

# Computability of measures of maximal entropy on coded shift spaces

Christian Wolf

Mississippi State University

Webinar in Dynamical Systems

Thermodynamic formalism, ergodic optimization, Gibbs measures, low complexity, random dynamical systems

March 17, 2026

# Outline of the talk

- 1 Measures of maximal entropy on coded shifts
- 2 Background from computability theory
- 3 Computability of measures of maximal entropy on coded shifts

# Papers

- Tamara Kucherenko, Martin Schmoll and C. W., Ergodic theory on coded shift spaces, *Advances in Mathematics* 457 (2024), Paper No. 109913, 43 pages.
- Tamara Kucherenko, Marco Lopez and C.W., Computability of measures of maximal entropy for coded shifts, arXiv:2601.15548

Setup: Let  $d \geq 2$  and let  $\mathcal{A} = \mathcal{A}_d = \{0, \dots, d-1\}$ . We denote by

$$\Sigma = \Sigma_d^\pm = \{x = (x_k)_{k \in \mathbb{Z}} : x_k \in \mathcal{A}\}$$

the two-sided full shift.

Given  $0 < \theta < 1$  we endow  $\Sigma$  with the  $\theta$ -metric  $d_\theta$  given by

$$d(x, y) = d_\theta(x, y) = \theta^{\min\{|k| : x_k \neq y_k\}}$$

which makes  $\Sigma$  into a compact metric space.

The shift map  $\sigma : \Sigma \rightarrow \Sigma$  is defined by  $\sigma(x)_k = x_{k+1}$ , and we note that  $\sigma$  is a homeomorphism. We call a closed shift-invariant set  $X \subset \Sigma$  a shift space, and we say that  $\sigma : X \rightarrow X$  is a subshift.

We denote the set of  $X$ -admissible words of length  $n$  by  $\mathcal{L}(X, n)$  and by  $\mathcal{L}(X) = \bigcup_{n=0}^{\infty} \mathcal{L}(X, n)$  the language of  $X$ . Recall that shift spaces and languages are equivalent objects.

We endow  $\mathcal{M} = \mathcal{M}(X) = \{\mu : f - \text{invariant Borel probability measure on } X\}$  with the weak\* topology which makes  $\mathcal{M}$  a compact, convex, metrizable space. Let  $\mathcal{M}_E \subset \mathcal{M}$  be the subset of ergodic measures, that is, if  $f^{-1}(A) = A$  then  $\mu(A) \in \{0, 1\}$ . Let  $X$  be a shift space. We call

$$h_{\text{top}}(X) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \text{card} \mathcal{L}(X, n)$$

the *topological entropy* of  $f$ .

### Theorem (Variational Principle)

$$h_{\text{top}}(X) = \sup_{\mu \in \mathcal{M}} h_{\sigma}(\mu).$$

If  $\mu$  attains the supremum we call  $\mu$  a measure of maximal entropy (mme).

## Definition

A shift space  $X$  is coded if there exists a  $\mathcal{G} = \{g_i : i \in \mathbb{N}\} \subset \mathcal{L}(X)$  (called a generating set) such that  $X = X(\mathcal{G})$  is the smallest shift space that contains all bi-infinite concatenations of generators, i.e.,

$$X_{\text{con}} = X_{\text{con}}(\mathcal{G}) = \{x \in \mathcal{A}^{\mathbb{Z}} : \exists (k_i)_{i \in \mathbb{Z}} \text{ s.th. } x[k_i, k_{i+1}) \in \mathcal{G}\},$$

and  $X = \overline{X_{\text{con}}}$ . We call  $X_{\text{con}}$  the concatenation set and  $X_{\text{res}} = X_{\text{res}}(\mathcal{G}) = X \setminus X_{\text{con}}$  the residual set of  $X$ . Note that the sets  $X_{\text{con}}$  and  $X_{\text{res}}$  depend on the choice of generating set.

See Blanchard-Hansel 1986, Fiebig-Fiebig 1992, Lind-Marcus 1995, Pavlov 2020, Burr-Das-W.-Yang 2022, Schmoll-Kucherenko-W. 2024 for details about coded shifts.

### Definition

A coded shift  $X$  is uniquely representable (also known as uniquely decomposable) if there exists a generating set  $\mathcal{G}$  such that  $X = X(\mathcal{G})$  and every  $x \in X_{\text{con}}$  can be uniquely written as an infinite concatenation of elements in  $\mathcal{G}$ . In this case, we say that  $\mathcal{G}$  uniquely represents of  $X_{\text{con}}$ .

