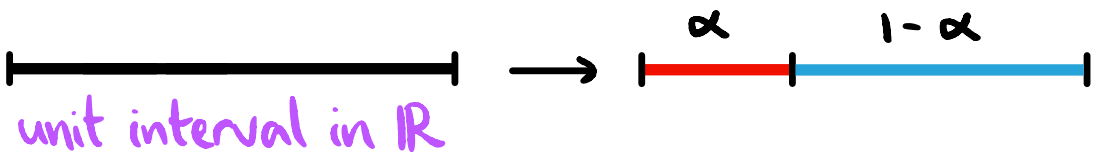
Uniformly Spread Kakutani Tilings of the Line

Yotam Smilansky, Manchester

THERMOGAMAS, 2026

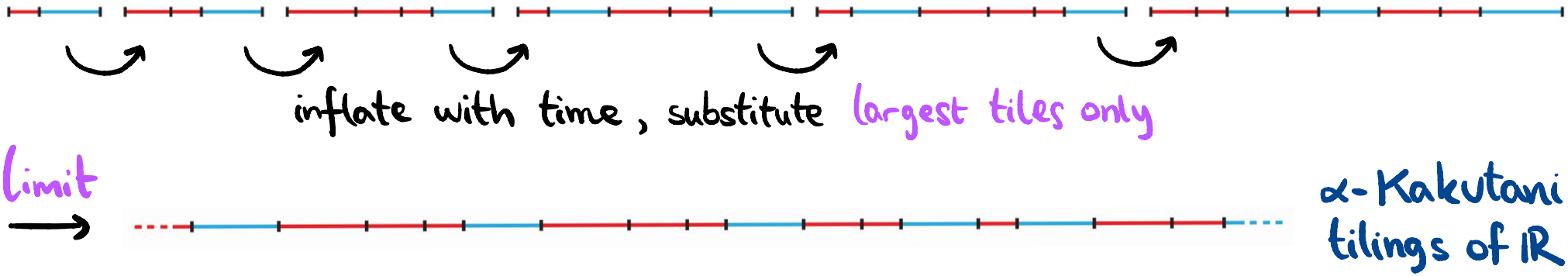
Multiscale Substitutions

Let $0<\alpha<1$ and consider the α-Kakutani substitution rule:



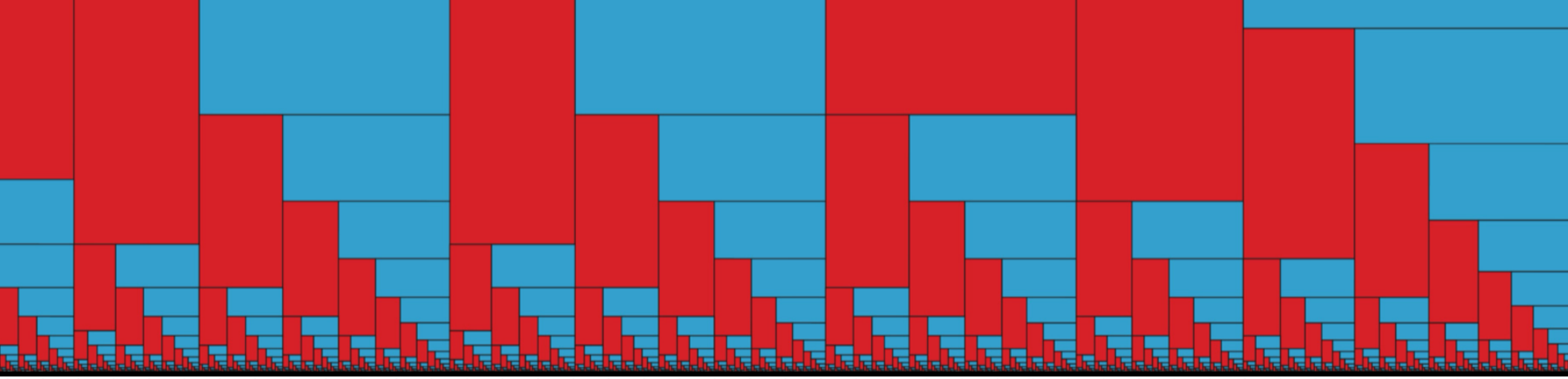
- multiscale substitution tilings [SS '21]

  a time t dependent substitution semi-flow F_t : inflate by e^t and subsitute tiles of volume > 1

