Tropical methods for ergodic control and zero-sum games

Minilecture, Part I

Stephane.Gaubert@inria.fr

INRIA and CMAP, École Polytechnique

Dynamical Optimization in PDE and Geometry
Applications to Hamilton-Jacobi
Ergodic Optimization, Weak KAM
Université Bordeaux 1, December 12-21 2011

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

So

- "$2 + 3$" $=$

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

So

- $\text{“}2 + 3\text{”} = 3$

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

So

- $\text{“}2 + 3\text{”} = 3$
- $\text{“}2 \times 3\text{”} =$

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

So

- “$2 + 3$” $= 3$
- “2×3” $=5$

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a + b\text{''} = \max(a, b) \qquad \text{“}a \times b\text{''} = a + b$$

So

- “$2 + 3$” $= 3$
- “2×3” $=5$
- “$5/2$” $=$

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

So

- "$2 + 3$" $= 3$
- "2×3" $=5$
- "$5/2$" $=3$

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$“a + b” = \max(a, b) \qquad “a \times b” = a + b$$

So

- $“2 + 3” = 3$
- $“2 \times 3” = 5$
- $“5/2” = 3$
- $“2^3” =$

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

So

- “$2 + 3$” $= 3$
- “2×3” $=5$
- “$5/2$” $=3$
- “2^3” $=$“$2 \times 2 \times 2$” $= 6$

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a+b\text{”} = \max(a,b) \qquad \text{“}a\times b\text{”} = a+b$$

So

- $\text{“}2+3\text{”} = 3$
- $\text{“}2\times 3\text{”} = 5$
- $\text{“}5/2\text{”} = 3$
- $\text{“}2^3\text{”} = \text{“}2\times 2\times 2\text{”} = 6$
- $\text{“}\sqrt{-1}\text{”} =$

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

So

- “$2 + 3$” $= 3$
- “2×3” $= 5$
- “$5/2$” $= 3$
- “2^3” $=$ “$2 \times 2 \times 2$” $= 6$
- “$\sqrt{-1}$” $= -0.5$

$$\text{“}\begin{pmatrix} 7 & 0 \\ -\infty & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}\text{”} =$$

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a+b\text{”} = \max(a,b) \qquad \text{“}a\times b\text{”} = a+b$$

So

- “$2+3$” $=3$
- “2×3” $=5$
- “$5/2$” $=3$
- “2^3” $=$ “$2\times 2\times 2$” $=6$
- “$\sqrt{-1}$” $=-0.5$

$$\text{“}\begin{pmatrix} 7 & 0 \\ -\infty & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\text{”} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$$

The notation $a \oplus b := \max(a, b)$, $a \odot b := a + b$, $\mathbb{0} := -\infty$, $\mathbb{1} := 0$ is also used in the tropical/max-plus litterature

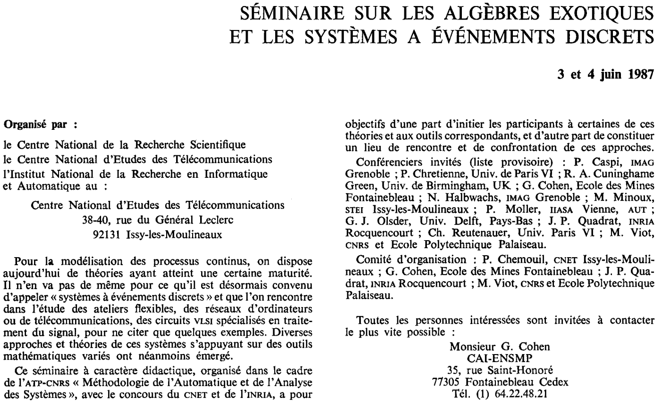
The sister algebra: min-plus

$$\text{“}a+b\text{”} = \min(a,b) \qquad \text{“}a\times b\text{”} = a+b$$

- “$2+3$” $=2$
- “2×3” $=5$

In the late 80's in France, the term "algèbres exotiques" was used

In the late 80's in France, the term "algèbres exotiques" was used


SÉMINAIRE SUR LES ALGÈBRES EXOTIQUES
ET LES SYSTÈMES A ÉVÉNEMENTS DISCRETS

3 et 4 juin 1987

Organisé par :

le Centre National de la Recherche Scientifique
le Centre National d'Etudes des Télécommunications
l'Institut National de la Recherche en Informatique et Automatique au :

Centre National d'Etudes des Télécommunications
38-40, rue du Général Leclerc
92131 Issy-les-Moulineaux

Pour la modélisation des processus continus, on dispose aujourd'hui de théories ayant atteint une certaine maturité. Il n'en va pas de même pour ce qu'il est désormais convenu d'appeler « systèmes à événements discrets » et que l'on rencontre dans l'étude des ateliers flexibles, des réseaux d'ordinateurs ou de télécommunications, des circuits VLSI spécialisés en traitement du signal, pour ne citer que quelques exemples. Diverses approches et théories de ces systèmes s'appuyant sur des outils mathématiques variés ont néanmoins émergé.

Ce séminaire à caractère didactique, organisé dans le cadre de l'ATP-CNRS « Méthodologie de l'Automatique et de l'Analyse des Systèmes », avec le concours du CNET et de l'INRIA, a pour objectifs d'une part d'initier les participants à certaines de ces théories et aux outils correspondants, et d'autre part de constituer un lieu de rencontre et de confrontation de ces approches.

Conférenciers invités (liste provisoire) : P. Caspi, IMAG Grenoble ; P. Chretienne, Univ. de Paris VI ; R. A. Cuninghame Green, Univ. de Birmingham, UK ; G. Cohen, Ecole des Mines Fontainebleau ; N. Halbwachs, IMAG Grenoble ; M. Minoux, STEI Issy-les-Moulineaux ; P. Moller, IIASA Vienne, AUT ; G. J. Olsder, Univ. Delft, Pays-Bas ; J. P. Quadrat, INRIA Rocquencourt ; Ch. Reutenauer, Univ. Paris VI ; M. Viot, CNRS et Ecole Polytechnique Palaiseau.

Comité d'organisation : P. Chemouil, CNET Issy-les-Moulineaux ; G. Cohen, Ecole des Mines Fontainebleau ; J. P. Quadrat, INRIA Rocquencourt ; M. Viot, CNRS et Ecole Polytechnique Palaiseau.