### Theorem (Béal, Perrin, Restivo, EJC 2023)

*Every coded shift  $X$  is uniquely representable. Moreover, the proof of the existence of a unique representation  $\mathcal{G}$  is constructive.*

Examples of coded shifts: Transitive SFTs and Sofic shifts, S-gap shifts, multiple gap shifts, Beta-shifts, etc.

An invariant probability measure  $\mu$  is  $\mathcal{G}$ -Bernoulli if there exist  $p_g \geq 0, g \in \mathcal{G}$  with  $\sum_{g \in \mathcal{G}} p_g = 1$  and  $c > 0$  such that

$$\mu(\llbracket g_0 \dots g_k \rrbracket) = \frac{1}{c} p_{g_0} \cdot \dots \cdot p_{g_k}$$

for all  $g_0, \dots, g_k \in \mathcal{G}$ . Here  $\llbracket g_0 \dots g_k \rrbracket$  denotes the  $\mathcal{G}$ -cylinder consisting of all points  $x = \dots g'_{-1} \cdot g'_0 \cdot \dots g'_k g'_{k+1} \dots \in X_{\text{con}}$  such that  $g'_j = g_j$  for  $j = 0, \dots, k$ .

Note that the  $\mathcal{G}$ -cylinder  $\llbracket g_0 \dots g_k \rrbracket$  differs from the classical cylinder  $[g_0 \dots g_k]$  defined by

$$[g_0 \dots g_k] = \{x = (x_k)_k \in X : x_0 \dots x_{|g_0 \dots g_k| - 1} = g_0 \dots g_k\}.$$

One can show that every invariant measure which puts full measure on  $X_{\text{con}}$  is completely determined by its mass on the  $\mathcal{G}$ -cylinders.

Define

$$h_{\text{con}}(X) = \sup\{h_{\sigma}(\mu) : \mu(X_{\text{con}}) = 1\}$$

and

$$h_{\text{res}}(X) = \sup\{h_{\sigma}(\mu) : \mu(X_{\text{res}}) = 1\}.$$

Theorem (Kucherenko, Schmoll, W., 2024)

*Let  $X = X(\mathcal{G})$  be a coded shift such that  $\mathcal{G}$  uniquely represents  $X_{\text{con}}$  and  $h_{\text{con}}(X) > h_{\text{res}}(X)$ . Then  $X$  has a unique measure of maximal entropy  $\mu_{\text{max}}$ . Moreover,  $\mu_{\text{max}}$  is  $\mathcal{G}$ -Bernoulli with  $p_g = \exp(-|g|h_{\text{top}}(X))$  and  $c = \sum_{g \in \mathcal{G}} |g| \exp(-|g|h_{\text{top}}(X))$ .*

We call  $c = \sum_{g \in \mathcal{G}} |g| \exp(-|g|h_{\text{con}}(X)) = \sum_{g \in \mathcal{G}} |g| \exp(-|g|h_{\text{top}}(X))$  the Vere-Jones parameter of the generating set  $\mathcal{G}$ .

## Basics from Computable Analysis:

## Definition

Let  $m \in \mathbb{N}$  and  $x \in \mathbb{R}^m$ . An *oracle* of  $x$  is a function  $\psi : \mathbb{N} \rightarrow \mathbb{Q}^m$  such that  $\|\psi(n) - x\| < 2^{-n}$ . Moreover,  $x$  is *computable* if there is a Turing Machine (a computer program for our purposes)  $\psi$  which is an oracle of  $x$ .

## Basic Facts:

- (i) Rational numbers, algebraic numbers, and some transcendental numbers such as  $e$  and  $\pi$  are computable real numbers.
- (ii) There are only countably many computable points in  $\mathbb{R}^m$ .

## Definition

Let  $S \subset \mathbb{R}^m$ . A function  $g : S \rightarrow \mathbb{R}$  is *computable* if there is a Turing machine  $\chi$  so that for any  $x \in S$ , any oracle  $\psi$  for  $x$  and any  $n \in \mathbb{N}$ ,  $\chi(\psi, n)$  is a rational number so that  $|\chi(\psi, n) - g(x)| < 2^{-n}$ .

