Strong natural measures and distributional convergence of empirical measures for intermittent maps

Joint work with Ian Melbourne and Amin Talebi

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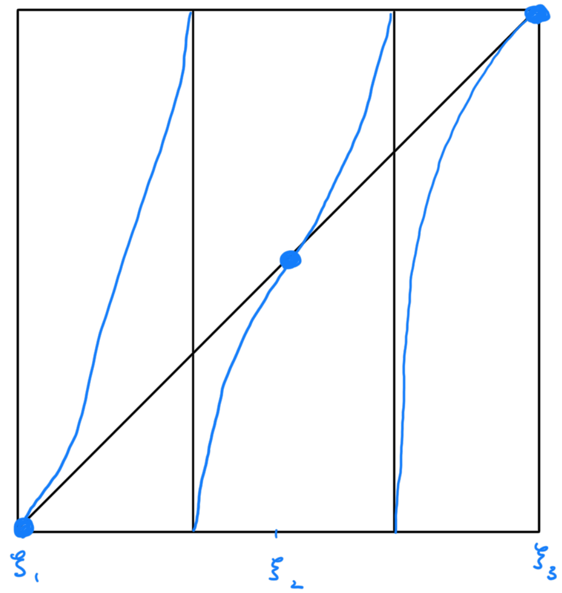
Thaler's intermittent maps

Let $X = [0,1]$, and consider $f : [0,1] \to [0,1]$ such that

- **$d \geq 2$ full branches:** $[0,1] = I_1 \cup \cdots \cup I_d$, sub-intervals with $f : \overline{I}_k \to [0,1]$ a orientation preserving C^2 diffeomorphism

So, f has d fixed points $\xi_1, ..., \xi_d$.

- **non-uniform expansion:** $f'(x) > 1, \quad x \notin \{\xi_1, ..., \xi_d\}$, and $f'(\xi_k) = 1$
- **nice-expansion near fixed points:** $\alpha \in (0,1)$, $f(x) - x \sim b_k(x - \xi_k)^{1+\frac{1}{\alpha}}$



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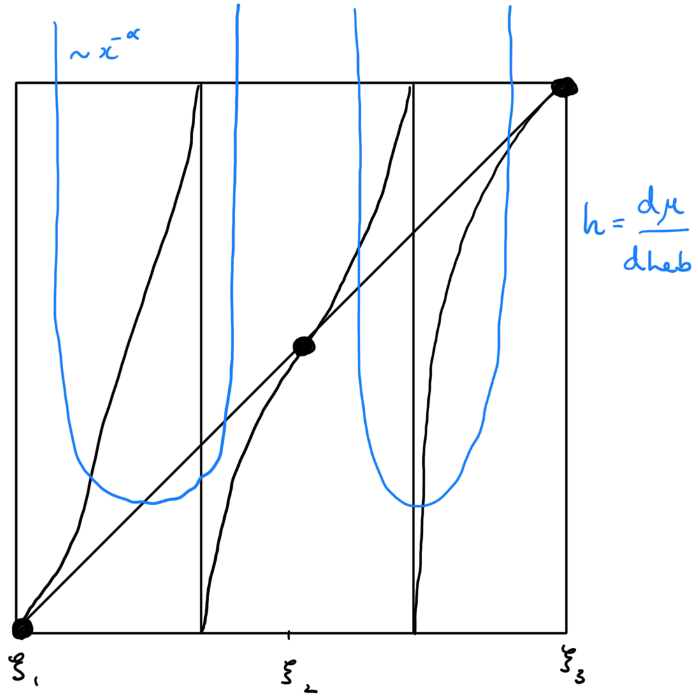
Thaler's intermittent maps

Theorem (Thaler): There exists a unique ergodic absolutely continuous σ-finite measure μ.



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Almost sure behaviour of the physical measures.

