On intrinsic ergodicity of bounded density shifts

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Motivation

Equilibrium states are a concept that comes from statistical mechanics. They are probability measures on spaces characterized by variational principles.

The idea is maximize the sum of an entropy and an energy like quantity.

Depending on the variational problem these measures can have several propierties.

Subshifts

Consider $\mathcal{A} = \{0, 1, \ldots, n\}$ a finite set of symbols. We endow $\mathcal{A}$ with the discrete topology and $\mathcal{A}^{\mathbb{Z}}$ with the product topology.

The set $\mathcal{A}^{\mathbb{Z}}$ is given by

$$\mathcal{A}^{\mathbb{Z}} = \{x = (x_i)_{i\in\mathbb{Z}} : x_i \in \mathcal{A} \text{ for all i } \in \mathbb{Z}\} \tag{1}$$

A typical point is given by

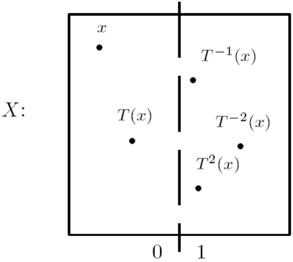
$$x = \cdots x_{-2}x_{-1}.x_0x_1x_2\cdots$$

And the shift map $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by $\sigma(x) = \cdots x_{-1}x_0.x_1x_2\cdots$. We call $\mathcal{A}^{\mathbb{Z}}$ the full $\mathcal{A}$-shift.

Subshift

A set $X \subset \mathcal{A}^{\mathbb{Z}}$ is a subshift if it is closed and $\sigma(X) \subseteq X$.

Subshifts can be seen as codifications of dynamical systems.



It is also possible to establish a connection between symbolic dynamics and number theory, for example β-shifts.

The golden mean shift

Example

If we restrict the word 11 in the appearance of our binary sequences, we have a subshift known as the golden mean shift $X_{\{11\}}$. We can see that

$$\begin{aligned}\mathcal{L}(X_{\{11\}}) &= \{\varepsilon, 0, 1, 00, 01, 10, 000, 001, 010, 100,\\ &\quad 101, 0000, 0001, 0010, 0101, 1000, 1001, 1010...\}\end{aligned}$$

Consider $w = 1010101$. Is straightforward verify that

$$\sum_{i=1}^{7} w_i \leq \frac{7+1}{2}.$$

In general, we can see that the inequality holds for every word in the language of $X_{\{11\}}$. i.e.

$$\sum_{i=1}^{n} w_i \leq \frac{n+1}{2}.$$

Bounded density shifts

The above motivates to definition of bounded density shifts

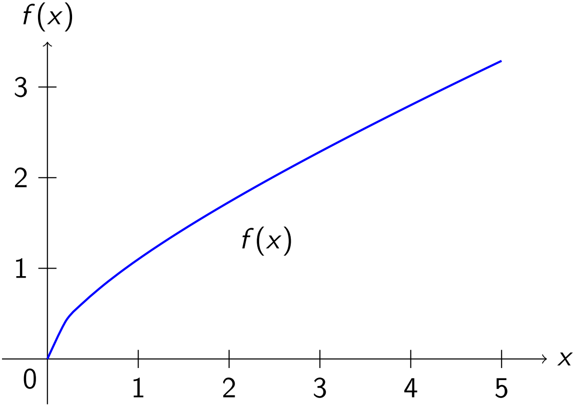
Definition

Let $f : \mathbb{N}_0 \to [0, \infty)$ be a function.
The **bounded density shift** (BDS) associated to a function f, is defined as follows:

$$X_f = \left\{ x \in (\mathbb{N}_0)^{\mathbb{Z}} : \forall p \in \mathbb{N} \text{ and } \forall i \in \mathbb{Z} \sum_{r=i}^{i+p-1} x_r \leq f(p) \right\}. \qquad (2)$$

The alphabet for bounded density shifts is $\{0, 1, \ldots, \lfloor f(1) \rfloor\}$.

We have that $X_{\{11\}} = X_f$ with $f(n) = \lfloor \frac{n+1}{2} \rfloor$.