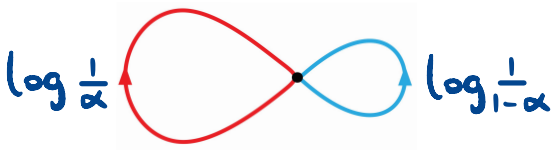


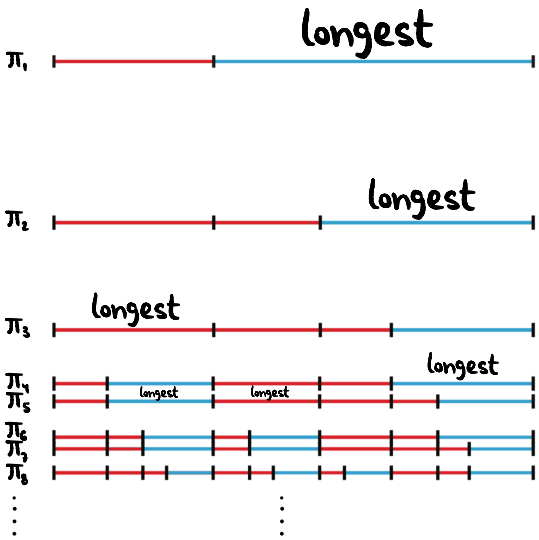
$\Lambda :=$ {endpoints of tiles} is a Delone set in $\mathbb{R}$