Toutes les personnes intéressées sont invitées à contacter le plus vite possible :

Monsieur G. Cohen
CAI-ENSMP
35, rue Saint-Honoré
77305 Fontainebleau Cedex
Tél. (1) 64.22.48.21

The term “exotic” appeared also in the User’s guide of viscosity solutions of Crandall, Ishii, Lions (Bull. AMS, 92)

The term "exotic" appeared also in the User's guide of viscosity solutions of Crandall, Ishii, Lions (Bull. AMS, 92)

of its properties are given there. See also [104]. Its "magical properties" can be seen as related to the Lax formula for the solution of

$$\frac{\partial w}{\partial t} - \frac{1}{2}|\nabla w|^2 = 0 \quad \text{for } x \in \mathbb{R}^N,\ t \geq 0,\ w|_{t=0} = v \text{ on } \mathbb{R}^N,$$

which is

$$w(x,t) = \sup_y \left\{ v(y) - \frac{1}{2t}|x-y|^2 \right\}.$$

Indeed, the coincidence of this solution formula and solutions produced by the method of charteristics leads to the properties used. Of course, this is a heuristic connection, since characteristic methods require too much regularity to be rigorous here.

The inf convolution can also be seen as a nonlinear analogue of the standard mollification when replacing the "linear structure of L^2 and its duality" by the "nonlinear structure of L^∞ or C." One can also interpret this analogy in terms of the so-called exotic algebra ($\mathbb{R}$, max, +).

The term "tropical" is in the honor of Imre Simon, 1943 - 2009

who lived in Sao Paulo (south tropic).

These algebras were invented by various schools in the world

- Cuninghame-Green 1960- OR (scheduling, optimization)
- Vorobyev ∼65 . . . Zimmerman, Butkovic; Optimization
- Maslov ∼ 80'- . . . Kolokoltsov, Litvinov, Samborskii, Shpiz. . . Quasi-classic analysis, variations calculus
- Simon ∼ 78- . . . Hashiguchi, Leung, Pin, Krob, . . . Automata theory
- Gondran, Minoux ∼ 77 Operations research
- Cohen, Quadrat, Viot ∼ 83- . . . Olsder, Baccelli, S.G., Akian initially discrete event systems, then optimal control, idempotent probabilities, combinatorial linear algebra
- Nussbaum 86- Nonlinear analysis, dynamical systems, also related work in linear algebra, Friedland 88, Bapat ˜94
- Kim, Roush 84 Incline algebras
- Fleming, McEneaney ∼00- max-plus approximation of HJB
- Del Moral ∼95 Puhalskii ∼99, idempotent probabilities.

now in **tropical geometry**, after Viro, Mikhalkin, Passare, Sturmfels and many.

Menu: connections between...

- tropical convexity
- dynamic programming / zero-sum games
- Perron-Frobenius theory
- metric geometry

Tropical convex sets and cones

Definition

A set C of functions $X \to \mathbb{R}_{\max}$ is a tropical convex set if $u, v \in C$, $\lambda, \mu \in \mathbb{R}_{\max}$, $\max(\lambda, \mu) = 0$ implies $\sup(\lambda + u, \mu + v) \in C$.

A tropical convex cone or semimodule is defined similarly, ommiting the requirement that $\max(\lambda, \mu) = 0$.

Semimodules are analogous both to classical convex cones and to linear spaces

They can also be defined and studied abstractly

Korbut 65, Vorobyev 65, Zimmermann 77-, Cuninghame-Green 79-, Butkovič, Hegedus 84, Helbig 88; idempotent functional analysis by Litvinov, Maslov, Samborski, Shpiz 92-; max-plus / abstract convexity Cohen, Gaubert, Quadrat 96-, Briec and Horvath 04-, Singer 04-, . . .

Tropical point of view in Develin and Sturmfels, 04-. Since that time, many works by some of the above authors and others Joswig, Santos, Yu, Ardila, Nitica, Sergeev, Schneider, Meunier, Werner. . .

the term tropical linear space is ambiguous, may refer to elements of the tropical Grassmanian of Speyer and Sturmfels which are special tropical convex cones.

Several examples of tropical convex sets

Motivations from optimal control

$$v(t,x) = \sup_{\mathbf{x}(0)=x,\ \mathbf{x}(\cdot)} \int_0^t L(\mathbf{x}(s),\dot{\mathbf{x}}(s))ds + \phi(\mathbf{x}(t))$$

Lax-Oleinik semigroup: $(S^t)_{t\geq 0}$, $S^t\phi := v(t,\cdot)$.

Superposition principle: $\forall \lambda \in \mathbb{R}$, $\forall \phi,\psi$,

$$\begin{aligned} S^t(\sup(\phi,\psi)) &= \sup(S^t\phi, S^t\psi) \\ S^t(\lambda+\phi) &= \lambda + S^t\phi \end{aligned}$$

So S^t is max-plus linear.

Motivations from optimal control

$$v(t,x) = \sup_{\mathbf{x}(0)=x,\ \mathbf{x}(\cdot)} \int_0^t L(\mathbf{x}(s),\dot{\mathbf{x}}(s))ds + \phi(\mathbf{x}(t))$$

Lax-Oleinik semigroup: $(S^t)_{t\geq 0}$, $S^t\phi := v(t,\cdot)$.

Superposition principle: $\forall\lambda\in\mathbb{R}, \forall\phi,\psi,$

$$\begin{aligned} S^t(\text{“}\phi+\psi\text{”}) &= \text{“}S^t\phi+S^t\psi\text{”} \\ S^t(\text{“}\lambda\phi\text{”}) &= \text{“}\lambda S^t\phi\text{”} \end{aligned}$$

So S^t is max-plus linear.

The function v is solution of the Hamilton-Jacobi equation

$$\frac{\partial v}{\partial t} = H(x, \frac{\partial v}{\partial x}) \qquad v(0, \cdot) = \phi$$

Max-plus linearity $\Leftrightarrow$ Hamiltonian convex in p

$$H(x,p) = \sup_u (L(x,u) + p \cdot u)$$

Hopf formula, when $L = L(u)$ concave:

$$v(t,x) = \sup_{y \in \mathbb{R}^n} tL(\frac{x-y}{t}) + \phi(y) \ .$$

The function v is solution of the Hamilton-Jacobi equation

$$\frac{\partial v}{\partial t} = H(x, \frac{\partial v}{\partial x}) \qquad v(0,\cdot) = \phi$$

Max-plus linearity $\Leftrightarrow$ Hamiltonian convex in p

$$H(x,p) = \sup_u (L(x,u) + p \cdot u)$$

Hopf formula, when $L = L(u)$ concave:

$$v(t,x) = \text{“}\int G(x-y)\phi(y)dy\text{”} \quad .$$

$$\mathcal{V}_\lambda := \{\phi \mid S^t\phi = \lambda t + \phi, \forall t > 0\}$$

is a max-plus or tropical cone in infinite dimension. The functions ϕ are the weak-KAM solutions of Fathi.