## Definition

Let  $(X, d)$  be a separable metric space and let  $S = \{s_i\} \subset X$  be countable dense. We say  $(X, d, S)$  is a computable metric space if  $d|_{S \times S}$  is uniformly computable, that is, if there exists a Turing machine  $\chi = \chi(i, j, n)$ , which on input  $i, j, n \in \mathbb{N}$  outputs a rational number such that  $|d(s_i, s_j) - \chi(i, j, n)| < 2^{-n}$ . We call  $S$  the set of ideal points of  $X$ .

Examples:

- (i)  $(\mathbb{R}^m, d_E, \mathbb{Q}^m)$  is a computable metric space;
- (ii) Let  $(X, d, S)$  be a computable metric space. Then the space  $X_C$  of all compact subsets of  $X$  together with the Hausdorff metric carries a computable metric space structure. Here the set of ideal points  $S_C$  are simply the finite subsets of  $S$ . E.g. Computability of Julia sets;
- (iii) Every shift space  $X$  is a computable metric space. The space of all shift spaces over a finite alphabet denoted by  $\Sigma$  carries a computable metric space structure;
- (iii) If  $(X, d, S)$  is a computable metric space then  $\mathcal{M}(X)$  the space of Borel probability measures on  $X$  carries a computable metric space structure.

Let  $X = X(\mathcal{G})$  be a coded shift with generating set  $\mathcal{G}$  such that  $\mathcal{G}$  uniquely represents  $X_{\text{con}}$ .

### Theorem (Kucherenko, Lopez, W., 2025)

*Suppose that there exists  $\mu \in \mathcal{M}_\sigma(X)$  with  $\mu(X_{\text{con}}) = 1$  and  $h_\sigma(\mu) = h_{\text{con}}(X)$ . Then this measure is unique, and is  $\mathcal{G}$ -Bernoulli with  $p_g = \exp(-|g|h_{\text{con}}(X))$  and  $c = \sum_{g \in \mathcal{G}} |g| \exp(-|g|h_{\text{con}}(X))$ . Moreover, if  $h_{\text{con}}(X)$  and the Vere-Jones parameter  $c$  are computable then  $\mu$  is computable based on having oracle access to  $\mathcal{G}$  and  $\mathcal{L}(X)$ .*

We define a parameter  $b(\mathcal{G})$  by

$$b(\mathcal{G}) = \sup \left\{ \frac{1}{n} \log \text{card}\{g \in \mathcal{G} : |g| = n\} \right\}$$

### Theorem

*Assume that  $h_{\text{con}}(X) > h_{\text{res}}(X)$  and that  $0 < \varepsilon < h_{\text{top}}(X) - b(\mathcal{G})$ . Then the Vere-Jones parameter  $c$  and the mme  $\mu_{\text{max}}$  are computable from oracles of  $\mathcal{L}(X)$ ,  $\mathcal{G}$  and  $\varepsilon$ .*

### Corollary

*Let  $\beta > 1$ . The the unique mme  $\mu_\beta$  of the beta shift  $X_\beta$  and the corresponding Vere-Jones parameter  $c$  are computable.*

### Corollary

*Let  $X$  be an  $S$ -gap shift or a generalized gap shift. Then the unique mme  $\mu_{\max}$  and the Vere-Jones parameter  $c$  are computable.*

### Corollary

*Let  $X$  be the standard Dyke shift given by pairs of brackets and parenthesis. Then the two ergodic mmes of  $X$  are computable.*

## Theorem

*There exists a coded shift  $X = X(\mathcal{G})$  such that for all  $N, M \in \mathbb{N}$  there is a coded shift  $X' = X(\mathcal{G}')$  satisfying the following:*

- $h_{\text{con}}(X) > h_{\text{res}}(X)$  and  $h_{\text{con}}(X') > h_{\text{res}}(X')$ ;
- $\mathcal{L}_N(X) = \mathcal{L}_N(X')$  and the first  $M$  elements of  $\mathcal{G}$  and  $\mathcal{G}'$  are the same;
- $|c(\mathcal{G}') - c(\mathcal{G})| \geq \frac{1}{2}$ .

*In particular, the Vere-Jones parameter of  $X$  is not computable based on the oracle access to the generating set and the language.*

## Theorem

*There exists a coded shift  $X = X(\mathcal{G})$  such that  $\mathcal{G}$  uniquely represents  $X_{\text{con}}$ , with the following properties:*

- $h_{\text{res}}(X) > h_{\text{con}}(X)$ ;
- $X$  has a unique mme  $\mu_{\text{max}}$  and the Vere-Jones parameter  $c$  is computable;
- But  $\mu_{\text{max}}$  is not computable.

---

Thank You!