# Geometric properties of probability spaces: disintegration of measures

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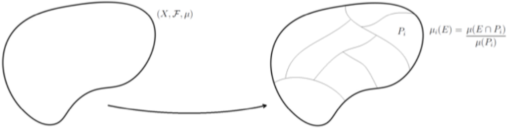
Based on joint works with Gomes, Münch, and Possobon

# Introduction

The disintegration of a measure over a partition of the space on which it is defined is a way to rewrite this measure as a combination of probability measures, which are concentrated on the elements of the partition.




Partition into a finite number of measurable subsets $P_1, \ldots, P_n$ with $\mu(P_i) > 0$, $i = 1, \ldots, n$.

$$\mu(E) = \sum_{i=1}^{n} \mu_i(E)\mu(P_i) = \sum_{i=1}^{n} \mu(P_i)\frac{\mu(E \cap P_i)}{\mu(P_i)}.$$

# Introduction

- The disintegration of measures plays an important role on the understanding of statistical properties in many areas such as Dynamical Systems, Geometry, and Probability Theory.
- Example: In Ergodic Theory, the possibility of disintegrating a probability measure is directly related to the ergodic decomposition of invariant measures, which are objects encoding the asymptotic behavior of dynamical systems.

# Disintegration of measures

**Definition: disintegration**

Consider:

- $(X, \mathcal{F}, \mu)$ probability space;
- $\mathcal{P}$ partition of $X$ into measurable subsets;
- $\pi$ the natural projection that associates each $x \in X$ to the element $P \in \mathcal{P}$ which contains $x$;
- $\hat{\mu} := \pi_* \mu := \mu \circ \pi^{-1}$.

A **disintegration** of $\mu$ with respect to $\mathcal{P}$ is a family $\{\mu_P : P \in \mathcal{P}\}$ of probability measures on $X$ such that:

1. $\mu_P(P) = 1$ for $\hat{\mu}$-almost every $P \in \mathcal{P}$;
2. $P \mapsto \mu_P(E)$ is measurable for all $E \in \mathcal{F}$;
3. $\mu(E) = \int \mu_P(E) \, d\hat{\mu}(P)$ for all $E \in \mathcal{F}$.

Geometric information is not taken into account while studying the disintegration of measures.

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# On optimal transport

Basic problem (G. Monge, 1781): moving a distribution like a pile of sand from a place to another at minimum cost.

Monge-Kantorovich Problem: $X$, $Y$ Radon spaces, $\mu \in \mathscr{P}(X)$, $\nu \in \mathscr{P}(Y)$, and $c : X \times Y \to [0, \infty]$ fixed Borel cost function, minimise

$$\gamma \mapsto \int_{X \times Y} c(x, y)\, d\gamma(x, y)$$

among all measures $\gamma \in \mathscr{P}(X \times Y)$ with marginals $\mu$ and $\nu$.

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# Wasserstein spaces

Assume that the Monge-Kantorovich Problem is defined for $\mu, \nu \in \mathscr{P}(M)$.

1. When $(M, d)$ is a metric space:

$$W(\mu, \nu) = \left( \inf_{\gamma \in \Pi(\mu \times \nu)} \int d(x_1, x_2)^p d\gamma(x_1, x_2) \right)^{1/p}$$

   $\Pi(\mu \times \nu)$ is the set of $\gamma$ with marginals $\mu$ and $\nu$.

   Constraining $W_p$ to a subset in which it takes finite values:

2. $(\mathscr{P}(M), W_p)$ is a metric space.
3. Moreover $\mathscr{P}(M)$ inherits properties of $M$.
4. We can study geometry on $(\mathscr{P}(M), W_2)$.

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How is the disintegration of measure associated with the Optimal Transport Theory?

**Theorem A - Disintegration of measures (Possobon, R. 2025)**

- $X$ and $Y$ locally compact and separable metric spaces;
- $\pi : X \to Y$ Borel map;
- $\mu$ in $\mathcal{M}_+(X)$;
- $\nu = \pi_* \mu \in \mathcal{M}_+(Y)$.

Then, there exists measures $\{\mu_y \in \mathcal{M}_+(X)\}_{y \in Y}$ such that:

1. $y \mapsto \mu_y$ is a Borel map;
2. $\mu_y$ is a probability on $X$ for $\nu$-almost every $y \in Y$;
3. $\mu = \nu \otimes \mu_y$, that is, $\mu(A) = \int_Y \mu_y(A)\, d\nu(y)$ for all measurable subset $A \subset X$;
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$$f : Y \to (\mathscr{P}(X), W_2)$$
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## Theorem B (Possobon, R. 2025)

- $X$, $Y$ be locally compact, complete, separable metric spaces;
- $\pi : X \to Y$ Borel map;
- $\mu \in \mathcal{M}_+(X)$;
- $\nu := \pi_* \mu$.

