

Rare events statistics for \mathbb{Z}^d map lattices

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A physical model

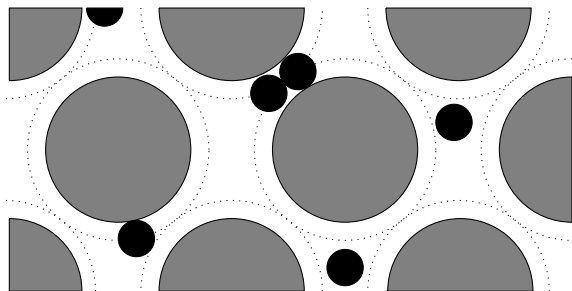


Figure: Gaspard-Gilbert [PRL, 08]. Grey circles represent obstacles, while the black circles are the moving balls.

When no collision happens, the state of each ball (described by its position and velocity) evolves according to classical billiard dynamics. When two moving balls collide, they exchange the components of their velocities that are parallel to the line through their centres at the moment of impact

Motivating figure

The setup

Results

Outline and goal of the talk

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- ▶ To present a simple coupled map lattice model, introduced by Keller and Liverani [CMP, 09], that has some 'vague' conceptual connections with the above physical model.
- ▶ Study statistics of collisions at a site on the lattice in the model of Keller and Liverani [CMP, 09]: laws for first collision time and for the number of collisions.

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The uncoupled dynamics

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- ▶ $\tau : I \rightarrow I$, where $I = [0, 1]$, is piecewise monotone, onto, C^2 , and uniformly expanding.

- ▶ $T_0 : I^{\mathbb{Z}^d} \rightarrow I^{\mathbb{Z}^d}$ as the product map

$$(T_0(x))_{\mathbf{p}} = \tau(x_{\mathbf{p}}),$$

$\mathbf{p} \in \mathbb{Z}^d$; i.e., T_0 is the uncoupled dynamics on the lattice.

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The coupled dynamics

- ▶ $V^+ := \{e_j\}$, standard basis of \mathbb{R}^d ; $V := V^+ \cup -V^+$.
- ▶ **Collision zones:** For $\epsilon > 0$, let $\{A_{\epsilon, -v}\}_{v \in V} \subset I$ be disjoint open intervals, each of length ϵ .
- ▶ Consider the coupling

$$(\Phi_\epsilon(x))_{\mathbf{p}} = \begin{cases} x_{\mathbf{p}+v} & \text{if } x_{\mathbf{p}} \in A_{\epsilon, v} \text{ and } x_{\mathbf{p}+v} \in A_{\epsilon, -v} \text{ for some } v \in V \\ x_{\mathbf{p}} & \text{otherwise} \end{cases}$$

- ▶ We define the coupled dynamics $T_\epsilon : I^{\mathbb{Z}^d} \rightarrow I^{\mathbb{Z}^d}$ as the composition $T_\epsilon := \Phi_\epsilon \circ T_0$.
- ▶ Keller & Liverani studied, for $\epsilon > 0$ small enough, existence, uniqueness of a physical invariant measure, and proved exponential mixing rates in a suitable Banach space [CMP, 09].

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Rare events on a site $\mathbf{p}^* \in \mathbb{Z}^d$: part (I)

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- ▶ In this work we study rare events at site $\mathbf{p}^* \in \mathbb{Z}^d$.
- ▶ For this purpose we use $\delta \in (0, \epsilon]$ to label collision sets related to \mathbf{p}^* : $A_{\delta, \mathbf{v}}$ and assume that $m(A_{\delta, \mathbf{v}}) = \delta$.
- ▶ All other collision intervals that are not involved with \mathbf{p}^* will be denoted, as before, by $A_{\epsilon, \mathbf{v}}$ and their size is independent of δ .
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$$X_{\delta}^{\mathbf{v}}(\mathbf{p}^*) := \{x \in I^{\mathbb{Z}^d} : x_{\mathbf{p}^*} \notin A_{\delta, \mathbf{v}} \text{ or } x_{\mathbf{p}^* + \mathbf{v}} \notin A_{\delta, -\mathbf{v}}\}$$

and define

$$X_{0, \delta}(\mathbf{p}^*) := \bigcap_{\mathbf{v} \in V} X_{\delta}^{\mathbf{v}}(\mathbf{p}^*), \quad \text{and} \quad H_{\delta}(\mathbf{p}^*) = I^{\mathbb{Z}^d} \setminus X_{0, \delta}(\mathbf{p}^*).$$

- ▶ First collision time at \mathbf{p}^*

$$t_{\delta}(x) = \inf\{n \geq 0 : T_{\epsilon, \delta}^n(x) \in H_{\delta}(\mathbf{p}^*)\}.$$

- ▶ Number of collisions at \mathbf{p}^*

$$Z_{\delta}(t) = \sum_{k=1}^{n(t)} \mathbf{1}_{H_{\delta}(\mathbf{p}^*)} \circ T_{\epsilon, \delta}^k.$$

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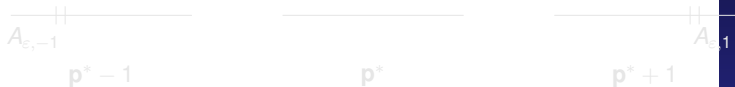
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Decoupling at \mathbf{p}^*

- ▶ We introduce a decoupling map $\Phi_{\epsilon, \mathbf{p}^*}$ which is identical to original coupling everywhere on the lattice except on \mathbf{p}^* where we ‘close’ the collision zones.
- ▶ Figure in the case of $d = 1$



- ▶ We call

$$T_{\epsilon, \mathbf{p}^*} := \Phi_{\epsilon, \mathbf{p}^*} \circ T_0$$

the decoupled dynamics at \mathbf{p}^* .