The accumulation function F_x of $x \in \mathcal{A}^{\mathbb{Z}}$ is defined as

$$F_x(n) = \sum_{i=1}^{n} x_i.$$

A point x is in the BDS if the acumulation function F_x associated to the point (and it's shifts) are below f.

A non BDS

Not every subshift can be seen as a BDS. Consider X the shift defined by having an even amount of 0 between consecutive symbols 1.

$$\begin{aligned} x &= \ldots 100100.00100001\ldots \\ y &= \ldots 100110.00000100\ldots \end{aligned}$$

Then, for example $u = 110$ is in the language of X but $v = 101$ is not. But

$$\sum_{i=1}^{|v|} v_i \leq \sum_{i=1}^{|u|} u_i \leq f(|u|).$$

BDS are part of a broader family of subshifts known as hereditary subshifts.

Definition

Let X be a subshift and $\leq$ a partial order on $\mathcal{A}$. We say X is $\leq -$**hereditary** (or simply hereditary) if for every $x \in \mathcal{A}^{\mathbb{Z}}$ such that there exists $y \in X$ such that $x \leq y$ (coordinate-wise) then $x \in X$.

Examples of hereditary shifts are: $\beta-$shifts, Bounded density shifts, $\mathcal{B}-$free systems and some S-gaps.

Similarly to β-shifts, BDS are characterized by an infinite one-sided maximal sequence. The one sided subshift $X_{\{11\}}$, is a $\beta-$shift X_β. With

$$y \in X_{\{11\}} \text{ if and only if } \sigma^n(y) \leq_{\text{lex}} 1010101010\ldots = (10)^\infty.$$

Where $\leq_{\text{lex}}$ denotes the lexicographical order between two one-sided sequences. BDS uses a different order, for $x, y \in \mathcal{A}^{\mathbb{N}}$, we say that $x \leq y$ if and only if

$$\text{For all } i \in \mathbb{N} \ \sum_{r=1}^{i} x_r \leq \sum_{r=1}^{i} y_r.$$

Then for X_f where $f(n) = \lfloor \frac{n+1}{2} \rfloor$. The maximal right infinite sequence is

$$x = 1010101010\ldots = (10)^\infty.$$

Connection with $\beta-$shifts.

Lemma (Stanley)

If there are two sequences x and y satisfyng $\sigma^n(y) \leq x$ for all $n \in \mathbb{N}$, then they satisfy $\sigma^n(y) \leq_{\text{lex}} x$ for all $n \in N$.

Theorem (Stanley)

If x is characteristic sequence for a transitive BDS, then x is the maximum sequence of a $\beta-$shift.

The converse is false. Let $x = 210(20)^\infty$, x is a maximum sequence of some $\beta-$shift, but $\sigma^3(x) \nleq x$.

Measure theoretic entropy

Given a subshift X we denote by $M(X,\sigma)$ the $\sigma-$invariant Borel probability measures. Let $\mu \in M(X,\sigma)$, the entropy of μ is defined by.

$$h_\mu(X) = \lim_{n\to\infty} \frac{-1}{n} \sum_{w\in\mathcal{L}_n(X)} \mu([w]) \log \mu([w]), \quad (3)$$

where terms with $\mu([w]) = 0$ are omitted.

By the well-known Variational Principle, the supremum of $h_\mu(X)$ over all $\mu \in M(X,\sigma)$ is the *topological entropy* $h_{top}(X)$ of X.

Topological entropy

For any subshift X, we have that

$$h_{top}(X) = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{L}_n(X)|. \quad (4)$$

Theorem

Let $X_{\mathcal{F}}$ a topological mixing SFT. Then the topological entropy is $\log(\lambda)$ where λ is the Perron eigenvalue of the matrix associated with the SFT $X_{\mathcal{F}}$.

A measure whose entropy attains the topological entropy is known as a **measure of maximal entropy**. If such measure is unique, the system is **intrinsically ergodic**.

Existence of MME

Theorem [Bowen]

Let X be a subshift, then X has at least one measure of maximal entropy.

To ensure intrinsic ergodicity we need

Theorem [Bowen]

Let X be a subshift that satisfies the specification property, then X has a unique measure of maximal entropy.