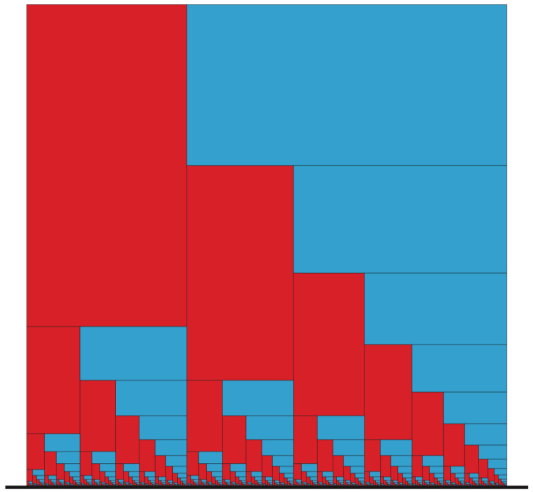
uniformly discrete relatively dense

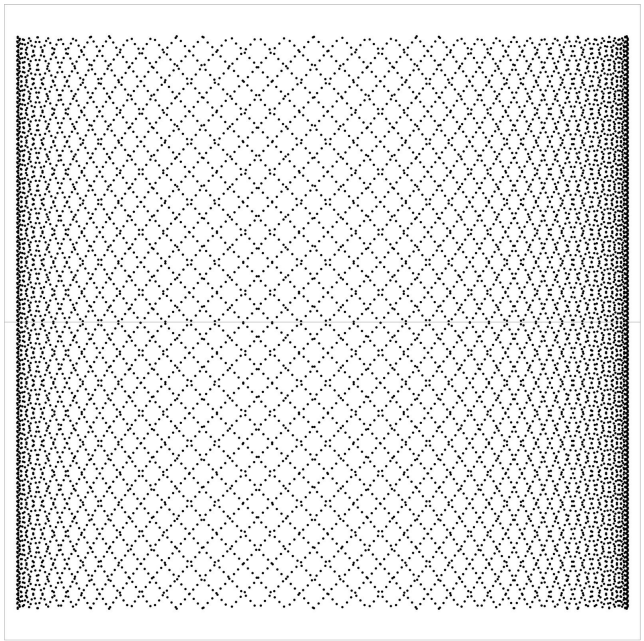


Other related constructions



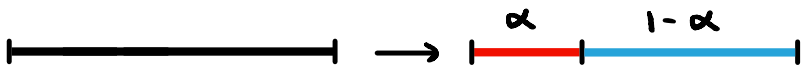


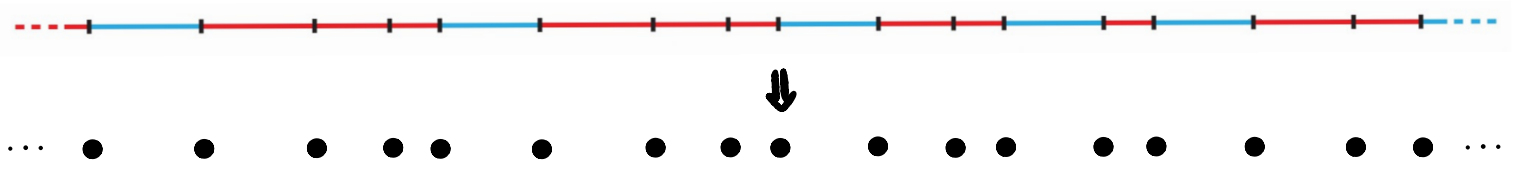




Tiling zeta function $\zeta_\sigma(s) = \dfrac{1}{1 - \alpha^s - (1-\alpha)^s}$

Uniform Spreadness





α-Kakutani tiling

$\Lambda \subseteq \mathbb{R}$ Delone

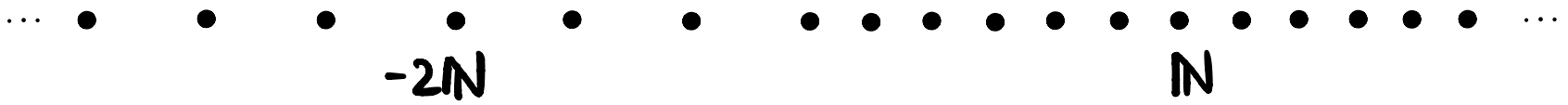
Λ is uniformly spread if $\exists\, c>0$ and bijection $\varphi: \Lambda \to c\mathbb{Z}$ with

bounded displacement

equivalent to a lattice

$$\sup_{x\in\Lambda} \|x-\varphi(x)\| < \infty$$

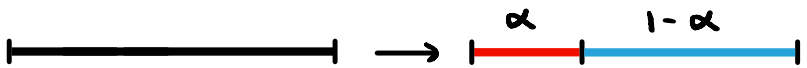
not all Delone sets are uniformly spread:



Theorem [S '25+] Let $0<\alpha\leq 1/2$, then Λ is uniformly spread iff

$$r_\alpha := \frac{\log \frac{1}{\alpha}}{\log \frac{1}{1-\alpha}} \in \{1, \tfrac{3}{2}, 2, 3, 4\}$$

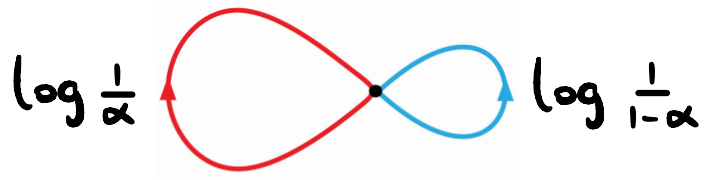
Counting Tiles



Caczkowich ('92) $\Lambda \subseteq \mathbb{R}$ with density d_Λ is uniformly spread iff

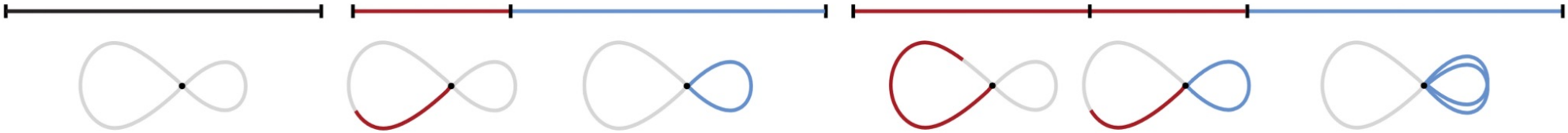
discrepancy $|\#(\Lambda \cap U) - d_\Lambda |U|| \leq C\,|(\partial U)^{+1}|$ $\forall$ bdd measurable $U \subseteq \mathbb{R}$

a directed weighted graph G_α:

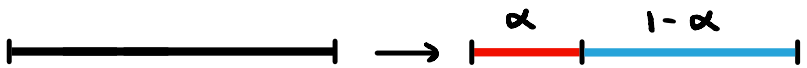


key observation: tiles in $F_t(I)$ $\longleftrightarrow$ walks on G_α

tile of length $|T|$ $\mapsto$ walk originating at vertex and terminating $\log \frac{1}{|T|}$ before the vertex



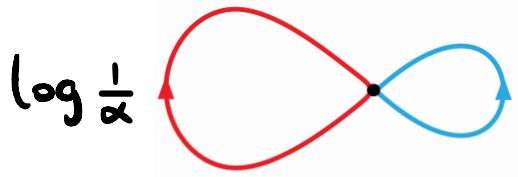
Counting Tiles



Caczkowich ('92) $\Lambda \subseteq \mathbb{R}$ with density d_Λ is uniformly spread iff

discrepancy $|\#(\Lambda \cap U) - d_\Lambda |U|| \leq C |(\partial U)^{+1}|$ $\forall$ bdd measurable $U \subseteq \mathbb{R}$

a directed weighted graph G_α:



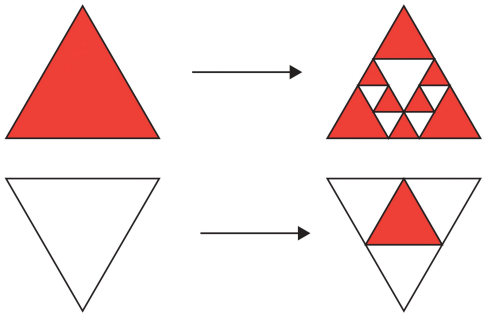
the substitution rule is incommensurable if $r_\alpha = \frac{\log \frac{1}{\alpha}}{\log \frac{1}{1-\alpha}} \notin \mathbb{Q}$

- tiles appear in a dense set of lengths in $[\alpha, 1]$
- # tiles in $F_t(I) \sim d_\Lambda e^t$ for $d_\Lambda = \frac{1}{-\alpha \log \alpha - (1-\alpha)\log(1-\alpha)}$
- # distinct lengths of tiles in $F_t(I) = \Omega(t)$

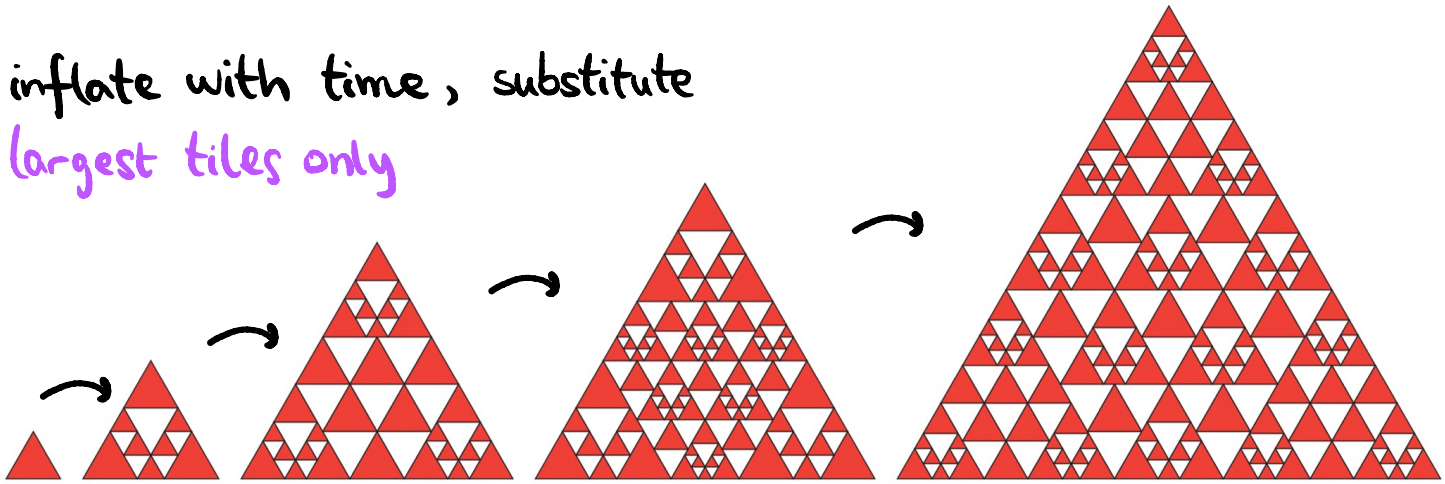
pigeonhole principle $\Rightarrow$ $\exists$ intervals U with discrepancy $\Omega\left(\frac{|U|}{\log |U|}\right)$