S^t is an instance of Moreau conjugacy:

$$S^t\phi(x) = \sup_y a(x,y) + \phi(y) \quad .$$

Metric geometry

(X, d) metric space.
1-Lip$:= \{u \mid u(x) - u(y) \leq d(x, y)\}$ is a tropical convex cone.
TFAE

$$u \text{ is } 1 - \mathrm{Lip}$$

$$u(y) = \max_{x \in X} -d(x, y) + u(x)$$

$$u(y) = \max_{x \in X} -d(x, y) + v(x), \quad \exists v$$

$$u \in \mathrm{Span}\{-d(x, \cdot) \mid x \in X\} \qquad \text{make picture!}$$

Question. What are the tropical extreme rays ?

We shall see they are precisely the maps $-d(x,\cdot)$, $x \in X$, together with the horofunctions associated with Busemann points (limits of infinite geodesics).

Gromov's horoboundary compactification of X.
Analogous to the probabilistic Martin boundary.
Related to results of Fathi and Maderna, Contreras, Ishii and Mitake for optimal control problems with noncompact state space.

Spaces of semiconvex functions

Fleming, McEneaney

$$C_\alpha := \{u : \mathbb{R}^n \to \mathbb{R} \mid u + \alpha\|x\|^2/2 \text{ is convex}\}$$

is a tropical convex cone.

Shapley operators

$X = \mathscr{C}(K)$, even $X = \mathbb{R}^n$; Shapley operator T,

$$T_i(x) = \max_{a \in A_i} \min_{b \in B_{i,a}} \Big(r_i^{ab} + \sum_{1 \leq j \leq n} P_{ij}^{ab} x_j \Big), \qquad i \in [n]$$

- $[n] := \{1, \ldots, n\}$ set of states
- a action of Player I, b action of Player II
- r_i^{ab} payment of Player II to Player I
- P_{ij}^{ab} transition probability $i \to j$

Shapley operators

$X = \mathscr{C}(K)$, even $X = \mathbb{R}^n$; Shapley operator T,

$$T_i(x) = \max_{a \in A_i} \min_{b \in B_{i,a}} \Big(r_i^{ab} + \sum_{1 \leq j \leq n} P_{ij}^{ab} x_j \Big), \qquad i \in [n]$$

T is order preserving and additively homogeneous:

$$x \leq y \implies T(x) \leq T(y)$$
$$T(\alpha + x) = \alpha + T(x), \qquad \forall \alpha \in \mathbb{R}$$

Shapley operators

$X = \mathscr{C}(K)$, even $X = \mathbb{R}^n$; Shapley operator T,

$$T_i(x) = \max_{a \in A_i} \min_{b \in B_{i,a}} \Big(r_i^{ab} + \sum_{1 \leq j \leq n} P_{ij}^{ab} x_j \Big), \qquad i \in [n]$$

Conversely, any order preserving additively homogeneous operator is a Shapley operator (Kolokoltsov), even with degenerate transition probabilities (deterministic) Gunawardena, Sparrow; Singer, Rubinov,

$$T_i(x) = \sup_{y \in \mathbb{R}} \Big(T_i(y) + \min_{1 \leq i \leq n} (x_i - y_i) \Big)$$

Variant. T is additively subhomogeneous if

$$T(\alpha + x) \leq \alpha + T(x), \qquad \forall \alpha \in \mathbb{R}_+$$

This corresponds to $1 - \sum_j P_{ij}^{ab} =$ death probability > 0.

Order-preserving + additively (sub)homogeneous $\implies$ sup-norm nonexpansive

$$\| T(x) - T(y)\|_\infty \leq \|x - y\|_\infty \ .$$

Variant. T is additively subhomogeneous if

$$T(\alpha + x) \leq \alpha + T(x), \qquad \forall \alpha \in \mathbb{R}_+$$

This corresponds to $1 - \sum_j P_{ij}^{ab} =$ death probability > 0.

Order-preserving + additively homogeneous $\iff$ top nonexpansive

$$\mathrm{t}(T(x) - T(y)) \leq \mathrm{t}(x - y), \qquad \mathrm{t}(z) := \max_i z_i \quad .$$

Variant. T is additively subhomogeneous if

$$T(\alpha + x) \leq \alpha + T(x), \qquad \forall \alpha \in \mathbb{R}_+$$

This corresponds to $1 - \sum_j P_{ij}^{ab} =$ death probability > 0.

Order-preserving + additively subhomogeneous $\iff$ top$^+$ nonexpansive

$$\mathrm{t}^+(T(x) - T(y)) \leq \mathrm{t}^+(x - y), \quad \mathrm{t}^+(z) := \max(\max_i z_i, 0) \ .$$

If T is order preserving and additively homogeneous, then the set of subsolutions

$$C = \{u \mid T(u) \geq u\}$$

(showing the game is superfair) is a tropical (max-plus) convex cone.

Similarly, if S^t is the semigroup of the Isaacs equation

$$v_t - H(x, Dv, D^2 v) = 0, \qquad H(x, p, \cdot) \text{ order preserving}$$

S^t is order preserving and additively homogeneous

$$C = \{u \mid S^t u \geq u, \ \forall t \geq 0\}$$

is a tropical convex cone.

Supersolutions constitute a min-plus convex cone.

Discounted case = tropical convex sets

If T is only order preserving and additively subhomogeneous

$$C = \{u \mid T(u) \geq u\}$$

is a tropical (max-plus) convex set.
Proof. If $u, v \in C, \beta \in \mathbb{R}_+$

$$\begin{aligned} T(\sup(u, -\beta + v)) &\geq \sup(T(u), T(-\beta + v)) \\ &\geq \sup(T(u), -\beta + T(v)) \\ &\geq \sup(u, -\beta + v) \end{aligned}$$

Population dynamics = games with exponential glasses

K closed convex pointed cone in a Banach space, say $K = \mathbb{R}^n_+$.

$$x \leq y \implies F(x) \leq F(y)$$

$$F(\lambda x) = \lambda F(x), \qquad \lambda > 0$$

If $K = \mathbb{R}^n_+$, or $\mathcal{C}(X)$, then:

$$T(x) = \log \circ F \circ \exp$$

is a Shapley operator.

