recall that a potential is

any Borel function $A : \Sigma \to \mathbb{R} \cup \{-\infty\}$ which is bounded from above.

Upper semi-continuity gives us a lot of freedom to obtain extension function on $\hat{\Sigma}$.

Proposition

Let $A : \Sigma \to \mathbb{R} \cup \{-\infty\}$ be an upper semi-continuous potential. Then, the potential $\hat{A} : \hat{\Sigma} \longrightarrow \mathbb{R} \cup \{-\infty\}$ given as

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$$(v, B_r, B_r, \ldots) \in \partial\Sigma \longmapsto \hat{A}\,(v, B_r, B_r, \ldots) = \limsup_{B_r \ni i \to \infty} \sup A|_{[vi]}$$

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We cannot expect continuous function on Σ to admit a continuous extension on $\hat{\Sigma}$.

Example. Let consider

- $(u, B_r, B_r, \ldots)$ be a point of $\partial\Sigma$
- countable partition of $B_r = \bigsqcup_{k\geq 1} C_k$ into infinite subsets.
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We introduce $A : [u] \to [0, 1]$ defined on the cylinder set $[u] \in \Sigma$ as

$$A(uax) = A(ua) = \begin{cases} 0, & \text{if } a \in \mathscr{A} \setminus B_r \\ q_k, & \text{if } a \in C_k \end{cases},$$

where $a \in \mathscr{A}$, $x \in \Sigma$, and $uax \in \Sigma$.

Due to Tietze extension Theorem, there is a bounded continuous $A : \Sigma \to [0, 1]$ which extends this locally constant function to Σ.

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Ergodic Optimization

Note the following convenient behavior on the class of coercive potentials.

Example. Assume that

- A is a coercive upper semi-continuous potential, $\lim_{i\to\infty} \sup A|_{[i]} = -\infty$;
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We have the following integrals

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$$\hat{\mathcal{M}}_{\max}(\bar{A}) = \left\{ \hat{\mu} \in \mathcal{M}(\hat{\Sigma}, \hat{\sigma}) \; : \; \int_{\hat{\Sigma}} \bar{A} \, d\hat{\mu} = \hat{\beta}(\bar{A}) \right\}$$

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$$\hat{\mathcal{D}}_{\max}(\bar{A}) = \left\{ \delta_{(B_r, B_r, \dots)} : \bar{A}(B_r, B_r, \dots) = \max \bar{A}|_{\hat{\mathcal{L}}_0} \right\}$$

The maximizing sets are associated according to

Proposition

1. If $\max \bar{A}|_{\hat{\mathcal{L}}_0} > \beta(A)$, then $\hat{\mathcal{M}}_{\max}(\bar{A}) = \mathrm{Conv}\left(\hat{\mathcal{D}}_{\max}(\bar{A})\right)$.

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Proposition

1. If $\max \bar{A}|_{\hat{\mathcal{L}}_0} = \beta(A)$, then $\hat{\mathcal{M}}_{\max}\big(\bar{A}\big) = \mathrm{Conv}\Big(\mathcal{M}_{\max}(A) \sqcup \hat{\mathcal{D}}_{\max}\big(\bar{A}\big)\Big)$;
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Sketch of the Proof of Existence Theorem.

1. The set of blur measures $\mathcal{M}\big(\hat{\Sigma}, \hat{\sigma}\big)$ is compact;
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