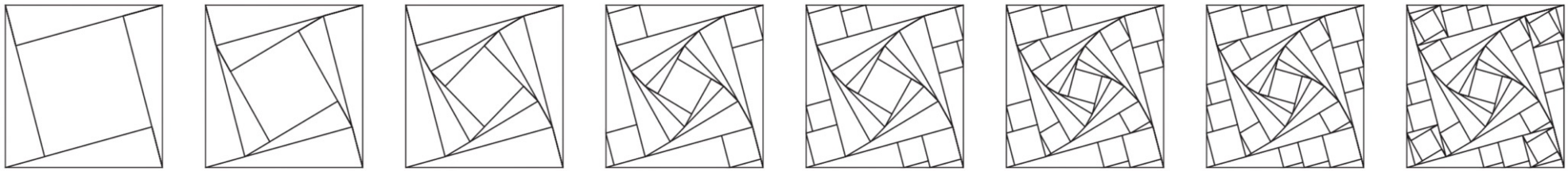
Multiscale Substitutions

- higher dimensions and additional prototiles



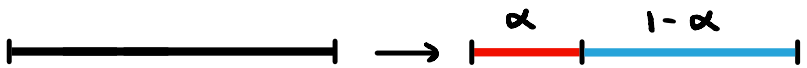
inflate with time, substitute largest tiles only





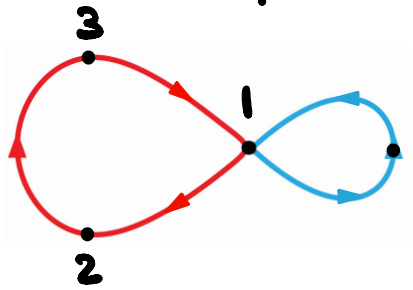
Ranzy's fractal (commensurable). Does incommensurability rule out fractal boundaries?

Commensurable Case



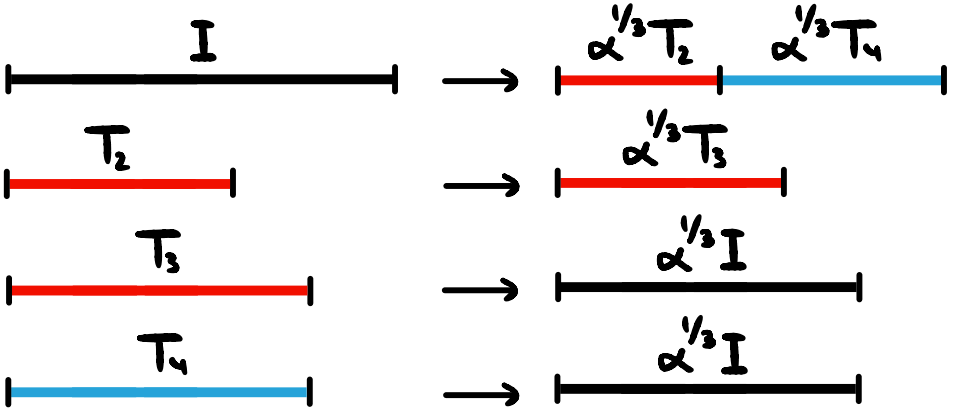
Assume now $r_\alpha = \frac{\log \frac{1}{\alpha}}{\log \frac{1}{1-\alpha}} = \frac{n}{m}$, $n > m \geq 1$ coprime. ($n=m \Rightarrow \alpha = \frac{1}{2}$, then Λ is a lattice)

