Thermodynamic formalism and Thue Morse subshift.

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Old version on arxiv (without Julien): false.

Substitutions

Consider a finite set $\mathcal{A}$. Then consider the monoid $\mathcal{A}^*$. A **substitution** is a morphism of this free monoid.

$$\begin{cases} 0 \to 01 \\ 1 \to 10 \end{cases}$$

- Thue Morse substitution.

A substitution σ defines a subshift $\mathbb{K} \subset \mathcal{A}^{\mathbb{N}}$. The sequence x is in $\mathbb{K}$ if for every integers n, k, the word $x_n \dots x_{n+k-1}$ appears in some $\sigma^p(a), a \in \mathcal{A}, p \in \mathbb{N}$.

For example $x = 000\dots$ does not appear in the subshift of the Thue Morse substitution.

$$\begin{cases} 0 \to 01 \to 0110 \to 01101001 \\ 1 \to 10 \to 1001 \to 10010110 \end{cases}$$

It defines a subshift of zero entropy and it is uniquely ergodic if the substitution is primitive.

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$$u = \lim_{+\infty} \sigma^n(a),$$

The word u is called a periodic point of the substitution. For Thue Morse we have for example

$$u = 01101001\dots$$

Remark that, in this case, the subshift is also equal to

$$\overline{\{S^n u, n \in \mathbb{N}\}}$$

Consider a finite set F of words in $\mathcal{A}^*$. A **subshift of finite type** (SFT) is a subshift X_F of $\mathcal{A}^{\mathbb{N}}$ such that no word of F appears in an element of X_F.

Examples with $\mathcal{A} = \{0, 1\}$.

- $F = \{11\}$
- $F = \emptyset, X_F = \mathcal{A}^{\mathbb{N}}$.

Entropy

Consider a finite set $\mathcal{A}$. Let S be the shift map on $\mathcal{A}^{\mathbb{N}}$.
Let $(\mathbb{K}, S)$ be a subshift and μ an invariant measure. We can define

- the **metric entropy** h_μ.
- the **topological entropy**:

$$h_{top} = \lim_{+\infty} \frac{\log p(n)}{n},$$

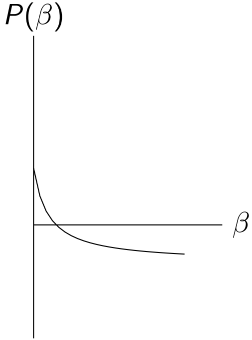
where $p(n)$ is the complexity function of the subshift.

$$\sup_\mu h_\mu = h_{top}$$

$V : \mathbb{K} \to \mathbb{R}_+$ potential.
Pressure function for $(\mathbb{K}, S)$:

$$\begin{array}{ccc} [0,+\infty) & \to & \mathbb{R} \\ \beta & \mapsto & \sup_{\mu}\left(h_\mu - \beta \int V d\mu\right) \end{array}$$

It is a decreasing function with the following properties:.



- For a fixed β, any measure which realizes the maximum is an **equilibrium state**.
- A **phase transition** is some β where the pressure function is not analytic.
- The function P has an asymptote of the form $-a\beta + b$ with $a = \inf\{\int V d\mu, \mu\}$.
- If P reaches its asymptote at some point, we speak of **freezing phase transition**.

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We find some class of potentials in order to have a freezing phase transition such that the equilibrium state after the transition is the measure of unique ergodicity of $\mathbb{K}$.

Potentials

Le $x \in \mathcal{A}^{\mathbb{N}}$ such that $x \notin \mathbb{K}$, then we define:

- The word w is the biggest prefix of x inside the language of $\mathbb{K}$. By definition, $d(x, \mathbb{K}) = 2^{-|w|}$.
- Let $\delta(x) = |w|$ and $\delta_k = \delta(S^k(x))$.

Let us denote Ξ_1 the set of potentials of the following form where $\delta(x) = n$.

$$V : \mathcal{A}^{\mathbb{N}} \to \mathbb{R}$$

$$V(x) = \frac{g(x)}{n+1} + o(\frac{1}{n})$$

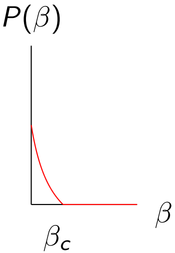
with $g > 0$ on $\mathbb{K}$ and g is a continuous functions.

Good example: $V_0(x) = \frac{1}{n+1}$.

Theorem (B-Cassaigne-Hubert-Leplaideur)

For the Thue Morse subshift $\mathbb{K}$, every potential $V \in \Xi_1$ fulfills:
There exists a phase transition at some β_c:

- *Before β_c the pressure is analytic and the equilibrium state has full support.*
- *After this point, the pressure is equal to zero and the unique ergodic measure of the substitution is the equilibrium state.*
- *Explicit upper bound (17) for β_c for the potential $\frac{1}{n+1}$.*


$P(\beta)$
β
β_c

Theorem (Bruin-Leplaideur 2013, 2015)

For the following substitutions we have

- *Thue-Morse:*
  - *If* $\alpha < 1$ *and* $V \in \Xi_\alpha$, *phase transition.*
  - *If* $\alpha > 1$ *and* $V \in \Xi_\alpha$, *no phase transition.*
- *Fibonacci substitution: For* $V \in \Xi_1$, *Phase transition.*

$$V(x) = \frac{g(x)}{(n+1)^\alpha} + o(\frac{1}{n^\alpha})$$

Proofs of Bruin-Leplaideur are not complete.

Recent results of

- Kucherenko-Quas 2023 ? but for two sided subshift and not for the same speed of convergence for the potential ($\frac{\log n}{n}$).
- See also Chazottes-Kucharenko-Quas 2025 ?