Parry Measure for the Golden Mean Shift

Returning to the golden mean shift, its Parry measure is a Markov measure $\mu_{\pi,P}$, where π is the distribution $\pi = \left(\varphi^2/(1+\varphi^2), 1/(1+\varphi^2)\right)$ and P is the stochastic matrix

$$P = \begin{bmatrix} \frac{1}{\varphi} & \frac{1}{\varphi^2} \\ 1 & 0 \end{bmatrix}.$$

$$\mu_{\pi,P}(w_1 w_2 \dots w_k) = \pi_{w_1} P_{w_1 w_2} \cdots P_{w_{k-1} w_k}.$$

Thus,

$$h_{\mu_{\pi,P}}(X) = h_{\text{top}}(X) = \log\left(\frac{1+\sqrt{5}}{2}\right).$$

Beyond specification

Theorem [Hofbauer, Walters]

Let X_β a $\beta-$shift. Then X_β is intrinsically ergodic.

Question: [Kwietniak]

Every hereditary subshift is intrinsically ergodic?

Answer: No. [Kułaga-Przymus, Lemańsky, Weiss]

They showed that in general, hereditary systems may not be intrinsically ergodic.

Definition

Let X_f be a bounded density shift, the limit

$$\lim_{n\to\infty} \frac{f(n)}{n} \tag{5}$$

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The existence of the limit is given by Fekete's lemma and the definition of canonical function; furthermore, the limit is an infimum, and so $f(n) \geq \alpha_f n$ for all n. In particular we are interested in BDS with $\alpha_f > 0$.

Denote by $\{{}^{\infty}0^{\infty}\}$ the trivial BDS given by the function $f = 0$.

More of topological entropy

Theorem

For a $\beta-$shift. The topological entropy is $\log(\beta)$.

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Theorem (Stanley)

Let X_f be a BDS with $\alpha_f > 0$ and x its charaqteristic sequence. Let X_β the $\beta-$shift with maximum sequence x. Then the topological entropy of X_f is non-zero and

$$\frac{\log(\beta)}{\lceil 2f(1)/\alpha_f \rceil} \leq h_{\text{top}}(X_f) \leq \log(\beta).$$

Characterization of specified BDS

Theorem [Stanley]

Let X_f be a bounded density shift and let $M \in \mathbb{N}$. Then the following are equivalent.

1. X_f is specified with specification constant M.
2. For all $m, p \in \mathbb{N}_0$ $f(m) + f(p) \leq f(m + p + M)$.
3. The word 0^M is an intrinsically synchronizing word.

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There exists bounded density shifts with $\alpha_f > 0$ but without the specification property.

Some results

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Theorem [García-Ramos, Pavlov, R.]

Let X_f be a bounded density shift over the alphabet $\{0,1\}$ with $\alpha_f > 1/2$. Then X_f is intrinsically ergodic.

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Let X_f be a bounded density shift over the alphabet $\{0,1\}$ with $\alpha_f > 1/2$. Then X_f is intrinsically ergodic.

Theorem [García-Ramos, Pavlov, R.]

Let X_f be a bounded density shift. If every measure of maximal entropy μ has the property that $\sum_{i=1}^{\lfloor f(1) \rfloor} i\mu([i]) < \alpha_f$. Then X_f is intrinsically ergodic and

$$\mu_n = \frac{1}{|Per(n)|} \sum_{x \in Per(n)} \delta_x$$

converges to the measure of maximal entropy in the weak* topology.

Theorem [García-Ramos, Pavlov, R.]

For bounded density shifts with $\alpha_f > 0$, every measure of maximal entropy is fully supported.

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Theorem [García-Ramos, Pavlov, R.]

Every synchronized BDS is intrinsically ergodic.

What about $\alpha_f < 1/2$?

Corollary

Let $\alpha_f > \sum_{i=1}^{\lfloor f(1) \rfloor} (i/(i+1))$. This implies that X_f is intrinsically ergodic.

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Conjecture

Is it true that for every BDS, we have that

$$\sum_{i=1}^{\lfloor f(1) \rfloor} i\mu([i]) < \alpha_f.$$

for every measure of maximal entropy?

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As a consequence.

Conjecture

All bounded density subshifts are intrinsically ergodic.

THANK YOU