Example: Perron-Frobenius $\subset$ stochastic control

$$F(X) = MX, \qquad M_{ij} \geq 0$$

$$x = \log X$$

$$T(x) = \sup_P \left(Px - S(P; M)\right)$$

where the sup is taken over the set of stochastic matrices, and S is the relative entropy

$$S_i(P; M) = \sum_j P_{ij} \log(P_{ij}/M_{ij})$$

The Perron eigenvector

$$F(U) = \mu U, \qquad U \in \operatorname{int} \mathbb{R}_+^n$$

corresponds to the additive eigenvector $u = \log U$,

$$\begin{aligned} T(u) &= \lambda + u, \qquad \lambda = \log \mu \\ &= \sup_P \left(Pu - S(P; M) \right) . \end{aligned}$$

So, the log of the Perron root μ is

$$\log \rho(M) = \sup_{P,m} -m \cdot S(P; M)$$

where the sup is over all invariant measures m of P.

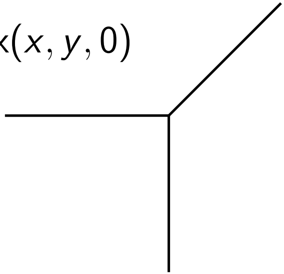
Some elementary tropical geometry

A tropical line in the plane is the set of (x, y) such that the max in

$$“ax + by + c”$$

is attained at least twice.

$\max(x, y, 0)$



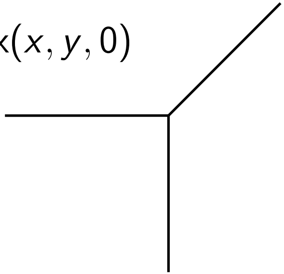
Some elementary tropical geometry

A tropical line in the plane is the set of (x, y) such that the max in

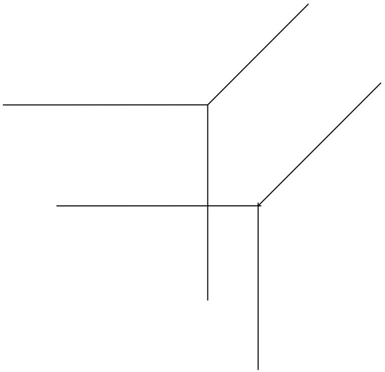
$$\max(a + x, b + y, c)$$

is attained at least twice.

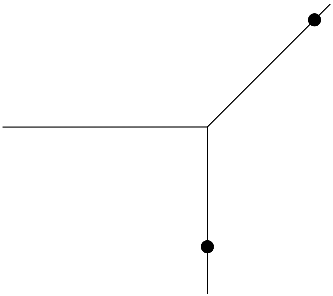
$\max(x, y, 0)$



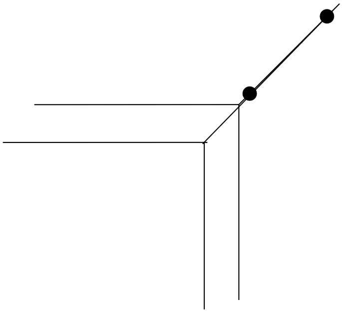
Two generic tropical lines meet at a unique point



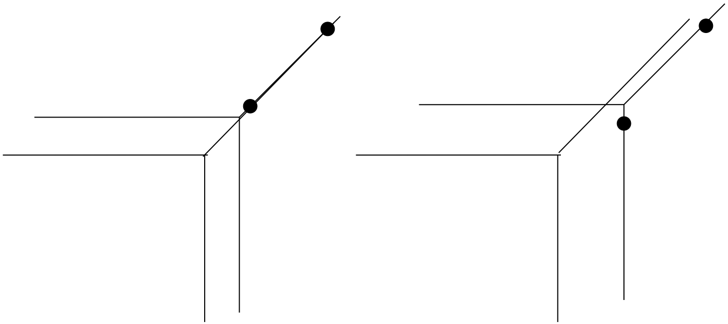
By two generic points passes a unique tropical line



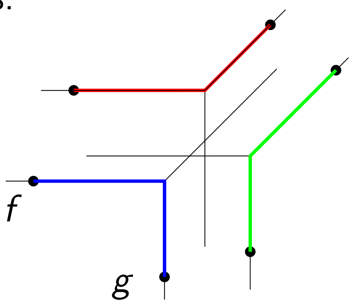
non generic case



non generic case resolved by perturbation



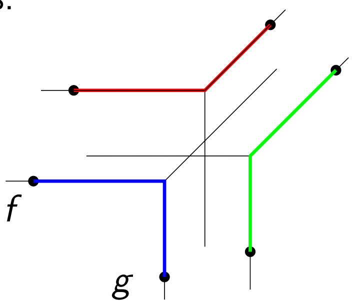
Tropical segments:



$$[f,g] := \{\text{“}\lambda f + \mu g\text{”} \mid \lambda,\mu \in \mathbb{R}\cup\{-\infty\},\ \text{“}\lambda+\mu = 1\text{”}\}.$$

(The condition “$\lambda, \mu \geq 0$” is automatic.)

Tropical segments:

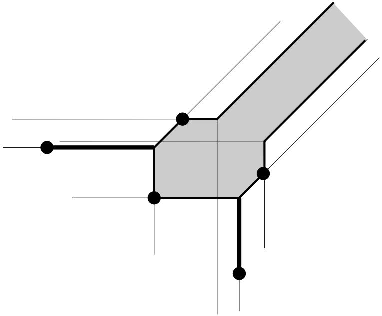


$[f,g] := \{\sup(\lambda + f, \mu + g) \mid \lambda, \mu \in \mathbb{R} \cup \{-\infty\},\ \max(\lambda,\mu) = 0\}.$

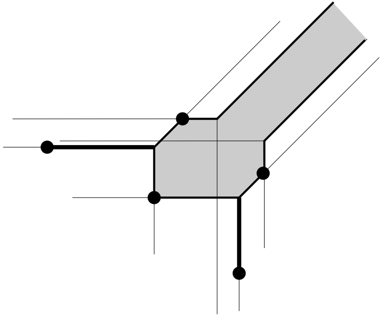
(The condition $\lambda, \mu \geq -\infty$ is automatic.)

Exercise: draw a convex set.