Theorem (Bowen 75, Ruelle 76, Sinai 72)

For an aperiodic irreducible SFT and for a Holder continuous potential, there is no phase transition. For every β, there is an equilibrium state which is a Gibbs measure.

Gibbs measure

Definition

The Gibbs measure of a potentiel V is an invariant measure μ such that there exist p, K such that for every $n \in \mathbb{N}$, for every cylinder C of length n and $x \in C$ we have

$$\frac{1}{K} \leq \frac{\mu(C)}{exp(S_n V(x) - np)} \leq K$$

Remark that $p = P_V$ and $P_V = h_\mu + \int V d\mu$.

Method of Ruelle

For the SFT and a potential V we define

$$L_V(f)(x) = \sum_{Sy=x} e^{V(y)} f(y)$$

$$L_V^n(f)(x) = \sum_{S^n y=x} e^{S^n V(y)} f(y)$$

Then L_V is an operator defined over continuous functions if V is.

Dual operator defined over measures

$$\mu \mapsto \nu,$$

$$\int f d\nu = \int L_V(f) d\mu$$

Theorem (Ruelle)

For an irreducible aperiodic SFT and for an Holder continuous potential V, there exists an unique eigenvalue $\lambda > 0$, an Holder eigenfunction $h > 0$ and a measure ν such that

- $L_V h = \lambda h$
- $L_V^* \nu = \lambda \nu$
- *Moreover $\mu = h\nu$ is an invariant measure and an equilibrium state.*
- *There exists $\theta < 1$ and $C > 0$ such that for every function f we have*
$$||\lambda^{-n} L_V^n f - h||_\infty \leq C\theta^n.$$
- *The measure μ is a Gibbs measure of pressure $\log \lambda$.*

Our method

We define a transfert operator:

Consider a word w_J outside the language of $\mathbb{K}$ and V be a potential constant on the cylinder $J = [w_J]$.

Let $\tau(x)$ be the return time on this cylinder for the shift and let g be a function from J to $\mathbb{R}$.

We define the **transfert operator** for $\beta > 0$ by :

$$(S_N V)(y) = \sum_{k=0}^{N-1} V \circ S^k(y)$$

$$\mathcal{L}_{z,\beta,V}(g)(x) = \sum_{n\in\mathbb{N}} \sum_{\substack{\tau(y)=n \\ S^n(y)=x}} e^{-\beta S_n V(y) - nz} g(y)$$

Theorem (first citation by Leplaideur in 2000).

Theorem

Assume there exists β_0 such that for $\beta > \beta_0$ and $x \in J$ we have:

$$\mathcal{L}_{0,\beta,V}(1_J)(x) < 1$$

Then $P(\beta) = z_c(\beta) = 0$ for $\beta > \beta_0$ and the unique equilibrium state is $\mu_{\mathbb{K}}$.

The construction does not depend on J !

Let $x \in J$, we need to compute for $\beta >> 1$:

$$\mathcal{L}_{0,\beta,V_0}(1_J)(x) = \sum_{n\in\mathbb{N}} \sum_{\substack{\tau(y)=n \\ S^n(y)=x}} e^{-\beta S_n V(y)}$$

Accident

Consider x with $\delta(x) = p$. If there exists some integer k such that $\delta(S^k x) > p - k$ and $\delta(S^i x) = p - i$ for all $i < k$, then we speak of **accident at time** k.

A **right special** word is a word which has several right extensions.

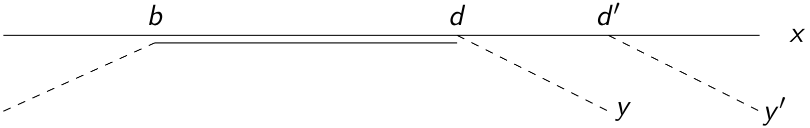
A **right special** word is a word which has several left extensions.

A **bispecial** word is a left and right special word.

Lemma

Let x be an infinite word outside $\mathbb{K}$. Assume $\delta(x) = d$ and that the first accident appears at $b \leq d$. Then we have:

- $x_b \ldots x_{d-1}$ *is a bispecial word of* $\mathcal{L}$.
- $x_0 \ldots x_{d-1}$ *is not a special word.*



If y has no accident, then it is easy to compute $S_N V(y)$:

$$S_N V(y) = \sum_{k=0}^{N-1} \frac{1}{p-k}.$$

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The problems come from accidents...

A **minimal forbidden word of** $\mathbb{K}$ is a word w which is not in L_{TM}, and has minimal length in the sense that each of its proper factors is in L_{TM}.

Lemma

The word w is a minimal forbidden word for L_{TM} if and only if it is a forbidden bilateral extension of a bispecial word.

Now let $R(w) = \{u \neq \varepsilon, uw \in w\mathcal{A}^*, uw \notin \mathcal{A}^+ w \mathcal{A}^+\}$, be the set of return words to w.

Proof in four lines

Assume w_J is a minimal forbidden word of L which defines J.

$$\mathcal{L}_{0,\beta,V_0}(1_J)(x) = \sum_{u \in R(w_J)} \prod_{k=0}^{|u|-1} (1 + \frac{1}{\delta(\sigma^k(uw_J))})^{-\beta}.$$

$$= \sum_{M \geq 0} \sum_{\substack{u \in R(w_J) \\ \text{M accidents}}} [\frac{(|u^0|+1)\dots(|u^{M-1}|+1)(|u^M|+1)}{(|v^1|+1)\dots(|v^M|+1)(|w_J|-1)}]^{-\beta}.$$

$$\sum_M S_M(w_J) = \sum_{M \geq 0} (A^{M+1})_{(a,v,b),(a,v,b)}$$

where A is an infinite matrix.