Let $e_n : [0,1] \to \mathcal{M}_1([0,1])$ denote the n^{th} empirical measure

$$e_n(x) = \frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k(x)}.$$

Non-statistical behaviour (C., Melbourne, Talebi, 24): For almost every $x \in [0,1]$

$$\{\text{limit points of } e_n(x)\} = \left\{p_1\delta_{\xi_1} + \cdots + p_d\delta_{\xi_d} : \sum_{i=1}^{d} p_i = 1, \text{and } p_i \geq 0\right\}.$$

- $e_n(x)$ does not converge for almost every x (non-statistical behaviour)
- no physical measures

Distributional behaviour of the empirical measures

- let $\mathcal{S} = \{(p_1, ..., p_d) \in \mathbb{R}^d_{\geq 0} : p_1 + \cdots + p_d = 1\}$
- for $p \in \mathcal{S}$ let $\nu_p = p_1\delta_{\xi_1} + \cdots + p_d\delta_{\xi_d}$, recall: $e_n(x)$ accumulates at $\{\nu_p : p \in \mathcal{S}\}$

Distributional convergence of e_n (C., Melbourne, Talebi, 24): There exists a random variable Z taking values in $\mathcal{S}$ so that

$$e_n \xrightarrow{d} \nu_Z,$$

i.e. for $A \subset \mathcal{M}_1([0,1])$

$$\mathrm{Leb}(x : e_n(x) \in A) \to \mathbb{P}(Z \in \pi(A)),$$

where $\pi : M_1([0,1]) \to \mathcal{S}$ is the natural identification.

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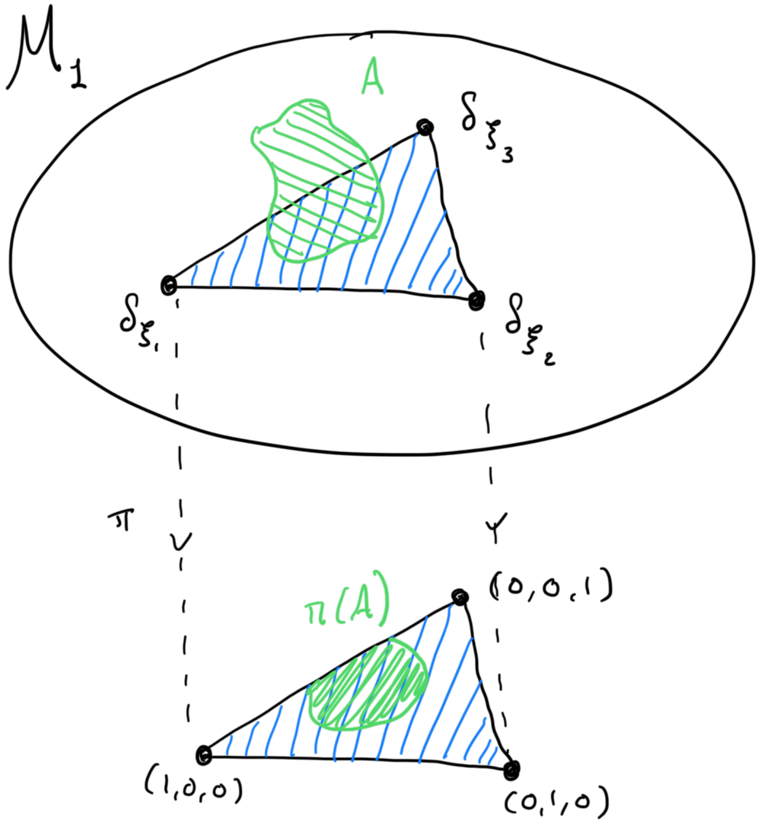
Distributional behaviour of the empirical measures

For $A \subset \mathcal{M}_1([0,1])$,

$$\mathrm{Leb}(x : e_n(x) \in A) \to \mathbb{P}(Z \in \pi(A)),$$

and $\mathbb{P}(Z \in \cdot)$ can be computed for nice sets.



Natural measures

Corollary: There exists a $p^* \in \mathcal{S}$ so that for every probability measure $\lambda \ll \mathrm{Leb}$

$$\frac{1}{n}\sum_{k=0}^{n-1} f_*^k \lambda \to \nu_{p^*}.$$

Strong natural measures (C., Melbourne, Talebi, 24): For every probability measure $\lambda \ll \mathrm{Leb}$

$$f_*^n \lambda \to \nu_{p^*}.$$

Decay of correlations

Corollary: For every bounded u continuous at the $\xi_1, \ldots, \xi_d$,

$$\int v \cdot u \circ f^n \, \mathrm{d}\,\mathrm{Leb} \to \int v \, \mathrm{d}\,\mathrm{Leb} \int u \, \mathrm{d}\nu_{p^*} \text{ for all } u \in L^1(\mathrm{Leb}),$$

or equivalently,

$$\int v \cdot u \circ f^n \, \mathrm{d}\mu \to \int v \, \mathrm{d}\mu \int u \, \mathrm{d}\nu_{p^*} \text{ for all } v \in L^1(\mu).$$

Thank you!