Tropical convex set: $f, g \in C \implies [f, g] \in C$



Tropical convex set: $f, g \in C \implies [f, g] \in C$



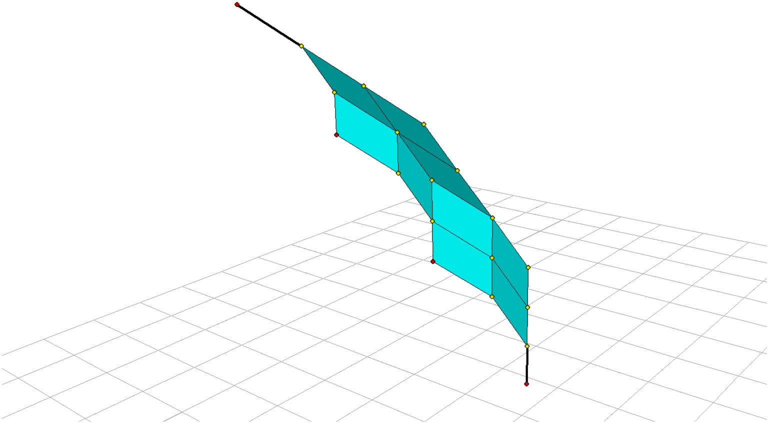
Tropical convex cone: ommit "$\lambda + \mu = 1$", i.e., replace $[f, g]$ by $\{\sup(\lambda + f, \mu + g) \mid \lambda, \mu \in \mathbb{R} \cup \{-\infty\}\}$

Homogeneization

A convex set C in $\mathbb{R}^n_{\max}$ corresponds to a convex cone $\hat{C}$ in $\mathbb{R}^{n+1}_{\max}$,

$$\hat{C} := \{(u, \lambda + u) \mid u \in C, \lambda \in \mathbb{R}_{\max}\}$$

A max-plus “tetrahedron”?

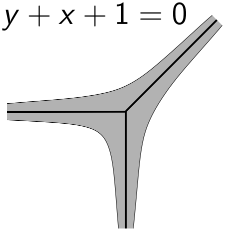


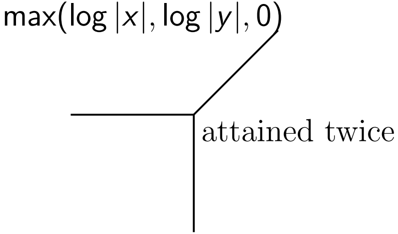
The previous drawing was generated by POLYMAKE of Gawrilow and Joswig, in which an extension allows one to handle easily tropical polyhedra. They were drawn with JAVAVIEW. See Joswig arXiv:0809.4694 for more information.

Why?

Gelfand, Kapranov, and Zelevinsky defined the amoeba of an algebraic variety $V \subset (\mathbb{C}^*)^n$ to be the "log-log plot"

$$A(V) := \{(\log|z_1|, \dots, \log|z_n|) \mid (z_1, \dots, z_n) \in V\} \ .$$

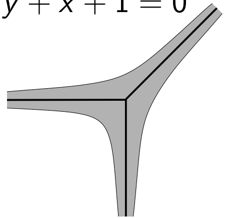




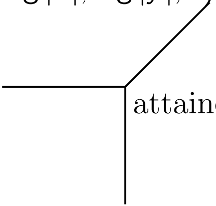
Gelfand, Kapranov, and Zelevinsky defined the amoeba of an algebraic variety $V \subset (\mathbb{C}^*)^n$ to be the “log-log plot”

$$A(V) := \{(\log|z_1|, \ldots, \log|z_n|) \mid (z_1, \ldots, z_n) \in V\} \ .$$

$y + x + 1 = 0$



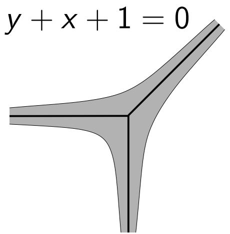
$\max(\log|x|, \log|y|, 0)$

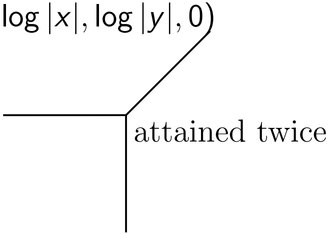


$|y| \leq |x| + 1$, $|x| \leq |y| + 1$, $1 \leq |x| + |y|$

Gelfand, Kapranov, and Zelevinsky defined the amoeba of an algebraic variety $V \subset (\mathbb{C}^*)^n$ to be the "log-log plot"

$$A(V) := \{(\log|z_1|, \ldots, \log|z_n|) \mid (z_1, \ldots, z_n) \in V\} \ .$$



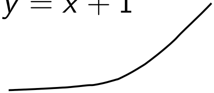


$X := \log|x|$, $Y := \log|y|$

$Y \leq \log(e^X + 1)$, $X \leq \log(e^Y + 1)$, $1 \leq e^X + e^Y$

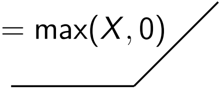
real tropical lines

$y = x + 1$



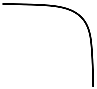
$X = \log(e^X + 1)$

$Y = \max(X, 0)$



real tropical lines

$x + y = 1$

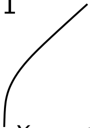


$\log(e^X + e^Y) = 1$

$\max(X, Y) = 0$

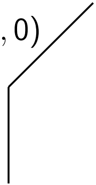
real tropical lines

$x = y + 1$



$X = \log(e^X + 1)$

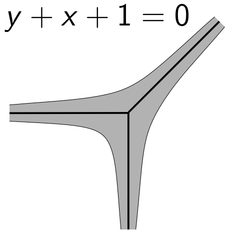
$X = \max(Y, 0)$

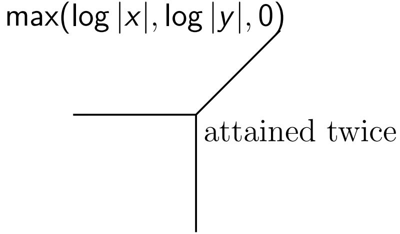


Viro's log-glasses, related to Maslov's dequantization

$$a +_h b := h \log(e^{a/h} + e^{b/h}), \qquad h \to 0^+$$

With h-log glasses, the amoeba of the line retracts to the tropical line as $h \to 0^+$





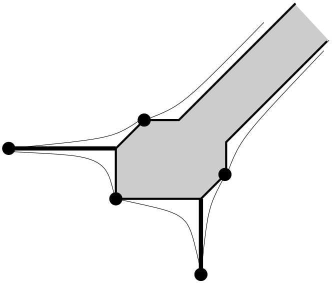
$$\max(a, b) \le a +_h b \le h \log 2 + \max(a, b)$$

Similar to convergence of p-norm to sup-norm

$$[a,b] := \{\lambda a +_p \mu b \mid \lambda,\mu \geq 0, \lambda +_p \mu = 1\}$$

$$a +_p b = (a^p + b^p)^{1/p}$$

The convex hull in the $+_h$ / $+_p$ sense converges to the tropical convex hull as $h \to 0$ / $p \to \infty$ (Briec and Horvath).



See Passare & Rullgard, Duke Math. 04 for more information on amoebas

Introduction to amoebas: lecture notes by Yger.