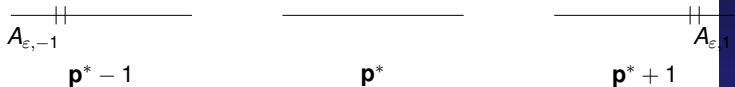
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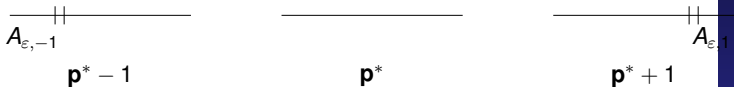
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Banach space

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Results

Let \mathcal{D} be the space of functions on \mathbb{Z}^d that depend only on a finite number of coordinates and are locally differentiable. Let

$$\mathcal{D}_1 := \{\varphi \in \mathcal{D} : |\varphi|_\infty \leq 1\}.$$

For any Borel complex measure μ defined on \mathbb{Z}^d , where \mathbb{Z}^d is equipped with the product topology, we define

$$|\mu| = \sup_{\varphi \in \mathcal{D}_1} \mu(\varphi)$$

$$\|\mu\| = \sup_{\mathbf{p} \in \mathbb{Z}^d} \sup_{\varphi \in \mathcal{D}_1} \mu(\partial_{\mathbf{p}} \varphi)$$

Statements of the main results (I)

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Recall

$$t_\delta(x) = \inf\{n \geq 0 : T_{\epsilon, \delta}^n(x) \in H_\delta(\mathbf{p}^*)\}.$$

Theorem 1 $T_{\epsilon, \mathbf{p}^*}$ admits a unique invariant measure $\mu_{\epsilon, \mathbf{p}^*} \in \mathcal{B}$. Moreover, $\exists C > 0, \theta \in (0, 1],^1, \xi_\delta > 0$, with $\lim_{\delta \rightarrow 0} \xi_\delta = \theta$, such that for all $t > 0$

$$\left| \mu_{\epsilon, \mathbf{p}^*} \left\{ t_\delta \geq \frac{t}{\xi_\delta \mu_{\epsilon, \mathbf{p}^*}(H_\delta(\mathbf{p}^*))} \right\} - e^{-t} \right| \leq (t \vee 1) e^{-t} C \delta^2 |\ln(\delta^2)|.$$

¹ θ has an explicit formula

Statements of the main results (II)

Recall

$$Z_\delta(t) := \left\lfloor \frac{t}{\mu_{\epsilon, \mathbf{p}^*}(H_\delta(\mathbf{p}^*))} \right\rfloor \sum_{k=1} \mathbf{1}_{H_\delta(\mathbf{p}^*)} \circ T_{\epsilon, \delta}^k,$$

Theorem 2 The process $Z_\delta(t)$ converges in law to a compound Poisson process

$$Z(t) = \sum_{i=1}^{N(t)} X_i,$$

where $(N(t))_t$ is a Poisson process of intensity θt , $\theta \in (0, 1]$ is as in Theorem 1, and $(X_i)_{i \in \mathbb{N}}$ is an iid sequence whose characteristic function is given by

$$\phi_X(\mathbf{s}) = \frac{\tilde{\theta}(\mathbf{s})(e^{i\mathbf{s}} - 1)}{\theta} + 1, \quad \tilde{\theta} : \mathbb{R} \rightarrow \mathbb{C}.$$

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Ideas of proofs (I)

- ▶ For Theorem 1, consider the transfer operator

$$\int \varphi d\hat{\mathcal{L}}_{\epsilon, \delta, \mathbf{p}^*}(\mu) = \int \varphi \circ T_{\epsilon, \mathbf{p}^*} \mathbf{1}_{X_{0, \delta}(\mathbf{p}^*)} d\mu,$$

and notice

$$\mu\{t_\delta \geq n\} = \int d\hat{\mathcal{L}}_{\epsilon, \delta, \mathbf{p}^*}^n(\mu).$$

- ▶ So prove a spectral gap on \mathcal{B} and find first order representation of the dominant eigenvalue as

$$e^{-\mu_{\epsilon, \mathbf{p}^*}(H_\delta(\mathbf{p}^*)) \cdot \theta(1+o(1))}.$$

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- ▶ For Theorem 2, consider the transfer operator

$$\int \varphi d\tilde{\mathcal{L}}_{\epsilon, \delta, \mathbf{p}^*}(\mu) = \int \varphi \circ T_{\epsilon, \delta}(e^{is1_{H_\delta(\mathbf{p}^*)}}) d\mu,$$

and notice for any $s \in \mathbb{R}$

$$\tilde{\mathcal{L}}_{\delta, s}^n(\cdot) = \mathcal{L}_{\epsilon, \delta}^n(e^{is \sum_{k=1}^n 1_{H_\delta(\mathbf{p}^*)} \circ T_{\epsilon, \delta}^{k-1}} \cdot).$$

- ▶ Then show spectral gap on \mathcal{B} and use it to show

$$\lim_{\delta \rightarrow 0} \int e^{isZ_\delta(t)} d\mu = e^{-(1-e^{is})\tilde{\theta}(s)t}.$$

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