Split edges of G_α to equal parts:

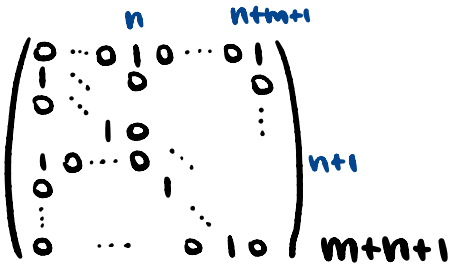


$n=3$, $m=2$

$\Rightarrow$ Primitive substitution rule



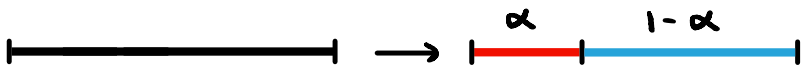
inflation constant $\xi = \alpha^{-1/n} > 1$, substitution matrix $M_\alpha =$



simultaneous substitution + inflation

key observation: corresponding tilings $\Leftrightarrow$ α-Kakutani tilings (up to rescaling)

Commensurable Case

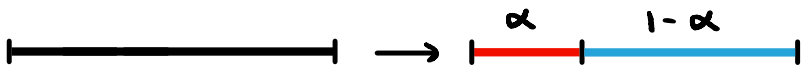


Solomon ('14) Let $j \geq 2$ be the minimal index with $V_{\lambda_j} \not\subseteq \mathbf{1}^\perp$ (eigenspace), where $\lambda_1 > |\lambda_2| \geq \dots \geq |\lambda_n|$ are the eigenvalues of M_α. Then if $|\lambda_j| < 1$ Λ is uniformly spread, if $|\lambda_j| > 1$ Λ is **NOT**.

- **linear algebra** $\lambda_1 = \alpha^{-1/n} > 1$. Nonzero λ_j satisfy $0 < |\lambda_j| \neq 1$, $V_{\lambda_j} \not\subseteq \mathbf{1}^\perp$.
- $p_\alpha(x) = x^{m-1}(\overbrace{x^n - x^{n-m} - 1}^{f_\alpha(x)})$. **Tverberg ('60)** $f_\alpha(x)$ is irreducible.
- Λ unif. spread $\Leftrightarrow$ $f_\alpha(x)$ is the min. poly. of a **PV number** (Pisot-Vijayaraghavan)
- **Dubickas, Jankauskas ('14)** then $f_\alpha(x)$ is one of the following:

$$x^2 - x - 1 \quad , \quad x^3 - x - 1 \quad , \quad x^3 - x^2 - 1 \quad , \quad x^4 - x^3 - 1$$

Commensurable Case



$f_\alpha(x) = x^3 - x - 1$ PV poly. of the plastic ratio $r_\alpha = \frac{3}{2} \Rightarrow \alpha^2 = (1-\alpha)^3$

smallest PV number

$f_\alpha(x) = x^2 - x - 1$ PV poly. of the golden ratio $r_\alpha = 2 \Rightarrow \alpha = (1-\alpha)^2$

$f_\alpha(x) = x^3 - x^2 - 1$ PV poly. of the supergolden ratio $r_\alpha = 3 \Rightarrow \alpha = (1-\alpha)^3$

$f_\alpha(x) = x^4 - x^3 - 1$ PV poly. of the 2nd smallest PV number $r_\alpha = 4 \Rightarrow \alpha = (1-\alpha)^4$

Thank You!