All the results of classical convexity have tropical analogues, sometimes more degenerate...

- generation by extreme points Helbig; SG, Katz 07; Butkovič, Sergeev, Schneider 07; Choquet Akian, SG, Walsh 09, Poncet 11 infinite dim.
- projection / best-approximation : Cohen, SG, Quadrat 01,04; Singer
- Hahn-Banach analytic Litvinov, Maslov, Shpiz 00; Cohen, SG, Quadrat 04; geometric Zimmermann 77, Cohen, SG, Quadrat 01,05; Develin, Sturmfels 04, Joswig 05
- cyclic projections Butkovic, Cuninghame-Green TCS03; SG, Sergeev 06
- Radon, Helly, Carathéodory, Colorful Carathéodory, Tverberg: SG, Meunier DCG09

This lecture

Tropical convexity is equivalent to dynamic programming (zero-sum games).

This lecture

Tropical convexity is equivalent to dynamic programming (zero-sum games).

- finite dimensional convex sets (cones) $\sim$ stochastic games with finite state spaces

This lecture

Tropical convexity is equivalent to dynamic programming (zero-sum games).

- finite dimensional convex sets (cones) $\sim$ stochastic games with finite state spaces
- leads to: equivalence (computational complexity) results, algorithms, approximation methods, ...

Some results and techniques . . .

Residuation / Galois correspondences in lattices

Let $A \in M_{dp}(\mathbb{R}_{\max})$. Then, for $x \in \mathbb{R}^p_{\max}$ and $b \in \mathbb{R}^d_{\max}$,

$$Ax \leq b \iff x \leq A^\sharp b$$

where

$$(A^\sharp b)_j = \min_{1 \leq i \leq d} -A_{ij} + b_j, \qquad 1 \leq j \leq p$$

$$AA^\sharp A = A \qquad A^\sharp A A^\sharp = A^\sharp$$

The row and column spaces of A are anti-isomorphic semi-lattices, $x \mapsto (A(-x))^T$, $y \mapsto ((-y)A)^T$, general residuation result (infinite dim OK, Cohen, SG, Quadrat 01,04), different proof by Develin and Sturmfels, 04.

The tropical sesquilinear form

$$x/v := \max\{\lambda \mid \text{“}\lambda v\text{”} \leq x\}$$
$$= \min_i (x_i - v_i) \quad \text{if } x, v \in \mathbb{R}^n \ .$$

$$\delta(x,y) = \text{“}(x/y)(y/x)\text{”} = \min_i (x_i - y_i) + \min_j (y_j - x_j)$$

$d = -\delta$ is the (additive) Hilbert's projective metric

$$d(x,y) = \|x - y\|_H, \quad \|z\|_H := \max_{1\leq i\leq d} z_i - \min_{1\leq i\leq d} z_i \ .$$

Projection on a tropical cone

If $C \subset \mathbb{R}^d_{\max}$ is a tropical convex cone stable by sups (closed in Scott topology -non-Haussdorf-):

$$P_C(x) = \max\{v \in C \mid v \leq x\}$$
$$= \max_{u \in U} (x/u) + u \ .$$

for any generating set U of C.

Compare with

$$P_C(x) = \sum_{u \in U} \langle x, u \rangle u$$

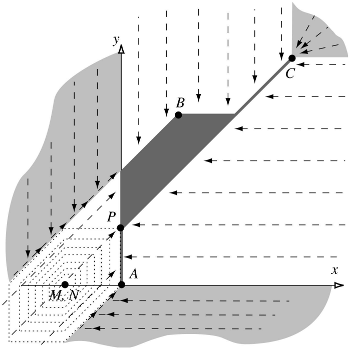
if U is a Hilbert base of a Hilbert space.

When $C = \mathrm{Col}(A)$, $P_C(x) = AA^\sharp x$.

Best approximation in Hilbert's projective metric

Prop.(Cohen, SG, Quadrat, in Bensoussan Festschrift 01)

$$d(x, P_{\mathcal{V}}(x)) = \min_{y \in \mathcal{V}} d(x, y) \ .$$


y
x
A
B
C
P
M, N

Separation

Goes back to Zimmermann 77, simple geometric construction in Cohen, SG, Quadrat in Ben01, LAA04.
C closed linear cone of $\mathbb{R}^d_{\max}$, or complete semimodule
If $y \notin C$, then, the tropical half-space

$$\mathcal{H} := \{v \mid y/v \leq P_C(y)/v\}$$

contains C and not y.
Compare with the optimality condition for the projection on a convex cone C: $\langle y - P_C(y), v\rangle \leq 0, \forall v \in C$

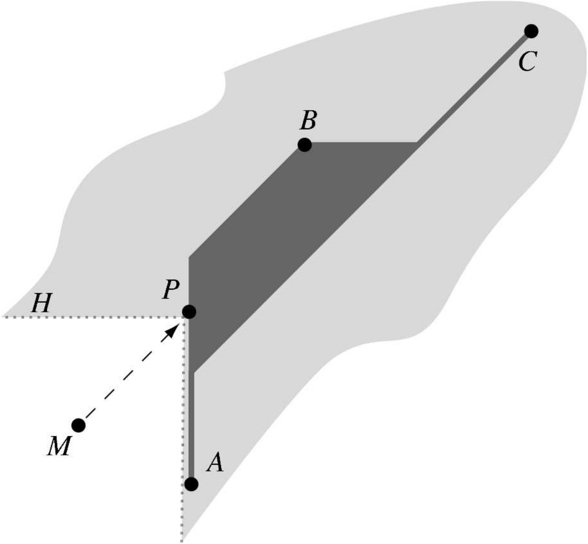
Separation

Goes back to Zimmermann 77, simple geometric construction in Cohen, SG, Quadrat in Ben01, LAA04.
C closed linear cone of $\mathbb{R}^d_{\max}$, or complete semimodule
If $y \not\in C$, then, the tropical half-space

$$\mathcal{H} := \{v \mid y/v \leq P_C(y)/v\}$$

contains C and not y.
Let $\bar{y} := P_C(y)$ and $I := \{i \mid y_i = \bar{y}_i\}$. Then,

$$\mathcal{H} = \{v \mid \max_{i \in I^c} v_i - \bar{y}_i \leq \max_{i \in I} v_i - \bar{y}_i\}$$