If the disintegration map of $\mu$ with respect to $\nu$ is weakly continuous and $Y$ is path connected, then given two points $y, y' \in Y$:

(i) there exists a path on $(\mathscr{P}_2(X), W_2)$, given by the disintegration map, connecting $\mu_y$ and $\mu_{y'}$, the respective conditional measures given by Theorem A;

(ii) if $X$ is a smooth compact Riemannian manifold equipped with a volume measure vol, $\mu \ll \mathrm{vol}$, $\pi$ is such that $\pi^{-1}(y)$ has $\mu$-positive measure for $\nu$-almost every $y$, the disintegration map is minimising invariant, and either $\mu_y$ or $\mu_{y'}$ is absolutely continuous w.r.t. vol, then all the measures $\mu_{y_t}$ on the path given by item (i) are absolutely continuous w.r.t. vol;

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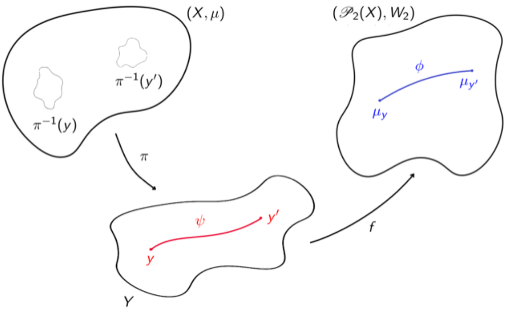


- Existence of $\phi$.
- $\mu \ll \mathrm{vol}$, $\mu(\pi^{-1}(y)) > 0$ for $\nu$-a.e. $y$, $f$ minimising invariant, and either $\mu_y \ll \mathrm{vol}$ or $\mu_{y'} \ll \mathrm{vol} \implies \mu_{y_t} \ll \mathrm{vol}$.

# On foliations and disintegration maps

- $X$, $Y$ locally compact separable metric spaces;
- $\pi : X \to Y$ Borel map;
- $\mu \in \mathcal{M}_+(X)$;
- $\nu := \pi_* \mu$;
- $\{\mu_y\}_{y \in Y}$ a disintegration of $\mu$ with respect to $\nu$ given by Theorem A;
- $f$ the disintegration map;

$$|\nabla f(y)|_p := \lim_{\varepsilon \to 0} \sup_{y', y'' \in B_\varepsilon(y)} \frac{W_p(\mu_{y'}, \mu_{y''})}{d_Y(y', y'')}$$

$$\mathcal{E}_p(f) := \|\nabla f\|_{\infty,p} = \sup_{y \in Y} |\nabla f(y)|_p.$$

- $|\nabla f(y)|_p \geq 1$ everywhere.
- $\min_f \ \mathcal{E}_p(f) = 1 \iff \|\nabla f\|_{\infty,p} = 1 \iff |\nabla f(y)| = 1$ for all $y$. In this case, we have an isometry $W_p = d_Y$.

**Theorem C (Münch, Possobon, R. 2025)**

- $(X, d)$ locally compact geodesic Polish space;
- Y compact Polish space;
- $\pi : X \to Y$ Borel map;
- $\mu \in \mathcal{M}_+(X)$;
- $\nu := \pi_* \mu$;
- $\{\mu_y\}_{y \in Y}$ disintegration of $\mu$ w.r.t. $\nu$ such that $\operatorname{supp}(\mu_y) = \pi^{-1}(y)$;
- $f_{\mu,\pi}$ the related disintegration map.

Then,

$$\min_{f_{\mu,\pi}} \mathcal{E}_p(f_{\mu,\pi}) = 1 \iff \{\pi^{-1}(y)\} \text{ "uniform" foliation on } X.$$

# Isometric group actions

## Definition: Isometric group actions

Given a group $G$ (with identity $e$) and a Riemannian manifold $M$, an **isometric group action** of $G$ on $M$, denoted by $G \curvearrowright M$, is a group homomorphism $\theta : G \to Iso(M)$, where $Iso(M)$ is the group of isometries of $M$. For $g \in G$ and $x \in M$ we denote $g \cdot x$, the action of $g$ on $x$. By the Myers-Steenrod Theorem, $G$ is actually a Lie group.

## Definition: Orbits and quotient spaces

Given $x \in M$ we define its **orbit** as the set $G \cdot x := \{g \cdot x : g \in G\}$, the **quotient space** of this action as

$$M/G := \{G \cdot x : x \in M\},$$

and endow it with the quotient topology under the projection $\pi : M \to M/G$ defined via $\pi(x) = G \cdot x$.

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**Theorem D (Gomes, R. 2023)**

Let $G \curvearrowright M$ be an isometric group action with group $G$ being compact and $M$ a complete Riemannian manifold equipped with geodesic distance $d$. Then, convexity properties of an entropy-like functional defined on the disintegrated measures on the orbits guarantees necessary and sufficient conditions to the Ricci curvature on directions related to the orbits to be bounded below by a constant $K \in \mathbb{R}$.

# References


- R. Possobon and C. Rodrigues; “Geometric properties of disintegration of measures” in *Ergodic Theory and Dynamical Systems* 45, 1619-1648 (2025)
- F. Münch, R. Possobon, and C. Rodrigues; “Classification of foliations via disintegration maps” in [arXiv:2509.14209]
- A. M. de S. Gomes, and C. Rodrigues; “Rigidity of curvature bounds of quotient spaces of isometric actions” in [arXiv:2310.15332]

Thanks for your attention!