C
B
P
H
M
A

Tropical half-spaces

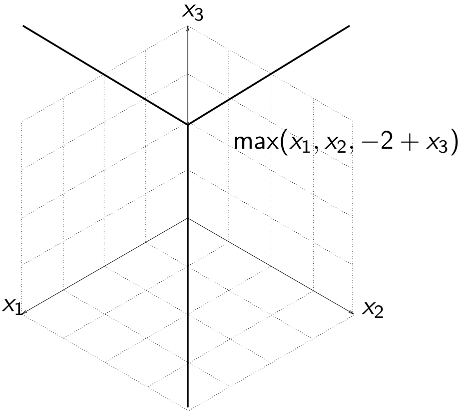
Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

$$H := \{x \in \mathbb{R}^n_{\max} \mid \text{“}ax \leq bx\text{”}\}$$

Tropical half-spaces

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

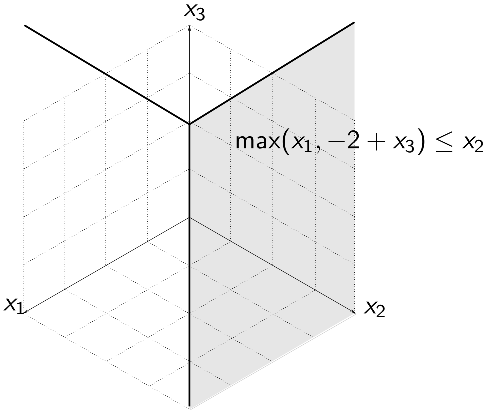
$$H := \{x \in \mathbb{R}^n_{\max} \mid \max_{1\leq i\leq n} a_i + x_i \leq \max_{1\leq i\leq n} b_i + x_i\}$$



Tropical half-spaces

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

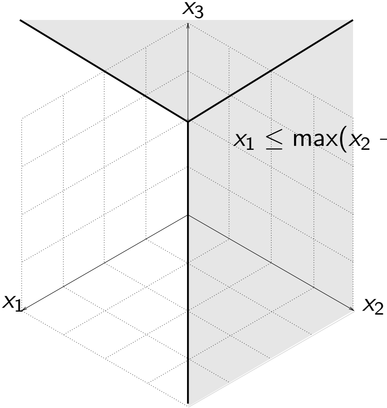
$$H := \{x \in \mathbb{R}^n_{\max} \mid \max_{1 \leq i \leq n} a_i + x_i \leq \max_{1 \leq i \leq n} b_i + x_i\}$$



Tropical half-spaces

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

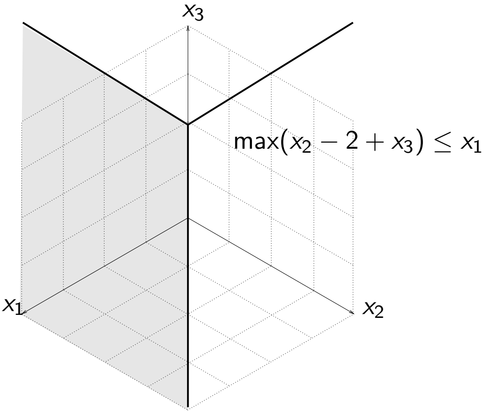
$$H := \{x \in \mathbb{R}^n_{\max} \mid \max_{1 \le i \le n} a_i + x_i \le \max_{1 \le i \le n} b_i + x_i\}$$



Tropical half-spaces

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

$$H := \{x \in \mathbb{R}^n_{\max} \mid \max_{1 \le i \le n} a_i + x_i \le \max_{1 \le i \le n} b_i + x_i\}$$



A halfspace can always be written as:

$$\max_{i\in I} a_i + x_i \leq \max_{j\in J} b_j + x_j, \qquad I \cap J = \varnothing \ .$$

Apex: $v_i := -\max(a_i, b_i)$.

If $v \in \mathbb{R}^n$, H is the union of sectors of the tropical hyperplane with apex v:

$$\max_{1\leq i\leq n} x_i - v_i \qquad \text{attained twice}$$

Halfspaces appeared in: Joswig 04; Cohen, Quadrat SG 00; Zimmermann 77, ...

Corollary (Zimmermann; Samborski, Shpiz; Cohen, SG, Quadrat, Singer; Develin, Sturmfels; Joswig...)

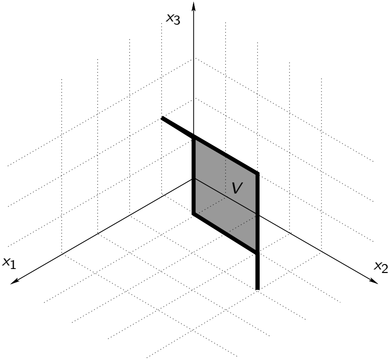
A tropical convex cone closed (in the Euclidean topology) is the intersection of tropical half-spaces.

$\mathbb{R}_{\max}$ is equipped with the topology of the metric $(x, y) \mapsto \max_i |e^{x_i} - e^{y_i}|$ inherited from the Euclidean topology by log-glasses.

The apex $-P_C(y)$ of the algebraic separating half-space $\mathcal{H}$ above may have some $+\infty$ coordinates, and therefore may not be closed in the Euclidean topology (always Scott closed). The proof needs a perturbation argument, this is where the assumption that C is closed (and not only stable by arbitrary sups = Scott closed) is needed.

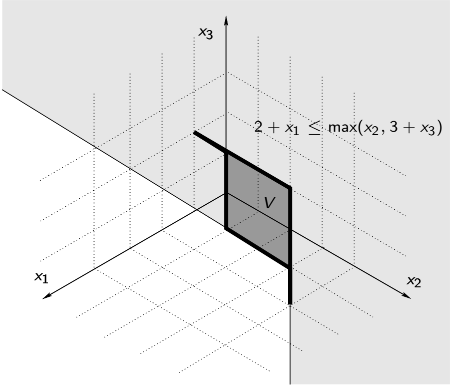
Tropical polyhedral cones

can be defined as intersections of finitely many half-spaces



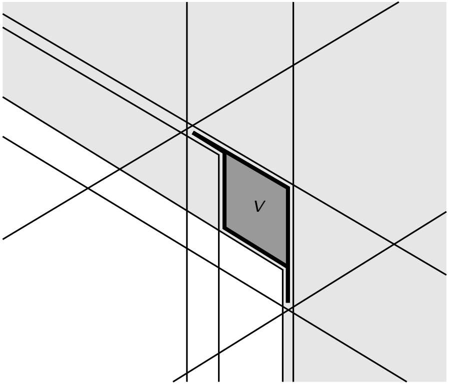
Tropical polyhedral cones

can be defined as intersections of finitely many half-spaces



Tropical polyhedral cones

can be defined as intersections of finitely many half-spaces



The Equivalence between tropical convexity and games. . .

Based on Akian, SG, Guterman arXiv:0912.2462 to appear in IJAC

Theorem (Equivalence, part I; Akian, SG, Guterman arXiv:0912.2462 → IJAC)

TFAE

- *C is a closed tropical convex cone*
- $C = \{u \mid u \leq T(u)\}$ *for some Shapley operator* T.

Recall $C \subset (\mathbb{R} \cup \{-\infty\})^n$ is a tropical convex cone if

$$u, v \in C, \lambda \in \mathbb{R} \cup \{-\infty\} \implies \sup(u, v) \in C, \quad \lambda + u \in C \ .$$

The Shapley operator $T : \mathbb{R}^n \to \mathbb{R}^n$ extends continuously $\mathbb{R}^n_{\max} \to \mathbb{R}^n_{\max}$,

$$T(x) = \inf_{y \geq x,\ y \in \mathbb{R}^n} T(y) \ .$$

Easy implication: T order preserving and additively homogeneous $\implies$ $\{u \mid u \leq T(u)\}$ is a closed tropical convex cone

Recall $C \subset (\mathbb{R} \cup \{-\infty\})^n$ is a tropical convex cone if

$$u, v \in C, \lambda \in \mathbb{R}\cup\{-\infty\} \implies \sup(u, v) \in C, \quad \lambda + u \in C \quad .$$

The Shapley operator $T : \mathbb{R}^n \to \mathbb{R}^n$ extends continuously $\mathbb{R}^n_{\max} \to \mathbb{R}^n_{\max}$,

$$T(x) = \inf_{y \geq x,\, y \in \mathbb{R}^n} T(y) \quad .$$

Easy implication: T order preserving and additively homogeneous $\implies$ $\{u \mid u \leq T(u)\}$ is a closed tropical convex cone
Remark: $\{u \mid u \geq T(u)\}$ is a dual tropical (min-plus) cone.

Conversely, any closed tropical convex cone can be written as

$$C = \bigcap_{i \in I} H_i$$

where $(H_i)_{i \in I}$ is a family of tropical half-spaces.

$$H_i : \text{"}A_i x \leq B_i x\text{"}$$

Conversely, any closed tropical convex cone can be written as

$$C = \bigcap_{i \in I} H_i$$

where $(H_i)_{i \in I}$ is a family of tropical half-spaces.

$$H_i : \max_{1 \leq j \leq n} a_{ij} + x_j \leq \max_{1 \leq k \leq n} b_{ik} + x_k , \qquad a_{ij}, b_{ij} \in \mathbb{R} \cup \{-\infty\}$$

$$[T(x)]_j = \inf_{i \in I} -a_{ij} + \max_{1 \leq k \leq n} b_{ik} + x_k \quad .$$

Conversely, any closed tropical convex cone can be written as

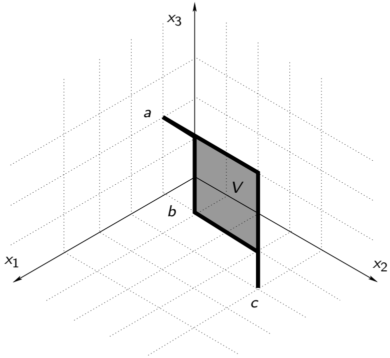
$$C = \bigcap_{i \in I} H_i$$

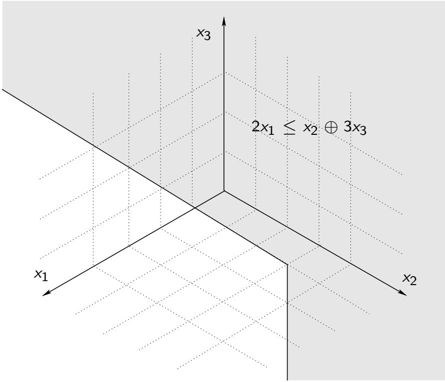
where $(H_i)_{i \in I}$ is a family of tropical half-spaces.

$$H_i : \max_{1 \le j \le n} a_{ij} + x_j \le \max_{1 \le k \le n} b_{ik} + x_k , \qquad a_{ij}, b_{ij} \in \mathbb{R} \cup \{-\infty\}$$

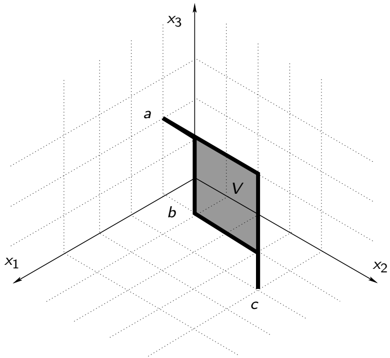
$$[T(x)]_j = \inf_{i \in I} -a_{ij} + \max_{1 \le k \le n} b_{ik} + x_k \ .$$

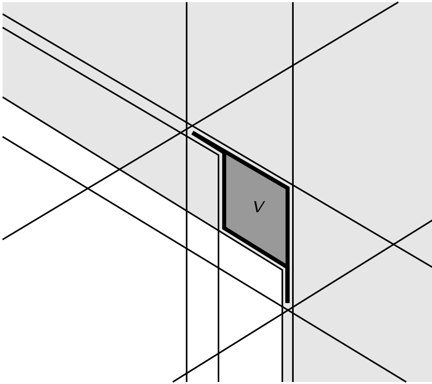
$$x \le T(x) \iff \max_{1 \le j \le n} a_{ij} + x_j \le \max_{1 \le k \le n} b_{ik} + x_k , \ \forall i \in I \ .$$





$$2 + x_1 \leq \max(x_2, 3 + x_3)$$





$$2 + x_1 \leq \max(x_2, 3 + x_3)$$

$$H_i : \max_{1\leq j\leq n} a_{ij} + x_j \leq \max_{1\leq k\leq n} b_{ik} + x_k$$

$$[T(x)]_j = \inf_{i\in I} -a_{ij} + \max_{1\leq k\leq n} b_{ik} + x_k \quad .$$

Interpretation of the game

- State of MIN: variable x_j, $j \in \{1, \dots, n\}$
- State of MAX: half-space H_i, $i \in I$
- In state x_j, Player MIN chooses a tropical half-space H_i with x_j in the LHS
- In state H_i, player MAX chooses a variable x_k at the RHS of H_i
- Payment $-a_{ij} + b_{ik}$.

Menu of the next lectures

- The mean payoff problem for repeated games
- Generalized Denjoy-Wolff theorem
- Deformation of Perron-Frobenius theory
- More combinatorics
- Extreme points of tropical polyhedra, Max-plus Martin Boundary
- Algorithms

Thank you!